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WEIGHTED BOUNDEDNESS OF THE FRACTIONAL MAXIMAL OPERATOR AND RIESZ POTENTIAL GENERATED BY GEGENBAUER DIFFERENTIAL OPERATOR

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Abstract. In the paper we study the weighted $(L_{p,\omega,\lambda}, L_{q,\omega,\lambda})$-boundedness of the fractional maximal operator M_G^α (G is a fractional maximal operator) and the Riesz potential (G is the Riesz potential) generated by the Gegenbauer differential operator

$$G_\lambda \equiv G = \left(x^2-1\right)^{\frac{1}{2}-\lambda} \frac{d}{dx} \left(x^2-1\right)^{\lambda+\frac{1}{2}} \frac{d}{dx}, \quad x \in (1,\infty), \quad \lambda \in \left(0, \frac{1}{2}\right).$$

1. Introduction

1.1. As is known the maximal functions, singular integrals and potentials generated by different differential operators are of importance in applications and in different questions of harmonic analysis and therefore their study is very topical. Non-accidentally, there exists extensive literature devoted to difference properties of the afore-mentioned object of harmonic analysis. We cite only those works that relate to the question considered in the paper: D. Adams [1], I. A. Aliev [2], A. P. Calderon [4], R. R. Coifman and C. Fefferman [5], C. Fefferman and E. Stein [8], A. I. Gadjiev [9], V. S. Guliev [11–13], G. H. Hardy and J. E. Littlewood [14], V. M. Kokilashvili, A. Kufner [20], V. M. Kokilashvili and S. Samko [21], I. A. Kipriyanov, M. N. Klyuchancev [16–19], L. N. Lyakhov [25, 26], G. Welland [30] and other.

In [12] E. V. Guliyev introduced analogous to Muckenhoupt classes generated by Gegenbauer differential operator and for the fractional maximal function and fractional integrals associated with the Bessel differential operator. He obtained some weighted inequalities, analogous to those given in Propositions 1.5 and 1.6 from [30].

We note that the measure of the homogeneous space satisfies the doubling properties $\mu E(x, 2r) \leq c\mu E(x,r)$, where the constant c is independent of x and r, and these properties are essentially used in proving many facts.

1.2. Based on our investigation, we introduce the Gegenbauer differential operator G_λ (see [7]). The shift operator $A^\lambda_{\operatorname{ch} t}$, generated by G_λ is given as follows (see [15]):

$$A^\lambda_{\operatorname{ch} t} f(\operatorname{ch} x) \equiv A_{\operatorname{ch} t}\ f(\operatorname{ch} x) = C_\lambda \int_0^\pi f\left(\operatorname{ch} x\ \operatorname{ch} t - \operatorname{sh} x\ \operatorname{sh} t\ \cos\varphi\right) (\sin\varphi)^{2\lambda-1} d\varphi,$$

where $C_\lambda = \dfrac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} = \left(\int\limits_0^\pi (\sin\varphi)^{2\lambda-1} d\varphi\right)^{-1}$ and possess all properties of the generalized shift operator from B.M. Levitan's monograph [22, 23].

Let $H(x,r) = (x-r, x+r) \cap [0,\infty)$, $r \in (0,\infty)$, $x \in \mathbb{R}_+ = [0,\infty)$. In this way,

$$H(x,r) = \begin{cases} (0,\ x+r), & 0 \leq x < r, \\ (x-r,\ x+r), & x \geq r. \end{cases}$$

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For any set $E \subset \mathbb{R}_+$, $\mu E = |E|_\lambda = \int_E \operatorname{sh}^{2\lambda} t dt$. The maximal functions (G-maximal functions) generated by the Gegenbauer differential operator [15] are defined as follows:

$$M_G f(\operatorname{ch} x) = \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r A_{\operatorname{ch} t} \big|f(\operatorname{ch} x)\big| d\mu_\lambda(t),$$

and

$$M_\mu f(\operatorname{ch} x) = \sup_{x\in\mathbb{R}_+, r>0} \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} \big|f(\operatorname{ch} t)\big| d\mu_\lambda(t),$$

where

$$d\mu_\lambda(t) = \operatorname{sh}^{2\lambda} t\, dt, \quad \big|H(x,r)\big|_\lambda = \int_{H(x,r)} \operatorname{sh}^{2\lambda} t\, dt, \quad \big|H(0,r)\big|_\lambda = \int_0^r \operatorname{sh}^{2\lambda} t\, dt.$$

The symbol $A \lesssim B$ denotes that there exists a constant C such that $0 < A \le CB$, and C may depend on some parameters. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

In [15], it is proved that

$$M_G f(\operatorname{ch} x) \lesssim M_\mu f(\operatorname{ch} x).$$

For any locally integrable function $f(\operatorname{ch} x)$, $x \in \mathbb{R}_+$, we denote the fractional maximal function (G-fractional maximal function) M_G^α generated by the Gegenbauer differential operator as follows:

$$M_G^\alpha f(\operatorname{ch} x) = \sup_{r>0} \big|H(0,r)\big|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int_0^r A_{\operatorname{ch} t} \big|f(\operatorname{ch} t)\big| d\mu_\lambda(t), \ 0 \le \alpha < 2\lambda + 1.$$

If $\alpha = 0$, then $M_G^0 f \equiv M_G f$.

We consider the Riezs potential (G-Riezs potential) generated by the Gegenbauer differential operator introduced in [15]

$$I_G^\alpha f(\operatorname{ch} x) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(\operatorname{ch} t) dr \right) A_{\operatorname{ch} t} f(\operatorname{ch} x) d\mu_\lambda(t), \tag{1.1}$$

where

$$h_r(\operatorname{ch} t) = \int_1^\infty e^{-\nu(\nu+2\lambda)r} P_\nu^\lambda(\operatorname{ch} t)(\nu^2 - 1)^{\lambda - \frac{1}{2}} d\nu$$

and $P_\nu^\lambda(\operatorname{ch} t)$ is an eigenfunction of the operator G.

We denote by $L_p(\mathbb{R}_+, G) \equiv L_{p,\lambda}(\mathbb{R}_+)$, $1 \le p \le \infty$, the space of functions measurable on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} \big|f(\operatorname{ch} t)\big|^p d\mu_\lambda(t) \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|f\|_{\infty,\lambda} = \|f\|_\infty = ess \sup_{t\in\mathbb{R}_+} \big|f(\operatorname{ch} t)\big|, \quad p = \infty.$$

We also denote by $WL_{p,\lambda}(\mathbb{R}_+)$ the weak space defined as a set of locally integrable functions on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{WL_{p,\lambda}(\mathbb{R}_+)} = \sup_{t>0} t \Big(\Big| \big\{ x \in \mathbb{R}_+ : |f(\operatorname{ch} x)| > t \big\} \Big|_\lambda \Big)^{\frac{1}{p}}.$$

The main objective of this paper is to obtain the results in [12], [29] and [30] for the operators M_G^α and I_G^α. The paper is organized as follows. In Section 2 we present some auxiliary results. We establish the equivalence of the G-fractional maximal functions M_G^α and M_μ^α in the form $M_G^\alpha\ f(\operatorname{ch} x) \approx M_\mu^\alpha\ f(\operatorname{ch} x)$. We obtain the Calderon–Zygmund decomposition of $\mathbb{R}_+$. In Section 3, we obtain the weighted analogue of the Fefferman-Stein inequality for the maximal function M_μ^α and give the Chebyshev-type inequality. We also introduce the Muckenhoupt-type class and obtain some properties of weights in order to use them in proving the inverse Hölder's inequality. We prove the weighted ($L_{p,w,\lambda}$, $L_{q,w,\lambda}$)- boundedness of the G-fractional maximal operator. In Section 4, we prove the weighted ($L_{p,w,\lambda}$, $L_{q,w,\lambda}$)- boundedness of the G-Riezs potential.

2. Auxiliary Results

In this section we will drive and prove some necessary fact, which we will need in the sequel.

