A NEW FACTOR THEOREM FOR GENERALIZED ABSOLUTE CESÁRO SUMMABILITY METHODS

HÜSEYİN BOR

Abstract. In [6], we have proved a main theorem dealing with \( \varphi - | C, \alpha |_k \) summability factors of infinite series. In this paper, we will generalize this result for the \( \varphi - | C, \alpha, \beta |_k \) summability method. Also, some new and known results are obtained.

1. INTRODUCTION

Let \( \sum a_n \) be a given infinite series. We denote by \( t_{n}^{\alpha, \beta} \) the nth Cesàro mean of order \((\alpha, \beta)\), with \( \alpha + \beta > -1 \), of the sequence \((na_n)\), that is (see [7]),

\[
t_{n}^{\alpha, \beta} = \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_{v}^{\beta} a_v,
\]

where \( A_{n}^{\alpha+\beta} = O(n^{\alpha+\beta}) \), \( A_{0}^{\alpha+\beta} = 1 \) and \( A_{-n}^{\alpha+\beta} = 0 \) for \( n > 0 \).

Let \( (\omega_{n}^{\alpha, \beta}) \) be a sequence defined by (see [3])

\[
\omega_{n}^{\alpha, \beta} = \begin{cases} |t_{n}^{\alpha, \beta}|, & \alpha = 1, \beta > -1, \\ \max_{1 \leq v \leq n} |t_{v}^{\alpha, \beta}|, & 0 < \alpha < 1, \beta > -1. \end{cases}
\]

Let \( (\varphi_n) \) be a sequence of complex numbers. The series \( \sum a_n \) is said to be summable \( \varphi - | C, \alpha, \beta |_k \), \( k \geq 1 \), if (see [4])

\[
\sum_{n=1}^{\infty} n^{-k} | \varphi_n t_n^{\alpha, \beta} |^k < \infty.
\]

In the special case for \( \varphi_n = n^{1-\frac{1}{k}} \), the \( \varphi - | C, \alpha, \beta |_k \) summability is the same as \( | C, \alpha, \beta |_k \) summability (see [8]). Also, if we take \( \varphi_n = n^{\delta+1-\frac{1}{k}} \), then \( \varphi - | C, \alpha, \beta |_k \) summability reduces to \( | C, \alpha, \beta; \delta |_k \) summability (see [5]). If we take \( \delta = 0 \), then we have \( \varphi - | C, \alpha |_k \) summability (see [1]). If we take \( \varphi_n = n^{1-\frac{1}{k}} \) and \( \delta = 0 \), then we get \( | C, \alpha |_k \) summability (see [9]). Finally, if we take \( \varphi_n = n^{\delta+1-\frac{1}{k}} \) and \( \delta = 0 \), then we obtain \( | C, \alpha; \delta |_k \) summability (see [10]).

2. The Known Results

The following theorems dealing with the \( \varphi - | C, \alpha |_k \) summability factors of infinite series are known.

**Theorem A** ([2]). Let \( 0 < \alpha \leq 1 \). Let \( (X_n) \) be a positive non-decreasing sequence and let there exist the sequences \((\beta_n)\) and \((\lambda_n)\) such that

\[
| \Delta \lambda_n | \leq \beta_n
\]

\[
\beta_n \to 0 \quad \text{as} \quad n \to \infty.
\]
\[ \sum_{n=1}^{\infty} n \mid \Delta \beta_n \mid X_n < \infty \quad (5) \]

\[ | \lambda_n | X_n = O(1) \text{ as } n \to \infty. \quad (6) \]

If there exists an \( \epsilon > 0 \) such that the sequence \( (n^{\epsilon-k} | \varphi_n|^k) \) is non-increasing and if the sequence \( (\omega_n^\alpha) \) defined by (see [12])

\[ \omega_n^\alpha = \begin{cases} \frac{|t_n^\alpha|}{n} & (\alpha = 1) \\ \max_{1 \leq v \leq n} |t_v^\alpha| & (0 < \alpha < 1) \end{cases} \quad (7) \]

satisfies the condition

\[ \sum_{n=1}^{m} \left( | \varphi_n | \frac{w_n^\alpha}{n^k} \right)^k = O(X_m) \text{ as } m \to \infty, \]

then the series \( \sum a_n \lambda_n \) is summable \( \varphi - |C, \alpha|_{k} \), \( k \geq 1 \) and \( (\alpha + \epsilon) > 1 \).

**Theorem B** ([6]). Let \( 0 < \alpha \leq 1 \). Let \( (X_n) \) be a positive non-decreasing sequence and the sequences \( (\beta_n) \) and \( (\lambda_n) \) such that conditions (3), (4), (5), (6) of Theorem A are satisfied. If there exists an \( \epsilon > 0 \) such that the sequence \( (n^{\epsilon-k} | \varphi_n|^k) \) is non-increasing and if the sequence \( (\omega_n^\alpha) \) defined by (7) satisfies the condition

\[ \sum_{n=1}^{m} \left( | \varphi_n | \frac{w_n^\alpha}{n^kX_n^{k-1}} \right) = O(X_m) \text{ as } m \to \infty, \]

then the series \( \sum a_n \lambda_n \) is summable \( \varphi - |C, \alpha|_{k} \), \( k \geq 1 \) and \( (1 + \alpha k + \epsilon - k) > 1 \).

3. **The Main Result**

The aim of this paper is to generalize Theorem B for \( \varphi - |C, \alpha, \beta|_{k} \) summability method. Now we shall prove the following theorem.

