A NOTE ON THE MAXIMAL OPERATORS OF THE NÖRLUND LOGARITHMIC MEANS OF VILENKIN-FOURIER SERIES

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Abstract. The main aim of this paper is to investigate the \((H_p, L_p)\)-type inequalities for the maximal operators of Nörlund logarithmic means for \(0 < p < 1\).

1. Introduction

It is well-known that (see e.g., [1], [8] and [16]) Vilenkin systems do not form bases in the Lebesgue space \(L_1(G_m)\). Moreover, there exists a function in the Hardy space \(H_1\) such that the partial sums of \(f\) are not bounded in \(L_1\)-norm.

In [19] (see also [21]), it was proved that the following is true:

**Theorem T1.** Let \(0 < p < 1\). Then the maximal operator

\[
\hat{S}_p f := \sup_{n \in \mathbb{N}} \frac{|S_n f|}{(n + 1)^{1/p - 1}}
\]

is bounded from the Hardy space \(H_p(G_m)\) to the space \(L_p(G_m)\). Here, \(S_n\) denotes the \(n\)-th partial sum with respect to the Vilenkin system. Moreover, it was proved that the rate of the factor \((n + 1)^{1/p - 1}\) is in a sense sharp.

In the case \(p = 1\), it was proved that the maximal operator \(\check{S}^*\) defined by

\[
\check{S}^* := \sup_{n \in \mathbb{N}} \frac{|S_n|}{\log(n + 1)}
\]

is bounded from the Hardy space \(H_1(G_m)\) to the space \(L_1(G_m)\). Moreover, the rate of the factor \(\log(n + 1)\) is in a sense sharp. Similar problems for the Nörlund logarithmic means in the case, where \(p = 1\), was considered in [15].

Móricz and Siddiqi [9] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of \(L_p(G_m)\) functions in \(L_p\)-norm. Fridli, Manchanda and Siddiqi [5] improved and extended the results of Móricz and Siddiqi [9] to the Martingale Hardy spaces. However, the case for \(\{q_k = 1/k : k \in \mathbb{N}_+\}\) was excluded, since the methods are not applicable to the Nörlund logarithmic means. In [6], Gt and Goginava proved some convergence and divergence properties of Walsh-Fourier series of the Nörlund logarithmic means of functions in the Lebesgue space \(L_1(G_m)\). In particular, they proved that there exists a function in the space \(L_1(G_m)\) such that

\[
\sup_{n \in \mathbb{N}} \|L_n f\|_1 = \infty.
\]

In [2] (see also [15,17]), it was proved that there exists a martingale \(f \in H_p(G_m)\), \((0 < p < 1)\) such that

\[
\sup_{n \in \mathbb{N}} \|L_n f\|_p = \infty.
\]

Analogous problems for the Nörlund means with respect to Walsh, Kaczmarz and unbounded Vilenkin systems were considered in Blahota, and Tephnadze, [3,4], Goginava and Nagy [7], Nagy and Tephnadze [10–12], Persson, Tephnadze and Wall [13,14], Tephnadze [18,20,21], Tutberidze [22].

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In this paper, we discuss the boundedness of the weighted maximal operators from the Hardy space $H_p (G_m)$ to the Lebesgue space $L_p (G_m)$ for $0 < p < 1$.

2. Definitions and Notation

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \ldots)$ denote a sequence of the positive integers, not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo $m_k$.

The elements of $G_m$ are represented by the sequences

$$x := (x_0, x_1, \ldots, x_m) \quad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of $G_m$,

$$J_n(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, \ n \in \mathbb{N})$$

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$.

If we define the so-called generalized number system based on $m$ in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N})$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$; where $n_j \in Z_{m_j} \ (j \in \mathbb{N})$ and only a finite number of $n_j$’s differs from zero. Let $|n| := \max\{j \in \mathbb{N}; \ n_j \neq 0\}$.

The norm (or quasi-norm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu \quad (0 < p < \infty).$$

The space $\text{weak} - L_p(G_m)$ consists of all measurable functions $f$ for which

$$\|f\|_{\text{weak} - L_p(G_m)} := \sup_{\lambda > 0} \lambda^p \mu (x : |f(x)| > \lambda) < +\infty.$$

Next, we introduce on $G_m$ an orthonormal system which is called the Vilenkin system. First we define the complex-valued function $r_k(x) : G_m \rightarrow C$, the generalized Rademacher functions as

$$r_k(x) := \exp (2\pi i x_k/m_k) \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as:

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}, \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if $m=2$.

The Vilenkin system is orthonormal and complete in $L_2 (G_m)$ \cite{1,23}.