Lemma 2.1 ([15]). *For $0<\lambda<\frac{1}{2}$, the following relation*

$$\left|H(0,r)\right|_{\lambda} \approx \begin{cases} \left(\operatorname{sh}\dfrac{r}{2}\right)^{2\lambda+1}, & 0<r<2,\\ \left(\operatorname{ch}\dfrac{r}{2}\right)^{4\lambda}, & 2\le r<\infty, \end{cases}$$

is valid. Note that this measure does not possess the doubling properties of the homogeneous space, i.e., $\mu E(x,2r)\le c\mu E(x,r)$. In this case, we have

$$\left|H(0,2r)\right|_{\lambda} \approx \begin{cases} \left(\operatorname{sh}2\cdot\dfrac{r}{2}\right)^{2\lambda+1} \ge 2^{2\lambda+1}\left(\operatorname{sh}\dfrac{r}{2}\right)^{2\lambda+1} \ge c_{\lambda}\left|H(0,r)\right|_{\lambda}, & 0<r<2,\\ \left(\operatorname{ch}2\cdot\dfrac{r}{2}\right)^{4\lambda} \ge \left(\operatorname{sh}2\cdot\dfrac{r}{2}\right)^{4\lambda} \ge 2^{4\lambda}\left(\operatorname{sh}\dfrac{r}{2}\right)^{4\lambda} \ge c_{\lambda}\left|H(0,r)\right|_{\lambda}, & 2\le r<\infty, \end{cases}$$

where c_λ depends on the parameter $\lambda>0$.

Lemma 2.2 ([15]). *Let $0<\lambda<\frac{1}{2}$, $0\le x<\infty$ and $0<r<\infty$. Then for $0<r<2$,*

$$\left|H(x,r)\right|_{\lambda} \lesssim \begin{cases} \left(\operatorname{sh}\ \dfrac{r}{2}\right)^{2\lambda+1}, & 0\le x<r,\\ \left(\operatorname{sh}\ \dfrac{r}{2}\right)(\operatorname{sh}\ x)^{2\lambda}, & r\le x<\infty, \end{cases}$$

and for $2\le r<\infty$, we have

$$\left|H(x,r)\right|_{\lambda} \lesssim \begin{cases} (\operatorname{ch}r)^{2\lambda}, & 0\le x<r,\\ (\operatorname{ch}x)^{2\lambda}(\operatorname{ch}r)^{2\lambda}, & 2\le r<\infty. \end{cases}$$

Lemma 2.3. *Let $0\le x<\infty$ and $0<r<\infty$. For any $\gamma>0$, the following relation*

$$\left|H(x,r)\right|_{\frac{\gamma}{2}} \approx \begin{cases} \left(\operatorname{sh}\dfrac{x+r}{2}\right)^{\gamma+1}, & 0<x+r<2,\\ \left(\operatorname{sh}\dfrac{x+r}{2}\right)^{2\gamma}, & 2\le x+r<\infty, \end{cases}$$

holds.

Proof. Let $0\le x<r$. We consider the case where $0<x+r<2$. Let $0<\gamma\le 1$. Then

$$\begin{aligned} |H(x,r)|_{\frac{\gamma}{2}} &= \int\limits_0^{x+r}(\operatorname{sh}\ t)^{\gamma}dt = \int\limits_0^{x+r}(sh\ t)^{\gamma-1}d(\operatorname{ch}\ t) = \int\limits_0^{x+r}\left(\operatorname{ch}^2 t-1\right)^{\frac{\gamma-1}{2}}d(\operatorname{ch}\ t)\\ &= \int\limits_0^{x+r}\frac{(\operatorname{ch}\ t-1)^{\frac{\gamma-1}{2}}}{(\operatorname{ch}\ t+1)^{\frac{1-\gamma}{2}}}d(\operatorname{ch}\ t) \ge (\operatorname{ch}(x+r)+1)^{\frac{\gamma-1}{2}}\int\limits_0^{x+r}(\operatorname{ch}\ t-1)^{\frac{\gamma-1}{2}}d(\operatorname{ch}\ t)\\ &\ge \frac{2}{\gamma+1}(1+\operatorname{ch}\ 2)^{\frac{\gamma-1}{2}}\left(\operatorname{ch}(x+r)-1\right)^{\frac{\gamma+1}{2}}\\ &= \frac{2}{\gamma+1}\left(2\operatorname{ch}^2 1\right)^{\frac{\gamma-1}{2}}\left(2\operatorname{sh}^2\frac{x+r}{2}\right)^{\frac{\gamma+1}{2}} = \frac{2^{\gamma+1}}{\gamma+1}(\operatorname{ch}1)^{\gamma-1}\left(\operatorname{sh}\frac{x+r}{2}\right)^{\gamma+1}. \end{aligned} \tag{2.1}$$

Let $\gamma > 1$. Then

$$\int_0^{x+r} (\mathrm{sh}\ t)^\gamma dt = \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} (\mathrm{ch}\ t+1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$\geq 2^{\frac{\gamma-1}{2}} \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$= \frac{2^{\frac{\gamma+1}{2}}}{\gamma+1} (\mathrm{ch}(x+r)-1)^{\frac{\gamma+1}{2}} \frac{2^{\frac{\gamma+1}{2}}}{\gamma+1} \left(2\,\mathrm{sh}^2 \frac{x+r}{2}\right)^{\frac{\gamma+1}{2}} = \frac{2^{\gamma+1}}{\gamma+1} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}. \tag{2.2}$$

On the other hand, for $0 < \gamma \leq 1$ and $0 < x+r < 2$, we have

$$\int_0^{x+r} (\mathrm{sh}\ t)^\gamma dt = \int_0^{x+r} \frac{(\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}}}{(\mathrm{ch}\ t+1)^{\frac{1-\gamma}{2}}} d(\mathrm{ch}\ t) \leq 2^{\frac{\gamma-1}{2}} \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$= \frac{2^{\frac{\gamma+1}{2}}}{\gamma+1} (\mathrm{ch}(x+r)-1)^{\frac{\gamma+1}{2}} = \frac{2^{\gamma+1}}{\gamma+1} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}. \tag{2.3}$$

For $\gamma > 1$,

$$\int_0^{x+r} (\mathrm{sh}\ t)^\gamma dt \int_0^{x+r} (\mathrm{ch}\, t-1)^{\frac{\gamma-1}{2}} (\mathrm{ch}\ t+1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$\leq \frac{2}{\gamma+1} (\mathrm{ch}(x+r)+1)^{\frac{\gamma-1}{2}} (\mathrm{ch}(x+r)-1)^{\frac{\gamma+1}{2}}$$

$$\leq \frac{2}{\gamma+1} (\mathrm{ch}\ 2+1)^{\frac{\gamma-1}{2}} \left(2\,\mathrm{sh}^2 \frac{x+r}{2}\right)^{\frac{\gamma+1}{2}}$$

$$= \frac{2}{\gamma+1} \left(2\,\mathrm{ch}^2\, 1\right)^{\frac{\gamma-1}{2}} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1} = \frac{2^{\gamma+1}}{\gamma+1} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}. \tag{2.4}$$

Combining (2.1)–(2.4), we obtain

$$\big|H(x,r)\big|_{\frac{\gamma}{2}} \approx \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}, \quad \gamma > 0, \quad 0 < x+r < 2. \tag{2.5}$$

Consider now the case where $2 \leq x+r < \infty$ and $0 \leq x < r$.

Let $0 < \gamma \leq 1$. Then we get

$$\int_0^{x+r} (\mathrm{sh}\ t)^\gamma dt = \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} (\mathrm{ch}\ t+1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$\geq \big(\mathrm{ch}(x+r)+1\big)^{\frac{\gamma-1}{2}} \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$= \frac{2}{\gamma+1} \left(2\,\mathrm{ch}^2 \frac{x+r}{2}\right)^{\frac{\gamma-1}{2}} (\mathrm{ch}(x+r)-1)^{\frac{\gamma+1}{2}} = \frac{2}{\gamma+1} \frac{\left(2\,\mathrm{sh}^2\, \frac{x+r}{2}\right)^{\frac{\gamma+1}{2}}}{\left(2\,\mathrm{ch}^2\, \frac{x+r}{2}\right)^{\frac{1-\gamma}{2}}}$$

$$= \frac{2^{\gamma+1}}{\gamma+1} \frac{\left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}}{\left(\mathrm{ch}\, \frac{x+r}{2}\right)^{1-\gamma}} \geq \frac{2^{2\gamma}}{\gamma+1} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{2\gamma}, \tag{2.6}$$

since $\mathrm{ch}\, \frac{t}{2} \leq 2\,\mathrm{sh}\, \frac{t}{2}$ for $t \geq 2$.

