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NORM CONTINUITY AND COMPACTNESS PROPERTIES FOR SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

SAIFEDDINE GHNIMI

Abstract. In this work, we study the norm continuity and compactness properties to the solution operator for some partial functional integrodifferential equations. The results are established by using the resolvent operator theory suggested by Grimmer in [11].

1. Introduction

The purpose of this paper is to establish some properties of a solution operator for the following partial functional integrodifferential equations with a finite delay

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t B(t-s)\,u(s)ds + L(u_t) \quad \text{for} \quad t \geq 0, \\ u_0 = \varphi \in C = C\big([-r,0];X\big), \end{cases} \tag{1.1}$$

where $A : D(A) \to X$ is a closed linear operator on a Banach space X, for $t \geq 0$, $B(t)$ is a closed time-independent linear operator on X with domain $D(B) \supset D(A)$, L is a linear bounded operator from $C\big([-r,0];X\big)$ to X. $C([-r,0];X)$ is the Banach space of all continuous functions from $[-r,0]$ to X endowed with the uniform norm topology. For $u \in C\big([-r,+\infty),X\big)$ and for every $t \geq 0$, u_t denotes the history function of C defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for} \quad \theta \in [-r,0].$$

The theory of partial functional integrodifferential equations has been emerging as an important area of investigation in recent years. Many physical and biological models are represented by this class of equations. As a model, one may take the equation arising in the study of heat conductivity in materials with memory [14],

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} w(t,\xi) + \displaystyle\int_0^t h(t-s)\,\dfrac{\partial^2}{\partial \xi^2} w(s,\xi)ds \\ \qquad\qquad + \displaystyle\int_{-r}^0 F\big(w(t+\theta,\xi)\big)d\theta \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \xi \in [0,\pi], \\ w(t,0) = w(t,\pi) = 0 \qquad\qquad \text{for} \quad t \geq 0, \\ w(\theta,\xi) = w_0(\theta,\xi) \qquad\qquad \text{for} \quad \theta \in [-r,0] \quad \text{and} \quad \xi \in [0,\pi], \end{cases} \tag{1.2}$$

where r is a positive number, F, h are two continuous functions and w_0 is a given initial function. Other models arising in viscoelasticity and reaction diffusion problems are given in [4,5,12].

In [15], the authors considered equation (1.1) for $B = 0$. They established some results concerning the existence and stability, and the solutions are studied as a semigroup operator on $C([-r,0];X)$. Due to the importance of this semigroup operator, able to give some information on the stability and growth rate of solutions, many authors studied its properties. The works of Hale [13] for ordinary linear functional differential equations, Webb [16] for ordinary nonlinear functional differential equations, Wu

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[17] and Adimy et al. [1] for partial functional differential equations are worth mentioning. Recently, in [8], the authors established many results on the existence of solutions for equation (1.1). The solutions are studied via the variation of constant formula by using resolvent operators. Similarly, many works have been established in this direction; we refer to [9, 10]. However, the properties of the solution operator for equation (1.1) is an untreated topic and this is the main motivation of the present paper.

In this paper we use the theory of resolvent operators as developed by Grimmer [11] to define the solution operator $(V(t))_{t\geq 0}$ on $C([-r,0];X)$ which solves equation (1.1) in a mild sense (see Section 3). We then show the norm continuity and compactness properties of the solution operator. Our approach and results generalize some results for differential equations ($B=0$). See, for example, [13, 15, 17].

2. Resolvent operators

Throughout this work, we make the following assumptions:

(**H1**) A is a closed densely defined linear operator in a Banach space $\big(X, |\cdot|\big)$. Since A is closed, $D(A)$ equipped with the graph norm $\|x\| := |Ax| + |x|$ is a Banach space which is denoted by $\big(Y, \|\cdot\|\big)$.

(**H2**) $(B(t))_{t\geq 0}$ is a family of linear operators on X such that $B(t)$ is continuous from Y into X for almost all $t \geq 0$. Moreover, there is a locally integrable function $b : \mathbb{R}^+ \to \mathbb{R}^+$ so that $B(t)y$ is measurable and $\big|B(t)y\big| \leq b(t)\|y\|$ for all $y \in Y$ and $t \geq 0$.

