

## ON SINGULAR EXTENSIONS OF CONTINUOUS FUNCTIONALS FROM $C([0, 1])$ TO THE VARIABLE LEBESGUE SPACES

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**Abstract.** Valadier and Hensgen proved independently that the restriction of the functional  $\phi(x) = \int_0^1 x(t)dt$ ,  $x \in L^\infty([0, 1])$  on the space of continuous functions  $C([0, 1])$  admits a singular extension back to the whole space  $L^\infty([0, 1])$ . Some general results in this direction for the Banach lattices were obtained by Abramovich and Wickstead. In the present note we investigate analogous problem for the variable exponent Lebesgue spaces, namely, we prove that if the space of continuous functions  $C([0, 1])$  is a closed subspace in  $L^{p(\cdot)}([0, 1])$ , then every bounded linear functional on  $C([0, 1])$  is the restriction of a singular linear functional on  $L^{p(\cdot)}([0, 1])$ .

### 1. INTRODUCTION

In the theory of duality of function spaces, the investigation of the space of all singular linear functionals is of importance. It is well known that the topological dual of the Banach function space  $X$  can be represented in the form  $X^* = X_n^* \oplus X_s^*$ , where  $X_n^*$  is the order continuous dual of  $X$  and  $X_s^*$  (the disjoint complement of  $X_n^*$  in  $X^*$ ) is the space of all singular linear functionals on  $X$ .

The term “singular functional” is overused in the literature. According to [2], the space of singular functionals  $X_s^*$ , is defined, as we have mentioned above, as the band  $(X_n^*)^d$ , complementary to the band of order continuous functionals. Often the space of singular functionals for Banach function space  $X$  is defined as an annihilator

$$(X_a)^\perp = \{x^* \in X^*; x^*(x) = 0, \text{ for all } x \in X_a\},$$

where  $X_a$  is the space of order continuous elements in  $X$ . Although, the differences between these definitions are small and they often define the same objects (see, for instance, [1]), for variable exponent Lebesgue spaces the above-mentioned definitions are equivalent. (Note that for  $L^\infty([0, 1])$ , we have  $(L^\infty([0, 1]))_a = \{0\}$ ).

Let  $L^\infty([0, 1])$  and  $C([0, 1])$  denote, respectively, the Banach space of essentially bounded real-valued functions on the  $[0, 1]$  with the Lebesgue measure and its subspace of continuous functions. Valadier [12] and Hensgen [8] proved independently that the restriction of the functional

$$\phi(x) = \int_0^1 x(t)dt, \quad x \in L^\infty[0, 1]$$

to a fairly large subspace  $C([0, 1])$  of  $L^\infty([0, 1])$  admits a singular (“bad”) extension back to the  $L^\infty([0, 1])$ . Abramovich and Wickstead showed that every bounded linear functional on  $C([0, 1])$  is the restriction of a singular functional on  $L^\infty([0, 1])$ . They also generalized this result to the Banach lattice setting ([1], see Theorem 1 and remarks thereafter). Let a finitely additive measure  $\nu$  represent  $f \in (L^\infty([0, 1]))^*$  and  $\widehat{\nu}$  be the Borel measure representing  $f$  restricted on  $C([0, 1])$ . Many properties of  $\widehat{\nu}$  in terms  $\nu$  were recently investigated by Toland [11] and Wrobel [13].

Edmunds, Gogatishvili and Kopaliani [6] showed that there is a variable exponent space  $L^{p(\cdot)}([0, 1])$  with  $1 < p(t) < \infty$  a.e., which has in common with  $L^\infty([0, 1])$  the property that the space  $C([0, 1])$  is a closed linear subspace in it. Moreover, Kolmogorov’s and Marcinkiewicz examples of functions with a.e. divergent Fourier series belong to  $L^{p'(\cdot)}([0, 1])$ , where  $p'(\cdot)$  is a function, conjugate to  $p(\cdot)$ .

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2010 *Mathematics Subject Classification.* 42B35, 46A20, 46E30.

*Key words and phrases.* Variable Lebesgue spaces; Dual spaces; Singular functional.