On the other hand, for $0<\gamma\le 1$,

$$\int\limits_0^{x+r}(\text{sh }\ t)^\gamma dt=\int\limits_0^{x+r}\left(2\,\text{sh}\,\frac{t}{2}\,\text{ch}\,\frac{t}{2}\right)^\gamma dt=2^{\gamma+1}\int\limits_0^{x+r}\left(\text{sh}\,\frac{t}{2}\right)^\gamma\left(\text{ch}\,\frac{t}{2}\right)^{\gamma-1}d\left(\text{sh}\,\frac{t}{2}\right)$$

$$\le 2^{\gamma+1}\int\limits_0^{x+r}\left(\text{sh}\,\frac{t}{2}\right)^{2\gamma-1}d\left(\text{sh}\,\frac{t}{2}\right)=\frac{2^\gamma}{\gamma}\left(\text{sh}\,\frac{x+r}{2}\right)^{2\gamma}. \tag{2.7}$$

Let $\gamma>1$. Then

$$\int\limits_0^{x+r}(\text{sh }\ t)^\gamma dt=2^{\gamma+1}\int\limits_0^{x+r}\left(\text{sh}\,\frac{t}{2}\right)^\gamma\left(\text{ch}\,\frac{t}{2}\right)^{\gamma-1}d\left(\text{sh}\,\frac{t}{2}\right)$$

$$\ge 2^{\gamma+1}\int\limits_0^{x+r}\left(\text{sh}\,\frac{t}{2}\right)^{2\gamma-1}d\left(\text{sh}\,\frac{t}{2}\right)=\frac{2^\gamma}{\gamma}\left(\text{sh}\,\frac{x+r}{2}\right)^{2\gamma}, \tag{2.8}$$

and also

$$\int\limits_0^{x+r}(\text{sh }\ t)^\gamma dt=2^{\gamma+1}\int\limits_0^{x+r}\left(\text{sh}\,\frac{t}{2}\right)^\gamma\left(\text{ch}\,\frac{t}{2}\right)^{\gamma-1}d\left(\text{sh}\,\frac{t}{2}\right)$$

$$\le 2^{\gamma+1}\int\limits_0^{x+r}\left(\text{sh}\,\frac{t}{2}\right)^{2\gamma-1}d\left(\text{sh}\,\frac{t}{2}\right)=\frac{2^\gamma}{\gamma}\left(\text{sh}\,\frac{x+r}{2}\right)^{2\gamma}. \tag{2.9}$$

Combining (2.6)–(2.9), we obtain

$$\big|H(x,r)\big|_{\frac{\gamma}{2}}\approx\left(\text{sh}\,\frac{x+r}{2}\right)^{2\gamma},\ \ \gamma>0,\ \ 2\le x+r<\infty. \tag{2.10}$$

Now, from (2.5) and (2.10), for $0\le x<r$, we have

$$\big|H(x,r)\big|_{\frac{\gamma}{2}}=\int\limits_0^{x+r}(\text{sh }\ t)^\gamma dt\approx\begin{cases}\left(\text{sh}\ \dfrac{x+r}{2}\right)^{\gamma+1}, & 0<x+r<2,\\ \left(\text{sh}\ \dfrac{x+r}{2}\right)^{2\gamma}, & 2\le x+r<\infty.\end{cases} \tag{2.11}$$

Now, let $x\ge r$. Then from (2.3), (2.4), (2.7) and (2.9), for $\gamma>0$, we get

$$\big|H(x,r)\big|_{\frac{\gamma}{2}}=\int\limits_{x-r}^{x+r}(\text{sh }\ t)^\gamma dt\le\int\limits_0^{x+r}(\text{sh }\ t)^\gamma dt\lesssim\begin{cases}\left(\text{sh}\ \dfrac{x+r}{2}\right)^{\gamma+1}, & 0<x+r<2,\\ \left(\text{sh}\ \dfrac{x+r}{2}\right)^{2\gamma}, & 2\le x+r<\infty.\end{cases} \tag{2.12}$$

Now, let $\gamma>0$ and $0<x+r<2$. Then

$$\big|H(x,r)\big|_\lambda=\int\limits_{x-r}^{x+r}(\text{sh }\ t)^\gamma dt\ge\int\limits_{\frac{x+r}{2}}^{x+r}(\text{sh }\ t)^\gamma dt\ge\frac{x+r}{2}\left(\text{sh}\,\frac{x+r}{2}\right)^\gamma\approx\left(\text{sh}\,\frac{x+r}{2}\right)^{\gamma+1},$$

since $\text{sh}\,t\approx t$ for $0<t<1$.

On the other hand, from (2.3) and (2.4), for $\gamma>0$, we have

$$\big|H(x,r)\big|_\gamma=\int\limits_{x-r}^{x+r}(\text{sh }\ t)^\gamma dt\le\int\limits_0^{x+r}(\text{sh }\ t)^\gamma dt\le\frac{2^{\gamma+1}}{\gamma+1}\left(\text{sh}\ \frac{x+r}{r}\right)^{\gamma+1}.$$

In this way, by any $\gamma > 0$ and $0 < x+r < 2$,

$$\big|H(x,r)\big|_{\frac{\gamma}{2}} = \int\limits_{x-r}^{x+r} (\text{sh } t)^\gamma dt \approx \left(\text{sh } \frac{x+r}{2}\right)^{\gamma+1}. \tag{2.13}$$

Consider now the case where $2 \le x+r < \infty$. Let $\gamma > 0$, then

$$\begin{aligned}\big|H(x,r)\big|_{\frac{\gamma}{2}} &= \int\limits_{x-r}^{x+r} (\text{sh } t)^\gamma dt \ge \int\limits_{\frac{x+r}{2}}^{x+r} (\text{sh } t)^\gamma dt = \int\limits_{\frac{x+r}{2}}^{x+r} \frac{(\text{sh } t)^\gamma}{\text{ch } t} d(\text{sh } t)\\ &\ge \frac{1}{2}\int\limits_{\frac{x+r}{2}}^{x+r} (\text{sh } t)^{\gamma-1} d(\text{sh } t) = \frac{1}{2\gamma}\left(\text{sh}^\gamma (x+r) - \text{sh}^\gamma\left(\frac{x+r}{2}\right)\right)\\ &\ge \frac{1}{2\gamma}\left(\text{sh}^\gamma(x+r) - \frac{1}{2^\gamma}\text{sh}^\gamma(x+r)\right) = \frac{1}{2\gamma}\left(1-\frac{1}{2^\gamma}\right)\text{sh}^\gamma(x+r)\\ &= \frac{2^\gamma - 1}{2\gamma}\left(\text{sh}^\gamma \frac{x+r}{2}\,\text{ch}^\gamma \frac{x+r}{2}\right) \ge \frac{2^\gamma-1}{2\gamma}\left(\text{sh } \frac{x+r}{2}\right)^{2\gamma}.\end{aligned} \tag{2.14}$$

On the other hand, we can get from (2.7) and (2.9) for $\gamma > 0$

$$\big|H(x,r)\big|_{\frac{\gamma}{2}} = \int\limits_{x-r}^{x+r} (\text{sh } t)^\gamma dt \le \int\limits_{0}^{x+r} (\text{sh } t)^\gamma dt \le \frac{2^\gamma}{\gamma}\left(\text{sh } \frac{x+r}{2}\right)^{2\gamma}. \tag{2.15}$$

Thus from (2.14) and (2.15), for $x \ge r$, we obtain

$$\big|H(x,r)\big|_{\frac{\gamma}{2}} = \int\limits_{x-r}^{x+r} (\text{sh } t)^\gamma dt \approx \begin{cases} \left(\text{sh } \dfrac{x+r}{2}\right)^{\gamma+1}, & 0 < x+r < 2,\\ \left(\text{sh } \dfrac{x+r}{2}\right)^{2\gamma}, & 2 \le x+r < \infty. \end{cases} \tag{2.16}$$

The assertion of Lemma 2.1 follows from (2.11)–(2.16). □

Supposing $\gamma = 2\lambda$ in Lemma 2.3, we obtain

$$\big|H(x,r)\big|_{\lambda} \approx \begin{cases} \left(\text{sh } \dfrac{x+r}{2}\right)^{2\lambda+1}, & 0 < x+r < 2,\\ \left(\text{sh } \dfrac{x+r}{2}\right)^{4\lambda}, & 2 \le x+r < \infty. \end{cases} \tag{2.17}$$

Since $\text{sh } t \approx \text{ch}\, t$ for $t \ge 1$, then from this for $x = 0$ we, in particular, get Lemma 2.1.