Now we introduce analogues of the usual definitions in the Fourier analysis.
If \( f \in L_1(G_m) \), we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system \( \psi \) in the usual manner:

\[
\hat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu, \quad (k \in \mathbb{N}),
\]

\[
S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in \mathbb{N}_+, \quad S_0 f := 0),
\]

\[
D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+).
\]

Recall that (for details see e.g. [1])

\[
D_{M_n}(x) = \begin{cases} 
M_n & x \in I_n \\
0 & x \notin I_n.
\end{cases} \quad (1)
\]

The \( \sigma \)-algebra generated by the intervals \( \{I_n(x): x \in G_m\} \) will be denoted by \( F_n \ (n \in \mathbb{N}) \). Denote by \( f = (f_n: n \in \mathbb{N}) \) a martingale with respect to \( F_n \ (n \in \mathbb{N}) \) (for details see e.g. [24,25]). The maximal function of a martingale \( f \) is defined by

\[
f^* = \sup_{n \in \mathbb{N}} |f_n|.
\]

In the case, where \( f \in L_1 \), the maximal function is also given by

\[
f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|.
\]

For \( 0 < p < \infty \), the Hardy martingale spaces \( H_p(G_m) \) consist of all martingales for which

\[
\|f\|_{H_p} := \|f^*\|_p < \infty.
\]

If \( f \in L_1 \), then it is easy to show that the sequence \( (S_{M_n}f: n \in \mathbb{N}) \) is a martingale. If \( f = (f_n: n \in \mathbb{N}) \) is a martingale, then the Vilenkin-Fourier coefficients should be defined in a slightly different manner:

\[
\hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f_k \overline{\psi}_i d\mu.
\]

The Vilenkin-Fourier coefficients of \( f \in L_1(G_m) \) are the same as those of the martingale \( (S_{M_n}f: n \in \mathbb{N}) \) obtained from \( f \).

Let \( \{q_k: k > 0\} \) be a sequence of non-negative numbers. The \( n \)-th Nörlund means for the Fourier series of \( f \) is defined by

\[
\frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \quad \text{where} \quad Q_n := \sum_{k=1}^{n} q_k.
\]

If \( q_k = 1/k \), then we get the Nörlund logarithmic means

\[
L_n f := \frac{1}{l_n} \sum_{k=0}^{n-1} S_k f, \quad \text{where} \quad l_n = \sum_{k=0}^{n-1} \frac{1}{n-k} = \sum_{j=1}^{n} \frac{1}{j}.
\]

A bounded measurable function \( a \) is \( p \)-atom, if there exists a dyadic interval \( I \) such that

\[
\int_I a d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.
\]
3. Formulation of Main Results

**Theorem 1.** a) Let \( 0 < p < 1 \). Then the maximal operator

\[
\hat{L}_p^* f := \sup_{n \in \mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}}
\]

is bounded from the Hardy space \( H_p(G_m) \) to the space \( L_p(G_m) \).

b) Let \( 0 < p < 1 \) and \( \varphi : \mathbb{N}_+ \rightarrow [1, \infty) \) be a non-decreasing function satisfying the condition

\[
\lim_{n \to \infty} \frac{n^{1/p-1}}{\log n \varphi(n)} = +\infty.
\]

Then there exists a martingale \( f \in H_p(G_m) \) such that the maximal operator

\[
\sup_{n \in \mathbb{N}} \frac{|L_n f|}{\varphi(n+1)}
\]

is not bounded from the Hardy space \( H_p(G_m) \) to the space \( L_p(G_m) \).

4. Proof of the Theorem

**Proof.** Since

\[
\frac{|L_n f|}{(n+1)^{1/p-1}} \leq \frac{1}{(n+1)^{1/p-1}} \sup_{1 \leq k \leq n} |S_k f| \leq \sup_{1 \leq k \leq n} \frac{|S_k f|}{(k+1)^{1/p-1}} \leq \sup_{n \in \mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}},
\]

if we use Theorem T1, we obtain

\[
\sup_{n \in \mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}} \leq \sup_{n \in \mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}
\]

and

\[
\left\| \sup_{n \in \mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}} \right\|_p \leq \left\| \sup_{n \in \mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}} \right\|_p \leq c_p \| f \|_{H_p}.
\]

Now, prove part b) of the Theorem. Let

\[
f_{n_k} = D_{M_{2n_k+1}} - D_{M_{2n_k}}.
\]

It is evident that

\[
\hat{f}_{n_k}(i) = \begin{cases} 1, & \text{if } i = M_{2n_k}, \ldots, M_{2n_k} + 1 - 1, \\ 0, & \text{otherwise}. \end{cases}
\]