(**H3**) For any $y \in Y$, the map $t \to B(t)y$ belongs to $W^{1,1}_{loc}(\mathbb{R}^+, X)$ and

$$\Big|\frac{d}{dt}B(t)y\Big| \leq b(t)\|y\| \quad \text{for} \quad y \in Y \quad \text{and a.e.} \quad t \in \mathbb{R}^+.$$

(**H4**) L is a linear bounded operator from $C\big([-r,0];X\big)$ to X.

Now, we consider the following integrodifferential equation

$$\begin{cases} y'(t) = Ay(t) + \int_0^t B(t-s)\, y(s)\, ds \quad \text{for} \quad t \geq 0 \\ y(0) = y_0 \in X. \end{cases} \tag{2.3}$$

Definition 2.1 ([11]). A resolvent operator for equation (2.3) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$ having the following properties:

(a) $R(0) = I$ and $|R(t)| \leq Me^{\beta t}$ for some constants M and β.

(b) For each $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.

(c) $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y$, $R(\cdot)x \in C^1(\mathbb{R}^+;X) \cap C(\mathbb{R}^+;Y)$ and

$$\begin{aligned} R'(t)x &= A\,R(t)\,x + \int_0^t B(t-s)\,R(s)\,x\,ds \\ &= R(t)\,A\,x + \int_0^t R(t-s)\,B(s)\,x\,ds \quad \text{for} \quad t \geq 0. \end{aligned}$$

For the properties of resolvent operators, we refer the reader to the papers [3, 11]. The following theorem gives an existence result of the resolvent operator for equation (2.3).

Theorem 2.2 ([6]). *Assume that* (**H1**)–(**H3**) *hold. Then equation* (2.3) *admits a resolvent operator if and only if A generates a C_0-semigroup.*

From now, we suppose that

(**H5**) A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on the Banach space X.

Remark 2.3. It is worth noting that assumption (**H5**) ensures the existence of a resolvent operator for equation (2.3). This is a direct consequence of Theorem 2.2.

Lemma 2.4 ([6])**.** *Assume that* (**H1**)–(**H3**) *and* (**H5**) *hold. Then for all* $a > 0$ *there exists a constant* $H = H(a)$ *such that*

$$\big|R(s+h) - R(h)\,R(s)\big| \leq Hh \quad \text{for} \quad 0 < h \leq s \leq a.$$

Theorem 2.5 ([6])**.** *Assume that* (**H1**)–(**H3**) *and* (**H5**) *hold. Let* $T(t)$ *be compact for* $t > 0$*. Then the corresponding resolvent operator* $R(t)$ *of equation* (2.3) *is also compact for* $t > 0$*.*

The following theorem provides the necessary and sufficient conditions for the resolvent operator to be continuous in the uniform operator topology.

Theorem 2.6 ([7])**.** *Assume that* (**H1**)–(**H3**) *and* (**H5**) *are satisfied. Then* $T(t)$ *is norm continuous* (*or continuous in the uniform operator topology*) *for* $t > 0$ *if and only if the corresponding resolvent operator* $R(t)$ *of equation* (2.3) *is norm continuous for* $t > 0$*.*

3. Main Results

We state some relevant definitions and results taken from [8] for the case where L is autonomous.

Definition 3.1 ([8])**.** A continuous function $u : [-r, +\infty) \to X$ is said to be a mild solution of equation (1.1) if $u_0 = \varphi$ and

$$u(t) = R(t)\varphi(0) + \int_0^t R(t-s)L(u_s)ds \quad \text{for} \quad t \geq 0.$$