Lemma 2.4. *For a nonnegative function $f(\text{ch}\, x)$, $x \in \mathbb{R}_+$, the following relation*

$$\int\limits_0^r A^\lambda_{\text{ch}\, t} f(\text{ch } x)\, \text{sh}^{2\lambda} t\, dt \approx \int\limits_{H(x,r)} f(\text{ch}\, u)\, \text{sh}^{2\lambda} u\ du$$

holds.

Proof. In [15], it is proved that

$$J(x,r) = \int\limits_0^r A^\lambda_{\text{ch}\, t} f(\text{ch}\, x)\, \text{sh}^{2\lambda} t\ dt = C_\lambda \int\limits_{\text{ch}(x-r)}^{\text{ch}(x+r)} f(z)(z^2-1)^{\lambda - \frac{1}{2}} \int\limits_{\varphi(z,x,r)}^{1} (1-u^2)^{\lambda-1} du\ dz,$$

where

$$\varphi(z,x,r) = \frac{z \text{ ch } x - \text{ch } r}{\sqrt{z^2-1}\,\text{sh } x}, \quad -1 \le \varphi(z,x,r) \le 1, \quad C_\lambda = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma\left(\lambda\right)\Gamma\left(\frac{1}{2}\right)}.$$

Then

$$A(z,x,r)=C_\lambda\int\limits_{\varphi(z,x,r)}^{1}(1-u^2)^{\lambda-1}du\le C_\lambda\int\limits_{-1}^{1}(1-u^2)^{\lambda-1}du=1.$$

We now estimate the integral $A(z,x,r)$ from below. Let $-1\le\varphi(z,x,r)\le 0$. Then

$$\begin{aligned}A(z,x,r)&=C_\lambda\int\limits_{\varphi(z,x,r)}^{1}(1-u^2)^{\lambda-1}du\ge C_\lambda\int\limits_{0}^{1}(1-u^2)^{\lambda-1}du\\&\ge 2^{\lambda-1}C_\lambda\int\limits_{0}^{1}(1-u)^{\lambda-1}du=\frac{2^{\lambda-1}}{\lambda}\,C_\lambda.\end{aligned}$$

Now, let $0\le\varphi(z,x,r)\le 1$, then

$$\begin{aligned}A(z,x,r)&=C_\lambda\int\limits_{\varphi(z,x,r)}^{1}(1-u)^{\lambda-1}(1+u)^{\lambda-1}du\\&=C_\lambda\int\limits_{0}^{1-\varphi(z,x,r)}u^{\lambda-1}(2-u)^{\lambda-1}du=C_\lambda\int\limits_{\frac{1}{1-\varphi(z,x,r)}}^{\infty}u^{-\lambda-1}\left(2-\frac{1}{u}\right)^{\lambda-1}du\\&=C_\lambda\int\limits_{\frac{1}{1-\varphi(z,x,r)}}^{\infty}u^{-2\lambda}(2u-1)^{\lambda-1}du=2^{2\lambda-1}C_\lambda\int\limits_{\frac{2}{1-\varphi(z,x,r)}}^{\infty}u^{-2\lambda}(u-1)^{\lambda-1}du\\&=2^{2\lambda-1}C_\lambda\int\limits_{\frac{1-\varphi(z,x,r)}{1+\varphi(z,x,r)}}^{\infty}(u+1)^{-2\lambda}u^{\lambda-1}du=2^{2\lambda-1}\cdot C_\lambda\int\limits_{0}^{\frac{1+\varphi(z,x,r)}{1-\varphi(z,x,r)}}(1+u)^{-2\lambda}u^{\lambda-1}du\\&\ge 2^{2\lambda-1}C_\lambda\int\limits_{0}^{1}(1+u)^{-2\lambda}u^{\lambda-1}du\ge 2^{2\lambda-1}C_\lambda\int\limits_{0}^{1}\frac{u^{\lambda-1}}{(1+u)^{2\lambda}}du\ge\frac{C_\lambda}{2}\int\limits_{0}^{1}u^{\lambda-1}du=\frac{C_\lambda}{2\lambda}\ .\end{aligned}$$

Consequently,

$$A(z,x,r)=\int\limits_{\varphi(z,x,r)}^{1}(1-u^2)^{\lambda-1}du\approx 1$$

and

$$J(x,r)\approx\int\limits_{\operatorname{ch}(x-r)}^{\operatorname{ch}(x+r)}f(z)(z^2-1)^{\lambda-\frac{1}{2}}dz=\int\limits_{H(x,r)}f\left(\operatorname{ch}u\right)\operatorname{sh}^{2\lambda}u\,du.$$ □

Theorem 2.1. *For $0\le x<\infty$ and $0<r<\infty$, the relation*

$$M_G^\alpha f(\operatorname{ch}x)\approx M_\mu^\alpha f(\operatorname{ch}x)$$

is valid.

Proof. First, we prove that

$$M_G^\alpha f(\operatorname{ch}x)\lesssim M_\mu^\alpha f(\operatorname{ch}x).$$

We consider

$$A_{\operatorname{ch}t}\chi_{(0,r)}(\operatorname{ch}x)=C_\lambda\int\limits_{0}^{\pi}\chi_{(0,r)}(x,t)_\varphi(\sin\varphi)^{2\lambda-1}d\varphi,$$

where $(x,t)_\varphi = \operatorname{ch} x \ \operatorname{ch} t - \operatorname{sh} x \ \operatorname{sh} t \cos\varphi$, and $\chi_{(0,r)}$ is the characteristic function on the interval $(0,r)$, i.e.,

$$\chi_{(0,r)}(x,t)_\varphi = \begin{cases} 1, & (x,t)_\varphi \le r, \\ 0, & (x,t)_\varphi > r. \end{cases}$$

Since

$$x - t < \operatorname{ch}(x-t) \le (x,t)_\varphi \le \operatorname{ch}(x+t),$$

we have $x - t > r \Longrightarrow (x,t)_\varphi > r$.

From the inequality $|x-t| > r$ it follows that $A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x) = 0$. This shows that the carrier at the function $A_{\operatorname{ch} t}\chi_{(0,r)}$ belongs to $H(x,r)$.

More generally, $x,t \in \mathbb{R}_+$

$$A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x) = C_\lambda \int_0^\pi \chi_{(0,r)}(x,t)_\varphi (\operatorname{sh}\ \varphi)^{2\lambda-1} d\varphi \le C_\lambda \int_0^\pi (\sin\varphi)^{2\lambda-1} d\varphi = 1.$$

We estimate $A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x)$:

$$A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x) = C_\lambda \int_{\{\varphi\in[0,\pi]:(x,t)_\varphi<r\}} (\sin\varphi)^{2\lambda-1} d\varphi = A(x,t,r).$$

Making the substitution $\cos\varphi = y$, we obtain

$$A(x,t,r) = C_\lambda \int_{\max\left\{-1,\ \frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t}\right\}}^{1} (1-y^2)^{\lambda-1} dy.$$

For any $x,t \in \mathbb{R}_+$, we have

$$\frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t} \ge -1 \Leftrightarrow \operatorname{ch} x\ \operatorname{ch} t + \operatorname{sh} x\ \operatorname{sh} t \ge r \Leftrightarrow \operatorname{ch}(x+t) \ge r.$$

Then in the case for $\operatorname{ch}(x-t) < r < \operatorname{ch}(x+t)$, we obtain

$$\begin{aligned} A(x,t,r) &\lesssim \int_{\frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t}}^{1} (1-y^2)^{\lambda-1} dy \lesssim \int_{\frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t}}^{1} (1-y)^{\lambda-1} dy \\ &\lesssim \left(\frac{r - \operatorname{ch}(x-t)}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda \lesssim \left(\frac{\operatorname{ch}(x+t) - \operatorname{ch}(x-t)}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda = \left(\frac{2\ \operatorname{sh} x\ \operatorname{sh} t}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda = 2^\lambda. \end{aligned} \tag{2.18}$$