In [10], there is the necessary and sufficient condition on the decreasing rearrangement  $p^*$  of the exponent  $p(\cdot)$ , for the existence of equimeasurable exponent function of  $p(\cdot)$  whose corresponding variable Lebesgue space has the property that the space of continuous functions is closed in it. Indeed, let for the functions  $p(\cdot) : [0, 1] \rightarrow [1, \infty)$  we have

$$\limsup_{t \rightarrow 0^+} \frac{p^*(t)}{\ln(e/t)} > 0,$$

then there exists equimeasurable with  $p(\cdot)$  exponent function  $\tilde{p}(\cdot)$  such that the space  $C([0, 1])$  is a closed subspace in  $L^{\tilde{p}(\cdot)}([0, 1])$ .

Let the space  $C([0, 1])$  be a closed subspace of  $L^{p(\cdot)}([0, 1])$ . In this case, it is interesting to investigate the validity of the analogous Abramovich and Wickstead's theorem mentioned above. We have got the answer to this question. We prove the following

**Theorem 1.1.** *Let the space  $C([0, 1])$  be a closed subspace of  $L^{p(\cdot)}([0, 1])$ . Then every bounded linear functional on  $C([0, 1])$  is the restriction of a linear singular functional on  $L^{p(\cdot)}([0, 1])$ .*

## 2. SOME PROPERTIES OF SINGULAR FUNCTIONALS IN THE VARIABLE LEBESGUE SPACES

Let  $p(\cdot) : [0, 1] \rightarrow [1, \infty)$  be a measurable function. Define the modular

$$\rho_{p(\cdot)}(x) = \int_{[0,1]} |x(t)|^{p(t)} dt.$$

Given a measurable function  $x$ , we say that  $x \in L^{p(\cdot)}([0, 1])$  if there exists  $\lambda > 0$  such that  $\rho_{p(\cdot)}(x/\lambda) < \infty$ . This set becomes a Banach function space when equipped with the Luxemburg norm

$$\|x\|_{p(\cdot)} = \inf\{\lambda > 0; \rho_{p(\cdot)}(x/\lambda) \leq 1\}.$$

The variable Lebesgue spaces were first introduced by Orlicz. They have been widely studied for the past thirty years, both for their interest as function spaces and for their applications to PDEs and the calculus of variation (see [4], [5]).

Define the dual exponent  $p'(\cdot)$  pointwise by  $1/p(t) + 1/p'(t) = 1$ ,  $t \in [0, 1]$ .

In the case  $p_+ < \infty$ , where  $p_+ = \text{ess sup}_{t \in [0,1]} p(t)$ , the dual space of  $L^{p(\cdot)}([0, 1])$  can be completely characterized, it is isomorphic to  $L^{p'(\cdot)}([0, 1])$ . The problem characterizing the dual of  $L^{p(\cdot)}([0, 1])$  when  $p_+ = \infty$  was considered in [3]. The authors in this case give a decomposition of  $(L^{p(\cdot)}([0, 1]))^*$  as a direct sum of  $L^{p'(\cdot)}([0, 1])$  and the dual of a quotient space (we refer to the germ space and denote it by  $L_{\text{germ}}^{p(\cdot)}$ ). Note that the main aspect of this subject was made in a more general setting for Musielak-Orlicz spaces by Hudzik and Zbaszyniak (see [9]). First, we present some basic facts from mentioned paper for a variable Lebesgue setting. We will always assume without loss of generality that  $1 < p(t) < \infty$  a.e. (we are interested in characterizing the spaces  $L^{p(\cdot)}([0, 1])$  close to  $L^\infty([0, 1])$ ).

We define the closed subspace  $E^{p(\cdot)}([0, 1])$  of  $L^{p(\cdot)}([0, 1])$  by

$$E^{p(\cdot)}([0, 1]) = \{x : \rho_{p(\cdot)}(\lambda x) < \infty \text{ for any } \lambda > 0\}.$$

It is easy to see that  $E^{p(\cdot)}([0, 1])$  is the subspace of order continuous elements in  $L^{p(\cdot)}([0, 1])$ , i.e.,  $x \in L^{p(\cdot)}([0, 1])$  belongs to  $E^{p(\cdot)}([0, 1])$  if and only if for any sequence  $x_n$  of measurable functions on  $[0, 1]$  such that  $|x_n(t)| \leq |x(t)|$  for all  $n \in \mathbb{N}$  and  $|x_n| \rightarrow 0$  a.e. on  $[0, 1]$  there holds  $\|x_n\|_{p(\cdot)} \rightarrow 0$ . (For the definition of order continuous elements in Banach lattices, see [2]). Note that if  $p_+ < \infty$ , the spaces  $L^{p(\cdot)}([0, 1])$  and  $E^{p(\cdot)}([0, 1])$  coincide with each other (see [4], [5]).