Theorem 3.2 ([8])**.** *Assume that* (**H1**)–(**H5**) *hold. Then for each* $\varphi \in C$*, equation* (1.1) *has a mild solution* $u(\varphi)(\cdot)$ *on* $[-r, +\infty)$ *which is given by*

$$u(\varphi)(t) = \begin{cases} u(\varphi)(t) = R(t)\,\varphi(0) + \displaystyle\int_0^t R(t-s)\,L\big(u_s(\varphi)\big)ds & \textit{for } t \geq 0, \\ u_0(\varphi)(t) = \varphi(t) & \textit{for } t \in [-r, 0]. \end{cases} \tag{3.4}$$

For each $t \geq 0$ define the solution operator $V(t) : C \to C$ by

$$V(t)\varphi = u_t(\varphi).$$

Proposition 3.3. *The family* $\big(V(t)\big)_{t\geq 0}$ *satisfies the translation property*

$$\big(V(t)\varphi\big)(\theta) = \begin{cases} \big(V(t+\theta)\varphi\big)(0) & \textit{for} \quad t+\theta \geq 0, \\ \varphi(t+\theta) & \textit{for} \quad t+\theta \leq 0, \end{cases}$$

for $t \geq 0$, $\theta \in [-r, 0]$ *and* $\varphi \in C$*.*

Proof. For $t \geq 0$ and $\theta \in [-r, 0]$, it follows from (3.4) that

$$u_t(\varphi)(\theta) = \begin{cases} u(\varphi)(t+\theta) = R(t+\theta)\,\varphi(0) + \displaystyle\int_0^{t+\theta} R(t+\theta-s)\,L\big(u_s(\varphi)\big)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t+\theta \geq 0, \\ u_0(\varphi)(t+\theta) = \varphi(t+\theta) \qquad\qquad\qquad \text{for} \quad t+\theta \leq 0. \end{cases}$$

Hence, for $\varphi \in C$, we have

$$\big(V(t)\varphi\big)(\theta) = \big(u_t(\varphi)\big)(\theta) = \begin{cases} \big(V(t+\theta)\varphi\big)(0) & \text{for} \quad t+\theta \geq 0, \\ \varphi(t+\theta) & \text{for} \quad t+\theta \leq 0. \end{cases}$$

The proof of the above Proposition is completed. □

Let $B = \big\{\varphi \in C : |\varphi| \leq 1\big\}$. Take $N \geq 0$ such that $\big|L(V(s)\varphi)\big| \leq N$ for all $s \geq 0$ and $\varphi \in B$.

3.1. **Norm continuity of** $\big(V(t)\big)_{t\geq 0}$**.** To establish the norm continuity of the solution operator, we need the following

Lemma 3.4. *The map*

$$\begin{aligned}&\mathbb{R}^+\times C\to C\\&(t,\varphi)\to V(t)\,\varphi\quad \textit{is locally bounded with respect to } t \textit{ and } \varphi.\end{aligned}$$

Proof. Let $0\leq t\leq a$ and $\varphi\in B$. Then

$$\big|V(t)\,\varphi\big|=\sup_{-r\leq\theta\leq 0}\big|(V(t)\varphi)(\theta)\big|.$$

For $t+\theta\leq 0$, we have

$$\big|V(t)\varphi\big|=\sup_{-r\leq\theta\leq -t}\big|\varphi(t+\theta)\big|\leq\sup_{-r\leq\theta\leq 0}\big|\varphi(\theta)\big|\leq|\varphi|.$$

This implies that

$$\sup_{0\leq t\leq a,\,|\varphi|\leq 1}\big|V(t)\varphi\big|\leq 1.$$

For $t+\theta\geq 0$, we have

$$\begin{aligned}\big|(V(t)\varphi)(\theta)\big|&\leq\big|R(t+\theta)\,\varphi(0)\big|+\Big|\int_0^{t+\theta}R(t+\theta-s)\,L\big(V(s)\varphi\big)ds\Big|\\&\leq M_a|\varphi|+M_aN\int_0^t\big|V(s)\varphi\big|ds,\end{aligned}$$

where $M_a=\sup_{0\leq s\leq a}|R(s)|$. Thus

$$\big|V(t)\varphi\big|\leq M_a|\varphi|+M_aN\int_0^t\big|V(s)\varphi\big|ds.$$

By Gronwall's Lemma, we deduce that

$$\big|V(t)\varphi\big|\leq M_ae^{M_aN}|\varphi|.$$

Consequently,

$$\sup_{0\leq t\leq a,\,|\varphi|\leq 1}\big|V(t)\varphi\big|\leq M_ae^{M_aN},$$

and the proof of the lemma is completed. □

Theorem 3.5. *Assume that* (**H1**)–(**H5**) are satisfied. If $t\to T(t)$ is norm continuous for $t>0$. Then the solution operator $t\to V(t)$ is norm continuous on $t>r$.