Then we can write that

\[
S_i f_{n_k} = \begin{cases} D_i - D_{M_{2n_k}}, & \text{if } i = M_{2n_k}, \ldots, M_{2n_k} + 1 - 1, \\ f_{n_k}, & \text{if } i \geq M_{2n_k} + 1, \\ 0, & \text{otherwise}. \end{cases}
\]

From (1), we get

\[
\| f_{n_k} \|_{H_p} = \left\| \sup_{n \in \mathbb{N}} S_n f_{n_k} \right\|_p = \left\| D_{M_{2n_k+1}} - D_{M_{2n_k}} \right\|_p \leq \left\| D_{M_{2n_k+1}} \right\|_p + \left\| D_{M_{2n_k}} \right\|_p \leq c M_{1/p}^{1/p-1} < c < \infty.
\]

Let \( 0 < p < 1 \) and \( \{ \lambda_k : k \in \mathbb{N}_+ \} \) be an increasing sequence of the positive integers such that

\[
\lim_{k \to \infty} \varphi(\lambda_k) = \infty.
\]
Let \( \{ n_k : k \in \mathbb{N}_+ \} \subset \{ \lambda_k : k \in \mathbb{N}_+ \} \) such that
\[
\lim_{k \to \infty} \frac{(M_{2n_k} + 2)^{1/p-1}}{\log (M_{2n_k} + 2) \varphi (M_{2n_k} + 2)} \geq c \lim_{k \to \infty} \frac{\lambda_k^{1/p-1}}{\varphi (\lambda_k)} = \infty.
\]
According to (2), we can conclude that
\[
\frac{\lambda_{M_{2n_k}} + 2 f_{n_k}}{\varphi (M_{2n_k} + 2)} = \frac{|D_{M_{2n_k} + 1} - D_{M_{2n_k}}|}{l_{M_{2n_k} + 1} \varphi (M_{2n_k} + 1)}
\]
\[
= \frac{|\psi_{M_{2n_k}}|}{l_{M_{2n_k} + 2} \varphi (M_{2n_k} + 1)} = \frac{1}{l_{M_{2n_k} + 1} \varphi (M_{2n_k} + 2)}.
\]
Hence,
\[
\mu \left\{ x \in G_m : \frac{1}{l_{M_{2n_k} + 2} \varphi (M_{2n_k} + 2)} \geq \frac{1}{l_{M_{2n_k} + 2} \varphi (M_{2n_k} + 2)} \right\} = \mu (G_m) = 1.
\]
By combining (3) and (4), we get
\[
\frac{1}{l_{M_{2n_k} + 2} \varphi (M_{2n_k} + 2)} \left( \mu \left\{ x \in G_m : \frac{1}{l_{M_{2n_k} + 2} \varphi (M_{2n_k} + 2)} \geq \frac{1}{l_{M_{2n_k} + 2} \varphi (M_{2n_k} + 2)} \right\} \right)^{1/p}
\]
\[
\geq \frac{M_{2n_k}^{1/p-1}}{l_{M_{2n_k} + 2} \varphi (M_{2n_k} + 2)} \geq c \left( \frac{M_{2n_k} + 2}{\log (M_{2n_k} + 2) \varphi (M_{2n_k} + 2)} \right)^{1/p-1} \to \infty, \quad \text{as} \quad k \to \infty.
\]

**Open Problem.** For any \( 0 < p < 1 \), let us find a non-decreasing function \( \Theta : \mathbb{N}_+ \to [1, \infty) \) such that the following maximal operator
\[
\widetilde{L}_p f := \sup_{n \in \mathbb{N}} \frac{|L_n f|}{\Theta (n + 1)}
\]
is bounded from the Hardy space \( H_p (G_m) \) to the Lebesgue space \( L_p (G_m) \) and the rate of \( \Theta : \mathbb{N}_+ \to [1, \infty) \) is sharp, that is, for any non-decreasing function \( \varphi : \mathbb{N}_+ \to [1, \infty) \) satisfying the condition
\[
\lim_{n \to \infty} \frac{\Theta (n)}{\varphi (n)} = +\infty,
\]
there exists a martingale \( f \in H_p (G_m) \) such that the maximal operator
\[
\sup_{n \in \mathbb{N}} \frac{|L_n f|}{\varphi (n + 1)}
\]
is not bounded from the Hardy space \( H_p (G_m) \) to the space \( L_p (G_m) \).

**Remark 1.** According to Theorem 1, we can conclude that there exist absolute constants \( C_1 \) and \( C_2 \) such that
\[
\frac{C_1 n^{1/p-1}}{\log (n + 1)} \leq \Theta (n) \leq C_2 n^{1/p-1}.
\]

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References


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