On the other hand,

$$A(x,t,r) \lesssim \left(\frac{r - \operatorname{ch}(x-t)}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda \lesssim \left(\frac{\operatorname{ch}\ 2r - 1}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda = \left(\frac{2\ \operatorname{sh}^2 r}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda.$$

From this, for $t \ge x$, we have

$$A(x,t,r) \lesssim \left(\frac{\operatorname{sh}\ r}{\operatorname{sh}\ x}\right)^{2\lambda}. \tag{2.19}$$

We consider the case $0 < t < x$:

$$\begin{aligned} A(x,t,r) &\lesssim \int_{\frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t}}^{1} (1-y^2)^{\lambda-1} dy \lesssim \int_{\frac{\operatorname{ch} x\ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x\ \operatorname{sh} t}}^{1} (1-y)^{\lambda-1} dy \\ &\lesssim \left(1 - \frac{\operatorname{ch} x\ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda \lesssim \left(1 - \left(\frac{\operatorname{ch} x\ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x\ \operatorname{sh} t}\right)^2\right)^\lambda. \end{aligned} \tag{2.20}$$

We find the extremum of the function

$$
\begin{aligned}
f(\operatorname{ch} t) &= 1-\left(\frac{\operatorname{ch} x\ \operatorname{ch} t-\operatorname{ch} r}{\operatorname{sh} x\ \operatorname{sh} t}\right)^2, \\
(\operatorname{sh} t) f'(\operatorname{ch} t) &= -2\left(\frac{\operatorname{ch} x\ \operatorname{ch} t-\operatorname{ch} r}{\operatorname{sh} x\ \operatorname{sh} t}\right)\frac{\operatorname{sh}^2 t\ \operatorname{ch} x\ \operatorname{sh} x-\operatorname{sh} x\ \operatorname{ch} t\cdot(\operatorname{ch} x\ \operatorname{ch} t-\operatorname{ch} r)}{(\operatorname{sh} x\ \operatorname{sh} t)^2} \\
&= -2\left(\frac{\operatorname{ch} x\ \operatorname{ch} t-\operatorname{ch} r}{\operatorname{sh} t}\right)\left(\frac{\operatorname{ch} x\ \operatorname{sh}^2 t-\operatorname{ch}^2 t\ \operatorname{ch} x+\operatorname{ch} t\ \operatorname{ch} r)}{\operatorname{sh}^2 x\ \operatorname{sh}^2 t}\right) \\
&= \frac{2(\operatorname{ch} x\ \operatorname{ch} t-\operatorname{ch} r)(\operatorname{ch} x-\operatorname{ch} t\ \operatorname{ch} r)}{\operatorname{sh}^3 t\ \operatorname{sh}^2 x}.
\end{aligned}
$$

Then it follows that

$$
f'(\operatorname{ch} t)=\frac{2(\operatorname{ch} x\ \operatorname{ch} t-\operatorname{ch} r)(\operatorname{ch} x-\operatorname{ch} t\ \operatorname{ch} r)}{\operatorname{sh}^4 t\ \operatorname{sh}^2 x}.
$$

From this, for $x>r$, we have

$$
\begin{aligned}
f_{\max}\left(\frac{\operatorname{ch} x}{\operatorname{ch} r}\right) &= 1-\left(\frac{\operatorname{ch}^2 x-\operatorname{ch}^2 r}{\sqrt{\operatorname{ch}^2 x-\operatorname{ch}^2 r}\ \operatorname{sh} x}\right)^2 \\
&= 1-\frac{\operatorname{ch}^2 x-\operatorname{ch}^2 r}{\operatorname{sh}^2 x}=\frac{\operatorname{ch}^2 r-1}{\operatorname{sh}^2 x}=\left(\frac{\operatorname{sh} r}{\operatorname{sh} x}\right)^2.
\end{aligned}
$$

From this and (2.20), we obtain

$$
A(x,t,r)\lesssim\left(\frac{\operatorname{sh} r}{\operatorname{sh} x}\right)^{2\lambda},\quad 0<t<x,\ \ x>r. \tag{2.21}
$$

Thus from (2.18)–(2.21), we get

$$
A(x,t,r)\lesssim\min\left\{1,\left(\frac{\operatorname{sh} r}{\operatorname{sh} x}\right)^{2\lambda}\right\}.
$$

Consequently, for any $t\in H(x,r)$,

$$
A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x)\lesssim\min\left\{1,\ \left(\frac{\operatorname{sh} r}{\operatorname{sh} x}\right)^{2\lambda}\right\},\quad x>r. \tag{2.22}
$$

We now have

$$
\begin{aligned}
M_G^{\alpha} f(\operatorname{ch} x) &\le M_{G.1}^{\alpha} f(\operatorname{ch} x)+M_{G.2}^{\alpha} f(\operatorname{ch} x) \\
&= \sup_{0\le x<r<2}\left|H(0,r)\right|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1}\int\limits_{H(x,r)}\left|f(\operatorname{ch} t)\right|A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} t)d\mu_{\lambda}(t) \\
&+ \sup_{r\le x<2}\left|H(0,r)\right|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1}\int\limits_{H(x,r)}\left|f(\operatorname{ch} t)\right|A_{\operatorname{ch} t}\chi_{(0,r)}d\mu_{\lambda}(t).
\end{aligned}\tag{2.23}
$$

Let $0 \le x < r < 2$. From (2.22), it follows that $A_{\mathrm{ch}\, t}\chi_{(0,r)}(\mathrm{ch}\, x) \le 1$. From Lemmas 2.1 and 2.2, we obtain

$$
\begin{aligned}
M^{\alpha}_{G.1} f(\mathrm{ch}\, x) &\le \sup_{0\le x<r<2} |H(0,r)|_{\lambda}^{\frac{\alpha}{2\lambda+1}} \int\limits_{H(x,r)} \big|f(\mathrm{ch}\, t)\big| d\mu_\lambda(t) \\
&\le \sup_{0\le x<r<2} \frac{|H(0,r)|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} |H(x,r)|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1}}{|H(x,r)|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1}} \int\limits_{H(x,r)} |f(\mathrm{ch}\, t)| d\mu_\lambda(t) \\
&\lesssim \sup_{0\le x<r<2} \left(\frac{\mathrm{sh}\, \frac{r}{2}}{\mathrm{sh}\, \frac{r}{2}}\right)^{2\lambda+1-\alpha} |H(x,r)|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} \big|f(\mathrm{ch}\, t)\big| d\mu_\lambda(t) \\
&\lesssim M^{\alpha}_{\mu} f(\mathrm{ch}\, x). \qquad (2.24)
\end{aligned}
$$

From Lemmas 2.1 and 2.2, and also (2.21), for $r < x < 2$. we have

$$
\begin{aligned}
M^{\alpha}_{G.2} f(\mathrm{ch}\, x) &\lesssim \sup_{r<x<2} \left(\frac{\mathrm{sh}\, \frac{r}{2}\, \mathrm{sh}^{2\lambda} x}{\left(\mathrm{sh}\, \frac{r}{2}\right)^{2\lambda+1}}\right)^{1-\frac{\alpha}{2\lambda+1}} \left(\frac{\mathrm{sh}\;\, r}{\mathrm{sh}\;\, x}\right)^{2\lambda} M^{\alpha}_{\mu} f(\mathrm{ch}\, x) \\
&\lesssim \sup_{r<x<2} \left(\frac{\mathrm{sh}\;\, x}{\mathrm{sh}\, \frac{r}{2}}\right)^{2\lambda-\frac{2\lambda\alpha}{2\lambda+1}} \left(\frac{\mathrm{sh}\;\, r}{\mathrm{sh}\;\, x}\right)^{2\lambda} M^{\alpha}_{\mu} f(\mathrm{ch}\, x) \\
&\lesssim \sup_{r<x<2} \left(\frac{\mathrm{sh}\;\, \frac{r}{2}}{\mathrm{sh}\;\, x}\right)^{\frac{2\lambda\alpha}{2\lambda+1}} \left(2\, \mathrm{ch}\, \frac{r}{2}\right)^{2\lambda} M^{\alpha}_{\mu} f(\mathrm{ch}\, x) \\
&\lesssim \left(\frac{\mathrm{sh}\;\, \frac{r}{2}}{\mathrm{sh}\;\, r}\right)^{\frac{2\lambda\alpha}{2\lambda+1}} \left(2\, \mathrm{ch}\, \frac{r}{2}\right)^{2\lambda} M^{\alpha}_{\mu} f(\mathrm{ch}\, x) \lesssim \left(2\, \mathrm{ch}\, \frac{r}{2}\right)^{2\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)} M^{\alpha}_{\mu} f(\mathrm{ch}\, x) \\
&\lesssim (2\, \mathrm{ch}\, 1)^{2\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)} M^{\alpha}_{\mu} f(\mathrm{ch}\, x) \lesssim M^{\alpha}_{\mu} f(\mathrm{ch}\, x). \qquad (2.25)
\end{aligned}
$$