Proof. Let $t>r$ and $\theta\in[-r,0]$. Then

$$\big|V(t+h)-V(t)\big|=\sup_{|\varphi|\leq 1}\big|V(t+h)\varphi-V(t)\varphi\big|.$$

For $h<0$ to be sufficiently small, we have

$$\begin{aligned}\big|\big(V(t+h)\varphi\big)(\theta)-\big(V(t)\varphi\big)(\theta)\big|=&\big|R(t+h+\theta)\varphi(0)-R(t+\theta)\varphi(0)\big|\\\leq&\sup_{t-r\leq s\leq t}\big|R(s+h)-R(s)\big|\,\big|\varphi(0)\big|.\end{aligned}$$

Let us now fix t, h such that $0 < a < t - r < t + h < b$, then

$$\begin{aligned} \sup_{t-r\le s\le t} \big|R(s+h) - R(s)\big| \, \big|\varphi(0)\big| &\le \sup_{a\le s\le b} \big|R(s+h) - R(s)\big| \, \big|\varphi(0)\big| \\ &\le \sup_{|\varphi|\le 1} \sup_{a\le s\le b} \big|R(s+h) - R(s)\big| \, \big|\varphi(0)\big| \\ &\le \sup_{a\le s\le b} \big|R(s+h) - R(s)\big|. \end{aligned}$$

Theorem 2.6 implies that

$$\big|V(t+h)\varphi - V(t)\varphi\big|$$

tends to 0 as $h \to 0$ uniformly in $\varphi \in B$. Let $h > 0$ be such that $t + h - r > 0$. Then

$$\begin{aligned} \big(V(t+h)\varphi\big)(\theta) - \big(V(t)\varphi\big)(\theta) &= \int_0^{t+\theta} \big(R(t+\theta+h-s) - R(t+\theta-s)\big) L\big(V(s)\varphi\big) ds \\ &\quad + \int_{t+\theta}^{t+\theta+h} R(t+\theta+h-s)\, L\big(V(s)\varphi\big) ds. \end{aligned}$$

By virtue of Lemma 3.4, there exists $\tilde{C}$ such that

$$\begin{aligned} &\Big| \int_0^{t+\theta} R(t+\theta+h-s) - R(t+\theta-s) L\big(V(s)\varphi\big) ds \Big| \\ &\qquad \le \int_0^{t+\theta} \big|R(t+\theta+h-s) - R(t+\theta-s)\big| \, \tilde{C}\, ds. \end{aligned}$$

Thus, there exists $\theta_0 \in [-r, 0]$ such that

$$\begin{aligned} &\sup_{-r\le\theta\le 0} \int_0^{t+\theta} \big|R(t+\theta+h-s) - R(t+\theta-s)\big| ds \\ &\qquad = \int_0^{t+\theta_0} \big|R(t+\theta_0+h-s) - R(t+\theta_0-s)\big| ds, \end{aligned}$$

which implies that

$$\lim_{h\to 0} \Big| \int_0^{t+\theta} R(t+\theta+h-s) - R(t+\theta-s)\, L\big(V(s)\varphi\big) ds \Big| = 0.$$

On the other hand, using Definition 2.1 and Lemma 3.4, we deduce that there exists $\delta(h)$ such that

$$\Big| \int_{t+\theta}^{t+\theta+h} R(t+\theta+h-s) L(V(s)\varphi) ds \Big| \le MN\delta(h).$$