Let  $p_+ = \infty$ . Define the sets  $\Omega_n = \{t \in [0, 1] : p(t) \leq n\}$ ,  $n \in \mathbb{N}$ . We will always assume without loss of generality that  $|\Omega_n| > 0$  for  $n \in \mathbb{N}$ ,  $n \geq 2$  and  $|\Omega_1| = 0$ . For  $x \in L^{p(\cdot)}([0, 1])$ , define the functions  $x^{(n)} \in E^{p(\cdot)}([0, 1])$ ,  $n \in \mathbb{N}$  as  $x^{(n)} = x\chi_{\Omega_n}$  ( $\chi_{\Omega_n}$  denotes the characteristic function of the set  $\Omega_n$ ).

For any  $x \in L^{p(\cdot)}([0, 1])$ , define

$$\begin{aligned} d(x) &= \inf\{\|x - y\|_{p(\cdot)} : y \in E^{p(\cdot)}\}, \\ \theta(x) &= \inf\{\lambda > 0; \rho_{p(\cdot)}(\lambda x) < +\infty\}. \end{aligned}$$

For any  $x^* \in (L^{p(\cdot)}([0, 1]))^*$ , we define the norm in a dual space

$$\|x^*\| = \sup\{x^*(x) : \|x\|_{p(\cdot)} \leq 1\}.$$

The dual space of  $L^{p(\cdot)}([0, 1])$  is represented in the following way (see [9]):

$$(L^{p(\cdot)}([0, 1]))^* = L^{p'(\cdot)}([0, 1]) \oplus (L^{p(\cdot)}([0, 1]))_s^*,$$

i.e., every  $x^* \in (L^{p(\cdot)}([0, 1]))^*$  is uniquely represented in the form  $x^* = \xi_v + \varphi$ , where  $\xi_v$  is the regular functional defined by a function  $v \in L^{p'(\cdot)}([0, 1])$  by the formula

$$\xi_v(x) = \int_{[0,1]} v(t)x(t)dt, \quad x \in L^{p(\cdot)}([0, 1]), \quad (2.1)$$

and  $\varphi$  is a singular functional, i.e.,  $\varphi(x) = 0$  for any  $x \in E^{p(\cdot)}([0, 1])$  (for  $p_+ < \infty$ , we have  $L^{p(\cdot)}[0, 1]_s^* = \{0\}$ ).

**Proposition 2.1** ([9], Lemma 1.2). *For any  $x \in L^{p(\cdot)}([0, 1])$ , the equalities*

$$\lim_{n \rightarrow \infty} \|x - x_n\|_{p(\cdot)} = \theta(x) = d(x)$$

hold.

**Proposition 2.2** ([9], Lemma 1.3). *For any singular functional  $\varphi$ , the equalities*

$$\|\varphi\| = \sup\{\varphi(x) : \rho_{p(\cdot)}(x) < \infty\} = \sup_{x \in L^{p(\cdot)} \setminus E^{p(\cdot)}} \varphi(x)/\theta(x)$$

hold.

**Proposition 2.3** ([9], Lemma 1.4). *For any functional  $x^* = \xi_v + \varphi \in (L^{p(\cdot)}([0, 1]))^*$ , where  $\xi_v$  is defined by (2.1) and  $\varphi$  is a singular functional, the equality*

$$\|x^*\| = \|v\|_{p'(\cdot)} + \|\varphi\|$$

holds.

### 3. PROOF OF THEOREM 1.1

Let the space  $C([0, 1])$  be a closed subspace in  $L^{p(\cdot)}$ . Then there exists a positive constant  $c > 0$  such that

$$c \leq \|\chi_{(a,b)}\|_{p(\cdot)} \quad \text{whenever } 0 \leq a < b \leq 1, \quad (3.1)$$

(see [6]). It is obvious that for some constant  $C > 0$ ,

$$\|\chi_{(a,b)}\|_{p(\cdot)} \leq C \quad \text{whenever } 0 \leq a < b \leq 1. \quad (3.2)$$

From (3.1) and (3.2), we can deduce that for some constants  $c_1, c_2 > 0$  and for any  $x \in C([0, 1])$ ,

$$c_1 \|x\|_C \leq \|x\|_{p(\cdot)} \leq c_2 \|x\|_C \quad (3.3)$$

(for more details see [6]).