**Theorem.** Let \( 0 < \alpha \leq 1 \). Let \( (X_n) \) be a positive non-decreasing sequence and the sequences \( (\beta_n) \) and \( (\lambda_n) \) such that conditions (3), (4), (5), (6) of Theorem A are satisfied. If there exists an \( \epsilon > 0 \) such that the sequence \( (n^{\epsilon-k} | \varphi_n|^k) \) is non-increasing and if the sequence \( (\omega_n^{\alpha, \beta}) \) defined by (2) satisfies the condition

\[ \sum_{n=1}^{m} \left( | \varphi_n | \frac{w_n^{\alpha, \beta}}{n^{kX_n^{k-1}}} \right) = O(X_m) \text{ as } m \to \infty, \]

then the series \( \sum a_n \lambda_n \) is summable \( \varphi - |C, \alpha, \beta|_{k} \), \( k \geq 1 \) and \( (1 + (\alpha + \beta)k + \epsilon - k) > 1 \).

We need the following lemmas for the proof of our theorem.

**Lemma 1** ([3]). If \( 0 < \alpha \leq 1 \), \( \beta > -1 \), and \( 1 \leq v \leq n \), then

\[ \sum_{p=v}^{n} A_{n-p}^{\alpha-1} A_{p}^{\beta} a_p \leq \max_{1 \leq m \leq v} \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_{p}^{\beta} a_p. \]

**Lemma 2** ([11]). Under the conditions on \( (X_n) \), \( (\beta_n) \) and \( (\lambda_n) \) as taken in the statement of Theorem A, the conditions

\[ n \beta_n X_n = O(1) \text{ as } n \to \infty \quad (8) \]

\[ \sum_{n=1}^{\infty} \beta_n X_n < \infty. \]

hold, when (5) is satisfied.
4. Proof of the Theorem

Let \((T_n^{\alpha,\beta})\) be the \(n\)th \((C,\alpha,\beta)\) mean of the sequence \((na_n\lambda_n)\).
Then, by (1), we have

\[
T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.
\]

Applying first Abel’s transformation and then using Lemma 1, we have

\[
T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,
\]

\[
|T_n^{\alpha,\beta}| \leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} |\Delta \lambda_v| \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \right| + \left| \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \right|
\]

\[
\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \omega_v^{\alpha,\beta} |\Delta \lambda_v| + |\lambda_n| \omega_n^{\alpha,\beta} = T_n^{\alpha,\beta} + T_n^{\alpha,\beta}.
\]

To complete the proof of the theorem, by Minkowski’s inequality, it suffices to show that

\[
\sum_{n=1}^{\infty} n^{-k} |\varphi_n T_{n,r}^{\alpha,\beta}|^k < \infty, \quad \text{for } r = 1, 2.
\]

For \(k > 1\), applying first Hölder’s inequality with indices \(k\) and \(k'\), where \(\frac{1}{k} + \frac{1}{k'} = 1\), and then using (8), we obtain

\[
\sum_{n=2}^{m+1} n^{-k} |\varphi_n T_{n,1}^{\alpha,\beta}|^k \leq \sum_{n=2}^{m+1} n^{-k} (A_n^{\alpha+\beta})^{-k} |\varphi_n|^k \left( \sum_{v=1}^{n-1} A_v^{\alpha+\beta} \omega_v^{\alpha,\beta} \right)^{k-1}
\]

\[
\leq \sum_{n=2}^{m+1} \frac{1}{n^{\beta(n+\alpha)}} |\varphi_n|^k \sum_{v=1}^{n-1} v^{(\alpha+\beta)k} \omega_v^{\alpha,\beta} \beta_v^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta)k}} \sum_{v=1}^{n-1} v^{(\alpha+\beta)k} \omega_v^{\alpha,\beta} \beta_v^k
\]

\[
= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \omega_v^{\alpha,\beta} \beta_v^k \int_0^\infty \frac{dx}{x^{1+(\alpha+\beta)k+\varepsilon-k}}
\]

\[
= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \omega_v^{\alpha,\beta} \beta_v^k
\]

\[
= O(1) \sum_{v=1}^{m} |\Delta(v \beta_v)| X_v + O(1) m \beta_m X_m + O(1) m \beta_m X_m
\]

\[
= O(1) \text{ as } m \to \infty,
\]
by the hypotheses of the theorem and Lemma 2. Again, using (6), we have

\[
\sum_{n=1}^{m} n^{-k} |\varphi_n T_{n,2}^{\alpha,\beta}|^k = \sum_{n=1}^{m} n^{-k} |\varphi_n|^k |\lambda_n|^k |w_n^{\alpha,\beta}|^k = O(1) \sum_{n=1}^{m} |\lambda_n| \frac{(|\varphi_n| w_n^{\alpha,\beta})^k}{n^k X_n^{k-1}}
\]

\[
= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \left( |\varphi_v| w_v^{\alpha,\beta} \right)^k + O(1) |\lambda_m| \sum_{n=1}^{m} \left( |\varphi_n| w_n^{\alpha,\beta} \right)^k
\]

\[
= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| X_n + O(1) |\lambda_m| X_m
\]

\[
= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty,
\]

by the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

5. Conclusion

If we take \( \epsilon = 1 \) and \( \varphi_n = n^{1-\frac{1}{k}} \), then we obtain a new result concerning the \( |C, \alpha, \beta|_k \) summability factors of infinite series. If we take \( \epsilon = 1, \beta = 0 \) and \( \varphi_n = n^{\delta+1-\frac{1}{k}} \), then we have a new result dealing with the \( |C, \alpha; \delta|_k \) summability factors of infinite series. Also, if we take \( \beta = 0 \), then we obtain Theorem B.

References


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P. O. Box 121, TR-06502 BAŞÇELİEVLER, ANKARA, TURKEY
E-mail address: hhbor33@gmail.com