ON THE DOUBLE FOURIER–WALSH–PALEY SERIES OF CONTINUOUS FUNCTIONS

ROSTOM GETSADZE

Abstract. The following theorem is proved: There exists a continuous function $F$ on $[0, 1]^2$ such that the modulus of continuity

$$\omega(F; \delta) = O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right), \quad \delta \to 0^+,$$

and the double Fourier-Walsh-Paley series of $F$ diverges on a set of positive measure by rectangles.

1. Introduction

Kolmogorov [6] proved that there exists a function $f \in L([0, 2\pi])$ with the trigonometric Fourier series that diverges almost everywhere. Later, he constructed a function with everywhere divergent trigonometric Fourier series [7].

Stein [8] established that there exists a function $f \in L([0, 1])$ with the Fourier–Walsh–Paley series that diverges almost everywhere.

Carleson [3] proved that if $f \in L^2([0, 2\pi])$, then its trigonometric Fourier series converges almost everywhere. Billard [2] showed that if $g \in L^2([0, 1])$, then its Fourier–Walsh–Paley series converges almost everywhere.

Fefferman [4] proved that in the contrast to the Carleson’s theorem, there exists a continuous function of two variables on $[0, 2\pi]^2$ with the double trigonometric Fourier series that diverges almost everywhere by rectangles.

In [5], we have shown that there exists a continuous function on $[0, 1]^2$ the with double Fourier–Walsh–Paley series that diverges almost everywhere by rectangles.

The system of Rademacher functions $\{r_n(x)\}_{n=0}^\infty$ on $[0, 1)$ is defined as follows. Set

$$r_0(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} \leq x < 1. \end{cases}$$

We extend the function $r_0(x)$ on $(-\infty, \infty)$ with period 1. For $n \geq 1$, set

$$r_n(x) = r_0(2^n x).$$

The Walsh–Paley system of functions is defined as follows. Set $W_0(x) = 1$, for all $x \in [0, 1)$. For $n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_q}$, $(n \geq 1, m_1 > m_2 \cdots > m_q \geq 0)$, set

$$W_n(x) = r_{m_1}(x)r_{m_2}(x)\cdots r_{m_q}(x) \quad x \in [0, 1).$$

Let $f \in L([0, 1]^2)$ and let

$$\sum_{i=0}^\infty \sum_{j=0}^\infty a_{i,j}(f)W_i(x)W_j(y)$$

be the Fourier series of $f$ with respect to the double Walsh–Paley system $\{W_i(x)W_j(y)\}_{i,j=0}^\infty$ on $[0, 1)^2$.

The modulus of continuity $\omega(F; \delta)$ of a continuous function $F$ on $[0, 1]^2$ is defined by

$$\omega(F; \delta) = \sup_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \delta} \{|f(x_1, y_1) - f(x_2, y_2)|, \ (x_1, y_1), \ (x_2, y_2) \in [0, 1]^2\}.$$
We have established that we are able to achieve a certain smoothness in the divergence of the double Fourier–Walsh–Paley series of continuous functions. More precisely the following statement is true.

**Theorem 1.** There exists a continuous function \( F \) on \([0,1]^2\) such that

\[
\omega(F; \delta) = O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right), \quad \delta \to 0+,
\]

and the double Fourier–Walsh–Paley series of \( F \) diverges on a set of positive measure by rectangles.

Note that for the double trigonometric system, Bakhbokh and Nikishin [1] established stronger result where instead of \( O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right) \), \( \delta \to 0+ \), one has \( O\left(\frac{1}{\log_2 \frac{1}{\delta}}\right) \), \( \delta \to 0+ \). We note also that our method of proof in [5] allowed to achieve only a smoothness of order \( O\left(\frac{1}{\log_2 \log_2 \frac{1}{\delta}}\right) \), \( \delta \to 0+ \).

It is important to remark also that in the case of the trigonometric system the \( n \)-th kernel can be written as follows

\[
\sin nx \cot t + \cos nx,
\]

that is, as a sum of two terms, each of which is a product of a function in the trigonometric system \((\sin nx \text{ and } \cos nx)\) multiplied by a function that does not depend on \( n \) (\( \cot t \) and 1). This is not the case for the Walsh-Paley system and this fact complicates the proofs of the divergence results for this system.

**Acknowledgement**

Presented work was supported by the grant DI-18-118 of the Shota Rustaveli National Science Foundation of Georgia.

**References**


(Received 13.11.2019)

Department of Mathematics UpSala University, Box 480 751 06 UPPSALA, Sweden

E-mail address: rostom.getsadze@math.uu.se