Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 1, 29–32

A NEW FACTOR THEOREM FOR GENERALIZED ABSOLUTE CESÀRO SUMMABILITY METHODS

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Abstract. In [6], we have proved a main theorem dealing with $\varphi - \mid C, \alpha, \mid_k$ summability factors of infinite series. In this paper, we will generalize this result for the $\varphi - \mid C, \alpha, \beta \mid_k$ summability method. Also, some new and known results are obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α, β), with $\alpha + \beta > -1$, of the sequence (na_n), that is (see [7]),

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad \text{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \text{for} \quad n > 0.$$

Let $(\omega_n^{\alpha,\beta})$ be a sequence defined by (see [3])

$$\omega_n^{\alpha,\beta} = \begin{cases} \left| t_n^{\alpha,\beta} \right|, & \alpha = 1, \beta > -1, \\ \max\limits_{1 \le v \le n} \left| t_v^{\alpha,\beta} \right|, & 0 < \alpha < 1, \beta > -1. \end{cases} \tag{2}$$

Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \ge 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n t_n^{\alpha,\beta} \mid^k < \infty.$$

In the special case for $\varphi_n = n^{1-\frac{1}{k}}$, the $\varphi - \mid C, \alpha, \beta \mid_k$ summability is the same as $\mid C, \alpha, \beta \mid_k$ summability (see [8]). Also, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then $\varphi - \mid C, \alpha, \beta \mid_k$ summability reduces to $\mid C, \alpha, \beta; \delta \mid_k$ summability (see [5]). If we take $\beta = 0$, then we have $\varphi - \mid C, \alpha \mid_k$ summability (see [1]). If we take $\varphi_n = n^{1-\frac{1}{k}}$ and $\beta = 0$, then we get $\mid C, \alpha \mid_k$ summability (see [9]). Finally, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$ and $\beta = 0$, then we obtain $\mid C, \alpha; \delta \mid_k$ summability (see [10]).

2. The Known Results

The following theorems dealing with the $\varphi - \mid C, \alpha \mid_k$ summability factors of infinite series are known.

Theorem A ([2]). *Let $0 < \alpha \le 1$. Let (X_n) be a positive non-decreasing sequence and let there exist the sequences (β_n) and (λ_n) such that*

$$\mid \Delta \lambda_n \mid \le \beta_n \tag{3}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{4}$$

2010 *Mathematics Subject Classification.* 26D15, 40D15, 40F05, 40G05.

Key words and phrases. Cesàro mean; Absolute summability; Infinite series; Hölder's inequality; Minkowski's inequality.

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty \tag{5}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{6}$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\left|\varphi_n\right|^k)$ is non-increasing and if the sequence (ω_n^α) defined by (see [12])

$$\omega_n^\alpha = \begin{cases} |t_n^\alpha| & (\alpha = 1) \\ \max_{1\leq v\leq n} |t_v^\alpha| & (0 < \alpha < 1) \end{cases} \tag{7}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid w_n^\alpha)^k}{n^k} = O(X_m) \quad \text{as} \quad m \to \infty,$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C, \alpha \mid_k$, $k \geq 1$ and $(\alpha + \epsilon) > 1$.

Theorem B ([6]). *Let $0 < \alpha \leq 1$. Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) such that conditions (3), (4), (5), (6) of Theorem A are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\left|\varphi_n\right|^k)$ is non-increasing and if the sequence (ω_n^α) defined by (7) satisfies the condition*

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid w_n^\alpha)^k}{n^k {X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty,$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C, \alpha \mid_k$, $k \geq 1$ and $(1 + \alpha k + \epsilon - k) > 1$.

3. The Main Result

The aim of this paper is to generalize Theorem B for $\varphi - \mid C, \alpha, \beta \mid_k$ summability method. Now we shall prove the following theorem.

Theorem. *Let $0 < \alpha \leq 1$. Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) such that conditions (3), (4), (5), (6) of Theorem A are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\left|\varphi_n\right|^k)$ is non-increasing and if the sequence $(\omega_n^{\alpha,\beta})$ defined by (2) satisfies the condition*

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid w_n^{\alpha,\beta})^k}{n^k {X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty,$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$ and $(1 + (\alpha+\beta)k + \epsilon - k) > 1$.

We need the following lemmas for the proof of our theorem.

Lemma 1 ([3]). *If $0 < \alpha \leq 1$, $\beta > -1$, and $1 \leq v \leq n$, then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^\beta a_p\right| \leq \max_{1\leq m\leq v} \left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^\beta a_p\right|.$$

Lemma 2 ([11]). *Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of Theorem A, the conditions*

$$n\beta_n X_n = O(1) \quad as \quad n \to \infty \tag{8}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

hold, when (5) is satisfied.