Denote  $X = C([0, 1])$  and  $Y = E^{p(\cdot)}([0, 1])$  ( $X$  is the Banach space with both norms  $\|\cdot\|_C$  and  $\|\cdot\|_{p(\cdot)}$ ). We have  $X \cap Y = \{0\}$  (by (3.1)). Consider the Cartesian product  $X \times Y$ , equipped with coordinate-wise vector space operations. For this vector space we have the Banach norm

$$\|(u, v)\|_\infty = \max\{\|u\|_{p(\cdot)}, \|v\|_{p(\cdot)}\}.$$

Denote the  $X \times Y$  vector space equipped with the norm  $\|(\cdot, \cdot)\|_\infty$  as  $(X \times Y)_\infty$ . Obviously,  $(X \times Y)_\infty$  is the Banach space. Our main goal is to prove that the mapping  $(X \times Y)_\infty \rightarrow X + Y \subset L^{p(\cdot)}([0, 1]) : (u, v) \rightarrow u + v$  is a (topological) isomorphism. In this case, the vector space  $X + Y$  with the norm  $\|\cdot\|_{p(\cdot)}$  is the topological direct sum of the Banach spaces  $X$  and  $Y$ , and it is written as  $X + Y = X \oplus Y$ . From this fact we find that the vector space  $X + Y = C([0, 1]) \oplus E^{p(\cdot)}([0, 1])$  with the norm  $\|\cdot\|_{p(\cdot)}$  is the Banach subspace of  $L^{p(\cdot)}([0, 1])$ , and we have

$$\|x + y\|_{p(\cdot)} \approx \max\{\|x\|_{p(\cdot)}, \|y\|_{p(\cdot)}\}, \quad x \in X, y \in Y. \quad (3.4)$$

It is obvious that the vector space  $X + Y$  with the norm  $\|\cdot\|_{p(\cdot)}$  is the topological direct sum of Banach spaces  $X$  and  $Y$  if the linear projection  $P : X + Y \rightarrow X$  defined by  $P(x + y) = x$  is continuous when  $x \in X$  and  $y \in Y$ . Note that this condition is equivalent to the following: there exists a positive real number  $\delta$  such that  $\|x - y\|_{p(\cdot)} \geq \delta$  whenever  $x \in X, y \in Y$  and  $\|x\|_{p(\cdot)} = 1$ .

Let  $x \in X$  and  $\|x\|_{p(\cdot)} = 1$ . Take  $t_0 \in [0, 1]$  such that

$$|x(t_0)| = \max_{t \in [0, 1]} |x(t)| = \|x\|_C.$$

By (3.3), we have

$$1/c_2 \leq |x(t_0)| \leq 1/c_1. \quad (3.5)$$

We will prove that

$$d(x) = \inf_{y \in Y} \|x - y\|_{p(\cdot)} \geq \delta > 0$$

for some constant  $\delta$ , independent of  $x$ .

By Proposition 2.1, we have

$$d(x) = \lim_{n \rightarrow \infty} \|x - x^{(n)}\|_{p(\cdot)}, \quad (3.6)$$

where  $x^{(n)} = \chi_{\Omega_n}$ ,  $\Omega_n = \{t : p(t) \leq n\}$ .

Denote  $O_n = (t_0 - \varepsilon_n, t_0 + \varepsilon_n)$ , where the numbers  $\varepsilon_n > 0$  will be chosen later.