This implies that

$$\lim_{h\to 0} \Big| \int_{t+\theta}^{t+\theta+h} R(t+\theta+h-s)\, L(V(s)\varphi) ds \Big| = 0.$$

Thus,

$$\lim_{h\to 0} \big|V(t+h) - V(t)\big| = 0.$$

Hence the map $t \to V(t)$ is norm continuous for $t > r$. □

3.2. **Compactness of the solution operator.** To study the compactness of the solution operator, we introduce the Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on each bounded subset B of the Banach space X by

$$\alpha(B) = \inf\big\{d > 0;\ \ B \text{ can be covered by a finite number of sets of diameter } \ < d\big\}.$$

Some basic properties of $\alpha(\cdot)$ are given in the following.

Lemma 3.6 ([2]). *Let X be a Banach space and B, $C \subseteq X$ be bounded. Then*

(1) $\alpha(B) = 0$ *if and only if B is relatively compact;*

(2) $\alpha(B) = \alpha(\overline{B}) = \alpha(\overline{co}B)$, *where $\overline{co}B$ is the closed convex hull of B;*

(3) $\alpha(B) \le \alpha(C)$, *when $B \subseteq C$;*

(4) $\alpha(B + C) \le \alpha(B) + \alpha(C)$*;*

(5) $\alpha(B \cup C) \le \max\{\alpha(B), \alpha(C)\}$*;*

(6) $\alpha(B(0, r)) \le 2r$, *where $B(0, r) = \{x \in X : |x| \le r\}$.*

We need to add the following assumption:

(H6) the C_0-semigroup $T(t)$ is compact for $t > 0$.

Theorem 3.7. *Assume that* (**H1**)–(**H6**) are satisfied. Then the solution operator $V(t)$ is compact for $t > r$.

Proof. By the Ascoli-Arzela theorem we prove that $\{V(t)\varphi : \varphi \in B\}$ is relatively compact for each $r < t$. The proof is divided into two steps.

Step 1. We show that $\{(V(t)\varphi)(\theta) : \varphi \in B\}$ is relatively compact in X for every $\theta \in [-r, 0]$. Let $\theta \in [-r, 0]$. Then

$$\big(V(t)\varphi\big)(\theta) = R(t+\theta)\varphi(0) + \int_0^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds.$$

Since $t + \theta > 0$, by (**H5**) together with Theorem 2.5, we infer that $R(t + \theta)$ is compact. Thus Lemma 3.6 gives

$$\alpha\Big(\big\{R(t+\theta)\varphi(0) : \varphi \in B\big\}\Big) = 0. \tag{3.5}$$

Now we prove that $\left\{ \int_0^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds : \varphi \in B \right\}$ is relatively compact in X. Let $0 < \varepsilon < t+\theta$. Then

$$\begin{aligned}\int_0^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds &= \int_0^{t+\theta-\varepsilon} R(t+\theta-s)L\big(V(s)\varphi\big)ds \\ &\quad + \int_{t+\theta-\varepsilon}^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds \\ &= \int_0^{t\theta-\varepsilon} \big[R(t+\theta-s) - R(\varepsilon)R(t+\theta-s-\varepsilon)\big]L(V(s)\varphi)ds \\ &\quad + R(\varepsilon)\int_0^{t+\theta-\varepsilon} R(t+\theta-s-\varepsilon)L\big(V(s)\varphi\big)ds\end{aligned}$$

$$+\int_{t+\theta-\varepsilon}^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds.$$

By Lemma 2.4, we obtain

$$\Big|\int_0^{t+\theta-\varepsilon}\big[R(t+\theta-s)-R(\varepsilon)R(t+\theta-s-\varepsilon)\big]L\big(V(s)\varphi\big)ds\Big|$$
$$\le \int_0^{t+\theta-\varepsilon}\big|R(t+\theta-s)-R(\varepsilon)R(t+\theta-s-\varepsilon)\big|\big|L(V(s)\varphi)\big|ds$$
$$\le \varepsilon H\int_0^{t+\theta-\varepsilon}\big|L(V(s)\varphi)\big|ds \le \varepsilon(t-\varepsilon)HN.$$