Taking into account (2.24) and (2.25) in (2.23), we obtain

$$
M^{\alpha}_{G} f(\mathrm{ch}\, x) \lesssim M^{\alpha}_{\mu} f(\mathrm{ch}\, x), \quad 0 \le x < 2, \quad 0 < r < 2.
$$

Now, let $0 \le x < r$ and $2 \le r < \infty$. Then

$$
\begin{aligned}
M^{\alpha}_{G} f(\mathrm{ch}\, x) &\le M^{\alpha}_{G.1} f(\mathrm{ch}\, x) + M^{\alpha}_{G.2} f(\mathrm{ch}\, x) \\
&= \sup_{0\le x<r} |H(0,r)|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} |f(\mathrm{ch}\, t)|\, A_{\mathrm{ch}\, t}\chi_{(0,r)}(\mathrm{ch}\, t) d\mu_\lambda(t) \\
&+ \sup_{x\ge r} |H(0,r)|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} |f(\mathrm{ch}\, t)|\, A_{\mathrm{ch}\, t}\chi_{(0,r)}(\mathrm{ch}\, t) d\mu_\lambda(t). \qquad (2.26)
\end{aligned}
$$

Using Lemmas 2.1 and 2.2, for $2 \le r < \infty$, we get

$$
\begin{aligned}
M^{\alpha}_{G.1} f(\mathrm{ch}\, x) &\lesssim \left(\frac{|H(0,r)|_\lambda}{|H(x,r)|_\lambda}\right)^{\frac{\alpha}{2\lambda+1}-1} |H(x,r)|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} |f(\mathrm{ch}\, t)|\, d\mu_\lambda(t) \\
&\lesssim \left(\frac{\mathrm{ch}^{2\lambda} r}{\mathrm{ch}^{4\lambda} \frac{r}{2}}\right)^{1-\frac{\alpha}{2\lambda+1}} M^{\alpha}_{\mu} f(\mathrm{ch}\, x) = \left(\frac{4^{\lambda}\, \mathrm{ch}^{2\lambda} r}{(1+\mathrm{ch}\, r)^{2\lambda}}\right)^{1-\frac{\alpha}{2\lambda+1}} M^{\alpha}_{\mu} f(\mathrm{ch}\, x) \\
&\lesssim 4^{\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)} M^{\alpha}_{\mu} f(\mathrm{ch}\, x) \lesssim M^{\alpha}_{\mu} f(\mathrm{ch}\, x).
\end{aligned}
$$

Thus for $0 \le x < r$ and $2 \le r < \infty$,

$$
M^{\alpha}_{G.1} f(\mathrm{ch}\, x) \lesssim M^{\alpha}_{\mu} f(\mathrm{ch}\, x). \qquad (2.27)
$$

We consider the cases $r \le x < \infty$ and $2 \le r < \infty$. We investigate the function

$$f(\operatorname{ch} x) = \frac{\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x \ \operatorname{sh} t} = \frac{\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r}{\sqrt{\operatorname{ch}^2 t - 1}\operatorname{sh} x}.$$

Putting $u = cht$, we obtain

$$f(u) = \frac{u \ \operatorname{ch} x - \operatorname{ch} r}{\sqrt{u^2 - 1}\operatorname{sh} x}.$$

We will now find extremum of the function

$$\begin{aligned} f'(u) &= \frac{\sqrt{u^2-1}\operatorname{ch} x \ \operatorname{sh} x - u(u^2-1)^{-\frac{1}{2}}\operatorname{sh} x\,(u \ \operatorname{ch} x - \operatorname{ch} r)}{(u^2-1)\operatorname{sh}^2 x} \\ &= \frac{(u^2-1)\operatorname{ch} x \ \operatorname{sh} x - u^2 \operatorname{sh} x \ \operatorname{ch} x + u \ \operatorname{ch} r \ \operatorname{sh} x}{(u^2-1)^{\frac{3}{2}}\operatorname{sh}^2 x} = \frac{u \ \operatorname{ch} r - \operatorname{ch} x}{(u^2-1)^{\frac{3}{2}}\operatorname{sh} x} = 0 \ \Leftrightarrow \ u = \frac{\operatorname{ch} x}{\operatorname{ch} r}. \end{aligned}$$

By $u = \frac{\operatorname{ch} x}{\operatorname{ch} r}$, the function $f(u)$ has minimum

$$f_{\min}\left(\frac{\operatorname{ch} x}{\operatorname{ch} r}\right) = \frac{\operatorname{ch}^2 x - \operatorname{ch}^2 r}{\sqrt{\operatorname{ch}^2 x - \operatorname{ch}^2 r}\operatorname{sh} x}\frac{\sqrt{\operatorname{ch}^2 x - \operatorname{ch}^2 r}}{\operatorname{sh} x} = \frac{\operatorname{ch} x\sqrt{1 - \left(\frac{\operatorname{ch} r}{\operatorname{ch} x}\right)^2}}{\operatorname{sh} x} \sim \frac{\operatorname{sh} x}{\operatorname{ch} x},$$

as $x \to \infty$.

From this, by $x > r \ge 2$, we have

$$\begin{aligned} A(x,t,r) &\lesssim \int\limits_{\frac{\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x \ \operatorname{sh} t}}^{1} (1-y^2)^{\lambda-1}dy \lesssim \int\limits_{\frac{\operatorname{sh} x}{\operatorname{ch} x}}^{1} (1-y)^{\lambda-1}dy \\ &\lesssim \left(1 - \frac{\operatorname{sh} x}{\operatorname{ch} x}\right)^{\lambda} \lesssim \left(1 - \frac{\operatorname{sh}^2 x}{\operatorname{ch}^2 x}\right)^{\lambda} = (\operatorname{ch} x)^{-2\lambda}. \end{aligned}$$

Then by Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} M^{\alpha}_{G.2} f(\operatorname{ch} x) &\lesssim \sup_{r\ge 2}\left(\frac{\operatorname{ch}^{2\lambda} x \ \operatorname{ch}^{2\lambda} r}{\operatorname{ch}^{4\lambda}\frac{r}{2}}\right)^{1-\frac{\alpha}{2\lambda+1}} (\operatorname{ch} x)^{-2\lambda} M_\mu f(\operatorname{ch} x) \\ &\lesssim (\operatorname{ch} r)^{-\frac{2\lambda\alpha}{2\lambda+1}} M^{\alpha}_{\mu} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x), \quad 2 \le r < x < \infty. \end{aligned} \tag{2.28}$$

Taking into account (2.27) and (2.28) on (2.26), we have

$$M^{\alpha}_{G} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x), \quad 0 \le x < \infty, \ \ 2 \le r < \infty. \tag{2.29}$$

Combining (2.27) and (2.29), we obtain

$$M^{\alpha}_{G} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x), \ \ 0 \le x < \infty, \ \ 0 < r < \infty. \tag{2.30}$$

Now we are going to prove that

$$M^{\alpha}_{\mu} f(\operatorname{ch} x) \lesssim M^{\alpha}_{G} f(\operatorname{ch} x). \tag{2.31}$$

From (2.17), it follows that $\|H(x,r)|_\lambda \ge |H(0,r)|_\lambda$, then by Lemma 2.4, we have

$$\left|H(x,r)\right|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} \left|f(\operatorname{ch} u)\right| d\mu_\lambda(u) \lesssim \left|H(0,r)\right|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_0^r A_{\operatorname{ch} t} f(\operatorname{ch} x) d\mu_\lambda(t),$$

from which we get (2.31).