4. Proof of the Theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$.
Then, by (1), we have

$$T_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\lambda_v.$$

Applying first Abel's transformation and then using Lemma 1, we have

$$\begin{aligned}
T_n^{\alpha,\beta}=&\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p+\frac{\lambda_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v,\\
\mid T_n^{\alpha,\beta}\mid\leq&\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\mid\Delta\lambda_v\mid\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\right|+\frac{\mid\lambda_n\mid}{A_n^{\alpha+\beta}}\left|\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\right|\\
\leq&\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{(\alpha+\beta)}\omega_v^{\alpha,\beta}\mid\Delta\lambda_v\mid+\mid\lambda_n\mid\omega_n^{\alpha,\beta}=T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it suffices to show that

$$\sum_{n=1}^{\infty}n^{-k}\mid\varphi_nT_{n,r}^{\alpha,\beta}\mid^k<\infty,\quad\text{for}\quad r=1,2.$$

For $k>1$, applying first Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, and then using (8), we obtain

$$\begin{aligned}
\sum_{n=2}^{m+1}n^{-k}\mid\varphi_nT_{n,1}^{\alpha,\beta}\mid^k\leq&\sum_{n=2}^{m+1}n^{-k}(A_n^{\alpha+\beta})^{-k}|\varphi_n|^k\left\{\sum_{v=1}^{n-1}A_v^{\alpha+\beta}\omega_v^{\alpha,\beta}\beta_v\right\}^{k-1}\\
\leq&\sum_{n=2}^{m+1}\frac{1}{n}(A_n^{\alpha+\beta})^{-k}|\varphi_n|^k\sum_{v=1}^{n-1}(A_v^{\alpha+\beta})^k(\omega_v^{\alpha,\beta})^k\beta_v^k\times\left\{\frac{1}{n}\sum_{v=1}^{n-1}1\right\}^{k-1}\\
=&O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{1+(\alpha+\beta)k}}\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}(\omega_v^{\alpha,\beta})^k\beta_v^k\\
=&O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(\omega_v^{\alpha,\beta})^k\beta_v\beta_v^{k-1}\sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}|\varphi_n|^k}{n^{1+(\alpha+\beta)k+\epsilon-k}}\\
=&O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(\omega_v^{\alpha,\beta})^k\beta_v\frac{v^{\epsilon-k}|\varphi_v|^k}{v^{k-1}X_v^{k-1}}\int_v^{\infty}\frac{dx}{x^{1+(\alpha+\beta)k+\epsilon-k}}\\
=&O(1)\sum_{v=1}^{m}v\beta_v\frac{(\omega_v^{\alpha,\beta}|\varphi_v|)^k}{v^kX_v^{k-1}}\\
=&O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{(\omega_r^{\alpha,\beta}|\varphi_r|)^k}{r^kX_r^{k-1}}+O(1)m\beta_m\sum_{v=1}^{m}\frac{(\omega_v^{\alpha,\beta}|\varphi_v|)^k}{v^kX_v^{k-1}}\\
=&O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v+O(1)m\beta_mX_m\\
=&O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)m\beta_mX_m\\
=&O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}$$

by the hypotheses of the theorem and Lemma 2. Again, using (6), we have

$$\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha,\beta} \mid^k = \sum_{n=1}^{m} n^{-k} \left|\varphi_n\right|^k \left|\lambda_n\right| \left|\lambda_n\right|^{k-1} (\omega_n^{\alpha,\beta})^k = O(1) \sum_{n=1}^{m} \left|\lambda_n\right| \frac{(\mid \varphi_n \mid w_n^{\alpha,\beta})^k}{n^k {X_n}^{k-1}}$$
$$= O(1) \sum_{n=1}^{m-1} \Delta \left|\lambda_n\right| \sum_{v=1}^{n} \frac{(\mid \varphi_v \mid w_v^{\alpha,\beta})^k}{v^k {X_v}^{k-1}} + O(1) \left|\lambda_m\right| \sum_{n=1}^{m} \frac{(\mid \varphi_n \mid w_n^{\alpha,\beta})^k}{n^k {X_n}^{k-1}}$$
$$= O(1) \sum_{n=1}^{m-1} \left|\Delta\lambda_n\right| X_n + O(1) \left|\lambda_m\right| X_m$$
$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \left|\lambda_m\right| X_m = O(1) \quad \text{as} \quad m \to \infty,$$

by the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

5. Conclusion

If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a new result concerning the $\mid C, \alpha, \beta \mid_k$ summability factors of infinite series. If we take $\epsilon = 1$, $\beta = 0$ and $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we have a new result dealing with the $\mid C, \alpha; \delta \mid_k$ summability factors of infinite series. Also, if we take $\beta = 0$, then we obtain Theorem B.

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(Received 18.09.2019)

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