Let $t \le b$. Then Lemma 3.6 gives

$$\alpha\bigg(\bigg\{\int_0^{t+\theta-\varepsilon}\big[R(t+\theta-s)-R(\varepsilon)R(t+\theta-s-\varepsilon)\big]L\big(V(s)\varphi\big)ds : \varphi\in B\bigg\}\bigg)$$
$$\le 2\varepsilon(b-\varepsilon)HN. \tag{3.6}$$

Moreover, since $R(\varepsilon)$ is compact, we find that

$$\bigg\{R(\varepsilon)\int_0^{t+\theta-\varepsilon}R(t+\theta-s-\varepsilon)L\big(V(s)\varphi\big)ds : \varphi\in B\bigg\}$$

is relatively compact in X and, consequently

$$\alpha\bigg(\bigg\{R(\varepsilon)\int_0^{t+\theta-\varepsilon}R(t+\theta-s-\varepsilon)L\big(V(s)\varphi\big)ds : \varphi\in B\bigg\}\bigg)=0. \tag{3.7}$$

Note that

$$\Big|\int_{t+\theta-\varepsilon}^{t+\theta}R(t+\theta-s)L\big(V(s)\varphi\big)ds\Big|\le MN\delta(\varepsilon).$$

Therefore,

$$\alpha\bigg(\bigg\{\int_{t+\theta-\varepsilon}^{t+\theta}R(t+\theta-s)L\big(V(s)\varphi\big)ds : \varphi\in B\bigg\}\bigg)\le 2MN\delta(\varepsilon). \tag{3.8}$$

Combining (3.5)–(3.8) and using Lemma 3.6, we obtain

$$\alpha\Big(\big\{(V(t)\varphi)(\theta):\varphi\in B\big\}\Big)\le 2\varepsilon(b-\varepsilon)HN+2MN\delta(\varepsilon).$$

Letting $\varepsilon\to 0$, we deduce that

$$\alpha\Big(\big\{(V(t)\varphi)(\theta):\varphi\in B\big\}\Big)=0.$$

Consequently, $\big\{(V(t)\varphi)(\theta):\varphi\in B\big\}$ is relatively compact in X for all $\theta\in[-r,0]$.

Step 2. We show that $\{V(t)\varphi:\varphi\in B\}$ is equicontinuous on $[-r,0]$. To see this, let $-r\le\theta_1<\theta_2\le 0$. Then

$$\Big|\big(V(t)\varphi\big)(\theta_2)-\big(V(t)\varphi\big)(\theta_1)\Big|\le\Big|\big(R(t+\theta_2)-R(t+\theta_1)\big)\varphi(0)\Big|$$

$$
\begin{aligned}
&+ \int\limits_{t+\theta_1}^{t+\theta_2} \Big| R(t+\theta_2 - s) L\big(V(s)\varphi\big) \Big| ds \\
&+ \int\limits_{0}^{t+\theta_1} \Big| \big(R(t+\theta_2 - s) - R(t+\theta_1 - s)\big) L\big(V(s)\varphi\big) \Big| ds \\
&\leq \Big| R(t+\theta_2) - R(t+\theta_1) \Big| \, \Big| \varphi(0) \Big| + MN\delta(\theta_2 - \theta_1) \\
&+ N \int\limits_{0}^{t+\theta_1} \Big| R(t+\theta_2 - s) - R(t+\theta_1 - s) \Big| ds.
\end{aligned}
$$

Since

$$\big| R(t+\theta_2 - s) - R(t+\theta_1 - s) \big| \to 0 \quad \text{as} \quad \theta_2 \to \theta_1 \quad \text{for almost all} \quad s \neq t + \theta_1$$

and

$$\big| R(t+\theta_2 - s) - R(t+\theta_1 - s) \big| \leq M\big(e^{\beta(t+\theta_2 - s)} + e^{\beta(t+\theta_1 - s)}\big) \in L^1\big([0, t+\theta_1]\big),$$

the Lebesgue Dominated Convergence theorem ensures that

$$\int\limits_{0}^{t+\theta_1} \big| R(t+\theta_2 - s) - R(t+\theta_1 - s) \big| ds \to 0 \quad \text{as} \quad \theta_2 \to \theta_1.$$