The assertion of Theorem 2.1 follows from (2.30) and (2.31). □

Theorem 2.2 (Lebesgue differentiation theorem)**.** *Let f be a nonnegative monotone nondecreasing function and let $f \in L^{loc}_{p,\lambda}(\mathbb{R}_+)$, $1 \le p < \infty$. Then*

$$\lim_{r\to 0}\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)} f(\operatorname{ch} y)^p \operatorname{sh}^{2\lambda} y \ dy = f(\operatorname{ch} x)^p$$

for almost every $x \in \mathbb{R}_+$.

Proof. By the locality of the problem, we may assume that $f \in L_{1,\lambda}(\mathbb{R}_+)$. In a general case, one can multiply f by a characteristic function of the interval $[0, r)$ and obtain the required convergence almost everywhere interior of this interval. Then by tending r to infinity, one can obtain it for the whole interval $[0, \infty)$. Suppose for any $r > 0$ and for any $x \in [0, \infty)$,

$$f_r(\operatorname{ch} x) = \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} f(\operatorname{ch} y)^p \operatorname{sh}^{2\lambda} y \, dy$$

and

$$\Omega_f(\operatorname{ch} x) = \Big| \overline{\lim_{r\to 0}} f_r(\operatorname{ch} x) - \underline{\lim}_{r\to 0} f_r \ (\operatorname{ch} x) \Big|.$$

Then we have

$$\Omega_f(\operatorname{ch} x) \le 2 \sup_{r>0} \big| f_r(\operatorname{ch} x) \big| = 2 M_G \, f(\operatorname{ch} x).$$

First we show that for any $\beta > 0$,

$$\big| x \in \mathbb{R}_+ : \Omega_f(\operatorname{ch} x) > \beta \big|_\lambda = 0. \tag{2.32}$$

In fact, as is known, the set of all continuous functions with compact support in $\mathbb{R}_+$ is dense in $L_{p,\lambda}(\mathbb{R}_+)$ (see [21], Theorem 4.2).

Therefore for any number $\varepsilon > 0$, there exists a continuous function h with a compact support in $\mathbb{R}_+$ such that

$$\big\| f - h \big\|_{L_{p,\lambda}(\mathbb{R}_+)} < \varepsilon.$$

Suppose $g = f - h$, then $g \in L_{p,\lambda}(\mathbb{R}_+)$ and

$$\big\| g \big\|_{L_{p,\lambda}(\mathbb{R}_+)} < \varepsilon.$$

Thus, if $f \in L_{p,\lambda}(\mathbb{R}_+)$, then for any $\varepsilon > 0$, there exist a continuous function h with a compact support and a function $g \in L_{p,\lambda}(\mathbb{R})_+$, with the condition $\big\| g \big\|_{L_{p,\lambda}(\mathbb{R}_+)} < \varepsilon$ such that $f = h + g$. But $\Omega_f \le \Omega_h + \Omega_g$. If g is a continuous function with a compact support on $\mathbb{R}_+$, then g_r converges to g and, consequently, in this case we get $\Omega_g(\operatorname{ch} x) \equiv 0$. Therefore, for any $\beta > 0$ (see [15], Theorem 2.2),

$$\big| x \in \mathbb{R}_+ : \Omega_g(\operatorname{ch} x) > \beta \big|_\lambda \lesssim \frac{1}{\beta} \big\| g \big\|_{L_{1,\lambda}(\mathbb{R}_+)} \lesssim \frac{\varepsilon}{\beta}.$$

Since ε is arbitrarily small, we get (2.32), from which it follows that $\lim\limits_{r\to 0} f_r(\operatorname{ch} x)$ exists for almost everywhere on $\mathbb{R}_+$. Further, we have

$$\lim_{r\to 0} \big\| f_r - f \big\|_{L_{p,\lambda}} = \lim_{r\to 0} \left(\int\limits_{\mathbb{R}_+} \left| \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy - f(\operatorname{ch} x) \right|^p \operatorname{sh}^{2\lambda} x \, dx \right)^{\frac{1}{p}}.$$

By Lemma 2.3, we have $|H(x,r)|_\lambda \geq |H(0,r)|_\lambda$ and therefore, we find

$$\begin{aligned}
\lim_{r\to 0}\|f_r - f\|_{L_{p,\lambda}} &\lesssim \lim_{r\to 0}\left\{\int\limits_{\mathbb{R}^+}\left[\frac{1}{|H(x,r)|_\lambda}\int\limits_{x-r}^{x+r}\big|f(\operatorname{ch} y) - f(\operatorname{ch} x)\big|\operatorname{sh}^{2\lambda} y\,dy\right]^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}}\\
&\lesssim \lim_{r\to 0}\left\{\int\limits_{\mathbb{R}^+}\left[\frac{1}{|H(0,r)|_\lambda}\int\limits_{-r}^{r}\big|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)\big|\operatorname{sh}^{2\lambda}(x-y)dy\right]^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}}\\
&\lesssim \lim_{r\to 0}\left\{\int\limits_{\mathbb{R}^+}\left[\frac{\operatorname{sh}^{2\lambda} r}{|H(0,r)|_\lambda}\int\limits_{-r}^{r}\big|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)\big|dy\right]^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}}\\
&\lesssim \lim_{r\to 0}\frac{\operatorname{sh}^{2\lambda} r}{|H(0,r)|_\lambda}\int\limits_{-r}^{r}\left\{\int\limits_{\mathbb{R}^+}|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)|^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}}dy\\
&\lesssim \lim_{r\to 0}\frac{\operatorname{sh}^{2\lambda}\frac{r}{2}\operatorname{ch}^{2\lambda}\frac{r}{2}}{(\operatorname{sh}\frac{r}{2})^{2\lambda+1}}\int\limits_{-r}^{r}\left\{\int\limits_{\mathbb{R}^+}|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)|^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}}dy
\end{aligned}$$

(since by Lemma 2.1, $\operatorname{sh} r \approx r$)

$$\lesssim \lim_{r\to 0}\frac{1}{2r}\int\limits_{-r}^{r}\left\{\int\limits_{\mathbb{R}^+}\big|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)\big|^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}}dy = 0. \tag{2.33}$$

Further, by the monotonicity of the function f, we have

$$\begin{aligned}
f(\operatorname{ch}(x-y)) = f(\operatorname{ch} x\operatorname{ch} y - \operatorname{sh} x\operatorname{sh} y) &= \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)}\int\limits_0^\pi f(\operatorname{ch} x\operatorname{ch} y - \operatorname{sh} x\operatorname{sh} y)(\sin\varphi)^{2\lambda-1}d\varphi\\
&\leq \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)}\int\limits_0^\pi f\big(\operatorname{ch} x\operatorname{ch} y - \operatorname{sh} x\operatorname{sh} y\cos\varphi\big)(\sin\varphi)^{2\lambda-1}d\varphi = A_{\operatorname{ch} y} f(\operatorname{ch} x).
\end{aligned}$$

Then we have

$$\begin{aligned}
\lim_{r\to 0}\|f_r - f\|_{L_{p,\lambda}} &\lesssim \lim_{r\to 0}\frac{1}{2r}\int\limits_{-r}^{r}\left\{\int\limits_{\mathbb{R}^+}|A_{\operatorname{ch} y} f(\operatorname{ch} x) - f(\operatorname{ch} x)|^p\ \operatorname{sh}^{2\lambda} x\right\}^{\frac{1}{p}}dy\\
&\lesssim \lim_{r\to 0}\frac{1}{2r}\int\limits_{-r}^{r}\|A_{\operatorname{ch} y} f - f\|_{L_{p,\lambda}}\,dy = 0,
\end{aligned}$$

since $\sup\limits_{0<y\leq r}\big\|A^\lambda_{\operatorname{ch} y} f - f\big\|_{L_{p,\lambda}} = \omega_f(r)$ as $r\to 0$ (see [15], proof of Corollary 2.1).

