ON DOUBLE FOURIER SERIES WITH RESPECT TO THE CLASSICAL REARRANGEMENTS OF THE WALSH–PALEY SYSTEM

ROSTOM GETSADZE

Abstract. The following theorem is established: there exists a continuous function on $[0, 1]^2$ with a certain smoothness, whose double Fourier-Walsh series diverges by rectangles on a set of positive measure. Similar theorem is true also for the double Walsh–Kaczmarz system.

1. Introduction

There are two classical rearrangements of the Walsh–Paley system: (a) the Walsh system and (b) the Walsh–Kaczmarz system. It is well-known (see [3, 4]) these systems are systems of convergence. The system of Rademacher functions $\{r_n(x)\}_{n=0}^{\infty}$ on $[0, 1)$ is defined as follows. Set

$$r_0(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} \leq x < 1. \end{cases}$$

We extend the function $r_0(x)$ on $(-\infty, \infty)$ with period 1. For $n \geq 1$, we set $r_n(x) = r_0(2^n x)$.

For each $k \in N = \{0, 1, 2, \ldots\}$, we introduce a function $\alpha_k : [0, 1) \rightarrow \{0, 1\}$ defined by the dyadic expansion of $x$

$$x = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{2^{k+1}}.$$

If $x$ is a dyadic rational, then we suppose that its dyadic expansion contains infinitely many zeros.

The Walsh–Paley system of functions $\{W_n(x)\}_{n=0}^{\infty}$ on $[0, 1)$ is defined as follows. Set $W_0(x) = 1$ for all $x \in [0, 1)$. For $n \geq 1$, we consider the dyadic representation $n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_q}$, $(n \geq 1, m_1 > m_2 \cdots > m_q \geq 0)$ and set

$$W_n(x) = r_{m_1}(x) r_{m_2}(x) \cdots r_{m_q}(x) \quad x \in [0, 1).$$

The modulus of continuity $\omega(F; \delta)$ of a continuous function $F$ on $[0, 1]^2$ is defined by

$$\omega(F; \delta) = \sup_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \delta} \|F(x_1, y_1) - F(x_2, y_2)\|, (x_1, y_1), (x_2, y_2) \in [0, 1]^2.$$

Recently [2], we have proved the following

Theorem 1. There exists a continuous function $F$ on $[0, 1]^2$ such that

$$\omega(F; \delta) = O \left( \frac{1}{\sqrt{\delta \log_2 \frac{1}{\delta}}} \right), \quad \delta \rightarrow 0+,$$

and the Fourier series of $F$ with respect to the double Walsh–Paley system $\{W_m(x)W_n(y)\}_{m,n=0}^{\infty}$ diverges on a set of positive measure by rectangles.

2020 Mathematics Subject Classification. 42C10.

Key words and phrases. Walsh system; Rearrangements; Double Fourier series.
The Walsh system \( \{ \varphi_m(x) \}_{m=0}^{\infty} \) was introduced by Walsh (see, e.g., [4]) and defined as follows:

\[
\varphi_0(x) = 1, \quad \varphi_1(x) = \varphi_{2n}(x) = (-1)^{\alpha_{n-1}(x)+\alpha_n(x)}, \quad \varphi_{2n+k}(x) = \varphi_{2n}(x)\varphi_k(x), \quad k = 0, 1, \ldots, 2^n - 1; \quad n = 0, 1, \ldots,
\]

To define the Walsh–Kaczmarz system \( \{ h_m(x) \}_{m=1}^{\infty} \), we first introduce an auxiliary system of functions

\[
\psi_{n,i}(x) = r_{n-j_1-1}(x)r_{n-j_2-1}(x)\ldots r_{n-j_p-1}(x), \quad x \in [0, 1),
\]

where \( n, i \in \mathbb{N} \), \( 2 \leq i \leq 2^n, n \geq 1 \) and

\[
i - 1 = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_p},
\]

with \( j_1 > j_2 > \cdots > j_p \geq 0 \), is the dyadic expansion of the integer \( i - 1 \).

For \( i = 1 \) and \( n \geq 1 \), we set

\[
\psi_{n,1}(x) = 1, \quad x \in [0, 1).
\]

The Walsh–Kaczmarz system \( \{ h_m(x) \}_{m=1}^{\infty} \) on \( [0, 1) \) is defined as follows:

\[
h_1(x) = 1 \quad \text{and} \quad h_2(x) = r_0(x), \quad x \in [0, 1).
\]

For \( m = 2^n + i, n \geq 1, 1 \leq i \leq 2^n \), we set

\[
h_m(x) = h_{2^n+i}(x) = \psi_{n,i}(x)r_n(x), \quad x \in [0, 1).
\]

We establish the following two theorems.

**Theorem 2.** There exists a continuous function \( G \) on \([0, 1]^2\) such that

\[
\omega(G; \delta) = O \left( \frac{1}{\sqrt{\log_2 \delta}} \right), \quad \delta \to 0+,
\]

and the Fourier series of \( G \) with respect to the double Walsh system \( \{ \varphi_m(x)\varphi_n(y) \}_{m,n=0}^{\infty} \) diverges on a set of positive measure by rectangles.

**Theorem 3.** There exists a continuous function \( H \) on \([0, 1]^2\) such that

\[
\omega(H; \delta) = O \left( \frac{1}{\sqrt{\log_2 \delta}} \right), \quad \delta \to 0+,
\]

and the Fourier series of \( H \) with respect to the double Walsh–Kaczmarz system \( \{ h_m(x)h_n(y) \}_{m,n=1}^{\infty} \) diverges on a set of positive measure by rectangles.

A weaker result than Theorem 3 has been proved by us in [1].

**Acknowledgement**

The presented work was supported by grant No. DI-18-118 of the Shota Rustaveli National Science Foundation of Georgia.

**References**


(Received 11.12.2019)

**Department of Mathematics, Uppsala University, Box 480, 751 06 UPPSALA, Sweden**

*E-mail address:* rostom.getsadze@math.uu.se; rostom.getsadze@telia.com