Using Theorem 2.6, we obtain

$$\big| (V(t)\varphi)(\theta_2) - (V(t)\varphi)(\theta_1) \big| \to 0 \quad \text{as} \quad \theta_2 \to \theta_1,$$

uniformly in $\varphi \in B$. This implies that $\{V(t)\varphi : \varphi \in B\}$ is equicontinuous. Hence, $\{V(t)\varphi : \varphi \in B\}$ is relatively compact by the Ascoli Arzela theorem and so, $V(t)$ is compact for $t > r$. □

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References

1. M. Adimy, H. Bouzahir, K. Ezzinbi, Local existence and stability for some partial functional differential equations with infinite delay. *Nonlinear Anal. Theory, Methods and Applications* **48** (2002), no. 3, 323–348.
2. S. Banas, K. Goebel, *Mesure of Noncompactness in Banach Spaces.* Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 1980.
3. G. Chen, R. Grimmer, Semigroups and integral equations. *J. Integral Equations* **2** (1980), 133–154.
4. C. M. Dafermos, An abstrtact Volterra equation with applications to linear viscoelasticity. *J. Differential Equations* **7** (1970), 554–569.
5. W. Desch, R. Grimmer, W. Schappacher, Well-posedness and wave propagation for a class of integrodifferential equations in Banach space. *J. Differential Equations* **74** (1988), no. 2, 391–411.
6. W. Desch, R. Grimmer, W. Schappacher, Some considerations for linear integro-differential equations. *J. Math. Anal. Appl.* **104** (1984), no. 1, 219–234.
7. K. Ezzinbi, S. Ghnimi, M. A. Taoudi, Existence results for some partial integrodifferential equations with nonlocal conditions. *Glas. Mat. Ser. III* **51**(71) (2016), no. 2, 413–430.
8. K. Ezzinbi, S. Ghnimi, Local existence and global continuation for some partial functional integrodifferential equations. *Afr. Diaspora J. Math.* **12** (2011), no. 1, 34–45.
9. K. Ezzinbi, S. Ghnimi, Existence and regularity of solutions for neutral partial functional integrodifferential equations. *Nonlinear Anal., Real World Appl.* **11** (2010), 2335–2344.
10. K. Ezzinbi, S. Ghnimi, M. A. Taoudi, Existence and regularity of solutions for neutral partial functional integrodifferential equations with infinite delay. *Nonlinear Anal., Hybrid Syst.* **4** (2010), 54–64.
11. R. Grimmer, Resolvent operators for integral equations in a Banach space. *Trans. Amer. Math. Soc.* **273** (1982), no. 1, 333–349.
12. M. E. Gurtin, A. C. Pipkin, A general theory of heat conduction with finite wave speeds. *Arch. Rational Mech. Anal.* **31** (1968), no. 2, 113–126.

13. J. K. Hale, *Functional Differential Equations*. Applied Mathematical Sciences Series, vol. 3. Springer-Verlag, New York, New York-Heidelberg, 1971.
14. R. K. Miller, An integro-differential equation for rigid heat conductions with memory. *J. Math. Anal. Appl.* **66** (1978), no. 2, 313–332.
15. C. C. Travis, G. F. Webb, Existence and stability for partial functional differential equations. *Trans. Amer. Math. Soc.* **200** (1974), 395–418.
16. G. Webb, Autonomous nonlinear functional differential equations and nonlinear semigroups. *J. Math. Anal. Appl.* **46** (1974), 1–12.
17. J. Wu, *Theory and Applications of Partial Functional-Differential Equations*. Applied Mathematical Sciences, 119. Springer-Verlag, New York, 1996.

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Faculty of Sciences of Gafsa, Department of Mathematics, University of Gafsa, B. P. 2112, Gafsa, Tunisia

E-mail address: ghnimisaifeddine@yahoo.fr