We have

$$\begin{aligned} \|x - x^{(n)}\|_{p(\cdot)} &= \|x - x^{(n)}\chi_{[0, 1] \setminus O_n} - x^{(n)}\chi_{[0, 1] \cap O_n}\|_{p(\cdot)} \\ &\geq \left| \|x - x^{(n)}\chi_{[0, 1] \setminus O_n}\|_{p(\cdot)} - \|x^{(n)}\chi_{[0, 1] \cap O_n}\|_{p(\cdot)} \right|. \end{aligned} \quad (3.7)$$

Since for fixed  $n$  on the set  $\Omega_n$  we have  $p(t) \leq n$ , we may take  $O_n$  such that  $\|x^{(n)}\chi_{[0, 1] \cap O_n}\|_{p(\cdot)}$  is arbitrarily small. Using (3.1) and (3.5), we can choose  $O_n$  such small that

$$\|x\chi_{O_n}\|_{p(\cdot)} \geq \frac{1}{2}|x(t_0)|\|\chi_{O_n}\|_{p(\cdot)} \geq \frac{c}{2c_2}. \quad (3.8)$$

From (3.7) and (3.8), we obtain

$$\|x - x^{(n)}\|_{p(\cdot)} \geq \frac{c}{2c_2}$$

and, consequently, by (3.6), we have  $d(x) \geq \delta = \frac{c}{2c_2}$ .

Let  $x^*$  be any continuous linear functional from  $X^*$ . It is obvious that  $x^*$  is a continuous linear functional on the space  $X$  with the norm  $\|\cdot\|_{p(\cdot)}$  (by (3.3)). Since the space  $X \oplus Y$  is a Banach space with the norm  $\|\cdot\|_{p(\cdot)}$ , the trivial extension (i.e.,  $x^*(x) = 0$ , for  $x \in Y$ ) of  $x^*$  is also a continuous linear functional on  $X \oplus Y$  (see (3.4)). For the functional obtained in this way (defined on  $X \oplus Y$ ), there exists a continuous linear extension (non unique) on the whole space  $L^{p(\cdot)}([0, 1])$ . It is obvious that the obtained functional is singular (it is identically 0 on  $E^{p(\cdot)}([0, 1])$ ) on  $L^{p(\cdot)}([0, 1])$ .  $\square$

**Remark 1.** A closed subspace  $Y$  of the Banach space  $X$  is  $M$ -ideal in  $X$  if  $Y^\perp$  is the range of the bounded projection  $P : X^* \rightarrow X^*$  satisfying

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \quad \text{for all } x^* \in X^*.$$

For more details of the general  $M$ -ideal theory and their applications, we refer to [7]. If the subspace  $Y$  is  $M$ -ideal in  $X$ , then  $Y$  is proximal in  $X$  (see [7, p. 57, Proposition 1.1]), that is, for any  $x \in X$  there exists  $y \in Y$  such that

$$d(x) = \inf_{z \in Y} \|x - z\| = \|x - y\|.$$

From Proposition 2.3 we find that the space  $E^{p(\cdot)}([0, 1])$  is  $M$ -ideal in  $L^{p(\cdot)}([0, 1])$  and, consequently,  $E^{p(\cdot)}([0, 1])$  is proximal in  $L^{p(\cdot)}([0, 1])$ ; that is, for any  $x \in L^{p(\cdot)}([0, 1])$ , there exists  $y \in E^{p(\cdot)}([0, 1])$  such that  $d(x) = \|x - y\|_{p(\cdot)}$ .

**Remark 2.** Let  $C([0, 1])$  be a closed subspace in  $L^{p(\cdot)}([0, 1])$ . Denote  $I = L^\infty([0, 1]) \cap E^{p(\cdot)}([0, 1])$ . Note that if  $x_n \in I, n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \|x_n - x\|_\infty = 0$ , then  $x \in E^{p(\cdot)}([0, 1])$ . It is easy to show that  $I$  is an order ideal, which means that it is a closed subspace of  $L^\infty([0, 1])$  with the ideal property.

Note that there exists  $\delta > 0$  such that for  $x \in C([0, 1])$ ,  $\|x\|_C = 1$  and  $y \in I$ , we have  $\|x - y\|_\infty \geq \delta$ . The last inequality can be proved analogously as it has been done in the proof of Theorem 1.1. (It suffices to use the inequality  $\|x - y\|_\infty \geq \|x - y\|_{p(\cdot)}$  and the fact that  $\|x\|_{p(\cdot)} \approx 1$ ). Consequently, the vector space  $C([0, 1]) + I$  is the topological direct sum of Banach spaces  $C([0, 1])$  and  $I$  in  $L^\infty([0, 1])$ .

## ACKNOWLEDGEMENT

The research of the second author was supported by the Shota Rustaveli National Science Foundation of Georgia (SRNSFG) FR17\_589.

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(Received 01.12.2019)

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