From (2.33), it follows that there exists a subsequence r_k satisfying $r_k \to 0$ as $k\to\infty$ such that

$$\lim_{k\to\infty} f_{r_k}(\operatorname{ch} x) = f(\operatorname{ch} x)$$

for a.e. $x\in\mathbb{R}_+$. Because $\lim\limits_{r\to 0} f_r(\operatorname{ch} x)$ exists for a.e. $x\in\mathbb{R}_+$, thus

$$\lim_{r\to 0}\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)} f(\operatorname{ch} y)^p \operatorname{sh}^{2\lambda} y\,dy = f(\operatorname{ch} x)^p,$$

which is the desired conclusion. □

Applying the Lebesgue differentiation theorem, we may give a decomposition of $\mathbb{R}_+$, called as Calderon–Zygmund decomposition, which is extremely useful in harmonic analysis.

Theorem 2.3. *Suppose that f is a nonnegative integrable function on $\mathbb{R}_+$. Then for any fixed number $\beta > 0$, there exists a sequence $\{H_j(x_j, r_j)\} = \{H_j\}$ of disjoint intervals such that*

(1) $f(\operatorname{ch} x) \le \beta$, $x \notin \bigcup\limits_j H_j$;

(2) $\big|\bigcup\limits_j H_j\big|_\lambda \le \frac{1}{\beta}\|f\|_{1,\lambda}$;

(3) $\beta < \frac{1}{|H_j|_\lambda} \int\limits_{H_j} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \lesssim 2^{(2\lambda+1)n}\beta$, $n = 1, 2, \dots$.

Proof. Since $f \in L_{1,\lambda}(\mathbb{R}_+)$, we may decompose $\mathbb{R}_+$ into a net of equal intervals (by the Lindelof covering theorem this is possible (see [24])) such that for every H, from the net

$$\frac{1}{|H|_\lambda} \int\limits_H f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le \beta. \tag{2.34}$$

In fact, for any $\beta > 0$, $\exists \delta = \delta(\beta) > 0$ and for every H_j with measure $|H_j|_\lambda = |H|_\lambda < \delta$, we have

$$\int\limits_{H_j} f(\operatorname{ch} y)\, \operatorname{sh}^{2\lambda} y\, dy < \beta, \quad j = 1, 2, \dots,$$

where $H_j = (x_j - r, x_j + r)$ and $|H|_\lambda = |H_j|_\lambda = \int\limits_{x_j - r}^{x_j + r} \operatorname{sh}^{2\lambda} y\, dy$, $(j = 1, 2, \dots)$.

First, we prove (3).

Let $H_1 = (x_1 - r,\ x_1 + r)$ be a fixed interval in the net. Then by (2.33), we can write

$$\frac{1}{|H_1|_\lambda} \int\limits_{H_1} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le \beta. \tag{2.35}$$

We divide the interval H_1 into 2^n equal intervals and let $H_1' = \left(\frac{x_1 - r}{2^n}, \frac{x_1 + r}{2^n}\right)$ be one of those intervals. By (2.17), we have

$$|H_1'|_\lambda = \int\limits_{\frac{x_1 - r}{2^n}}^{\frac{x_1 + r}{2^n}} \operatorname{sh}^{2\lambda} y\, dy \approx \left(\operatorname{sh} \frac{x_1 + r}{2^{n+1}}\right)^{2\lambda+1}, \quad 0 < \frac{x_1 + r}{2^n} < 2.$$

Since for $0 < t < 1$, $\operatorname{sh} t \approx t$, we obtain

$$|H_1'|_\lambda \approx \left(\operatorname{sh} \frac{x_1 + r}{2^{n+1}}\right)^{2\lambda+1} \approx \left(\frac{x_1 + r}{2^{n+1}}\right)^{2\lambda+1} \approx \left(\frac{1}{2^n} \operatorname{sh} \frac{x_1 + r}{2}\right)^{2\lambda+1} = 2^{-(2\lambda+1)n} \big|tH_1\big|_\lambda. \tag{2.36}$$

There exist possibly two cases concerning H_1':

$$\text{(A)} \qquad \frac{1}{|H_1'|_\lambda} \int\limits_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy > \beta,$$

$$\text{(B)} \qquad \frac{1}{|H_1'|_\lambda} \int\limits_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le \beta.$$

In case (A), from (2.35) and (2.36), we obtain

$$\beta < \frac{1}{|H_1'|_\lambda} \int\limits_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \approx \frac{2^n}{|H_1|_\lambda} \int\limits_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy$$
$$\lesssim \frac{2^n}{|H_1|_\lambda} \int\limits_{H_1} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \lesssim 2^{(2\lambda+1)n}\beta.$$

Now, for H_1', we choose a sequence $\{H_j\}$.

We consider case (B). Suppose $H_1' = H_2(x_2 - r,\ x_2 + r)$. Dividing this interval into 2^n equal parts, we obtain

$$\beta < \frac{1}{|H_2'|_\lambda} \int\limits_{H_2'} f(\operatorname{ch} y)\, \operatorname{sh}^{2\lambda} y\; dy \lesssim \frac{2^n}{|H_1'|_\lambda} \int\limits_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\; dy \le 2^{(2\lambda+1)n}\beta,$$

where for H_2', we choose a sequence $\{H_j\}$. Continuing this process, we obtain a sequence of disjoint intervals $\{H_j\}$ such that

$$\beta < \frac{1}{|H_j|_\lambda} \int\limits_{H_j} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\; dy \lesssim 2^{(2\lambda+1)n}\beta, \quad (j = 1, 2, \dots).$$

Proof of (1). Taking into account (2.34), from Theorem 2.2, we have

$$f(\operatorname{ch} x) = \lim_{r\to 0} \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\; dy \le \beta$$

for a.e. $x \notin \bigcup_j H_j$.

Proof of (2). Passing to the limit by $n \to \infty$ in the inequality

$$\Big| \bigcup_{j=1,2,\dots,n} H_j(x_j, r_j)\Big|_\lambda \le \sum_{j=1}^n \big|H_j(x_j, r_j)\big|_\lambda \le \frac{1}{\beta} \sum_{j=1}^n \int\limits_{H_j(x_j, 3r_j)} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\; dy,$$

which is contained in the proof of Theorem 2.2, from [15], we obtain approval (2). □

Remark 2.1. The Caldeon-Zygmund decomposition stay valid if we replace $\mathbb{R}_+$ by a fixed interval $H_0(x_0, r_0)$ for $f \in L_{p,\lambda}(H_0)$.

3. Weighted $(L_{p,\omega,\lambda}, L_{q,\omega,\lambda})$-Boundedness of the Fractional Maximal Operator Generated by Gegenbauer Differential Operator

In this section, we prove the weighted $(L_{p,\omega,\lambda}, L_{q,\omega,\lambda})$-boundedness of the fractional maximal operator M_G^α (G-fractional maximal operator) generated by the Gegenbauer differential operator.

We need the following theorem.

Theorem (Marcinkiewicz interpolation theorem, [3, n.3.2., p. 43])**.** *Let $(\mathbb{R}_+, \varphi)$ and $(\mathbb{R}_+, \nu)$ be two measure spaces and let the sublinear operator T be both of weak type (p_0, p_0) and of weak type (p_1, p_1) for $1 \le p_0 < p_1 \le \infty$, that is, there exists a constant $C_0 > 0$ such that for any $\alpha > 0$,*

$$\text{(a)} \qquad \nu\Big(\Big\{x \in \mathbb{R}_+ : |Tf(\operatorname{ch} x)| > \alpha\Big\}\Big) \le \Big(\frac{C_0}{\alpha}\big\|f\big\|_{p_0,\varphi}\Big)^{p_0},$$

$$\text{(b)} \qquad \nu\Big(\Big\{x \in \mathbb{R}_+ : |Tf(\operatorname{ch} x)| > \alpha\Big\}\Big) \le \Big(\frac{C_0}{\alpha}\big\|f\big\|_{p_1,\varphi}\Big)^{p_1}, \quad p_1 < \infty.$$

If $p_1 = \infty$, then the weak type and strong type coincide by the definition

$$\big\|Tf\big\|_{\infty,\nu} \lesssim \big\|f\big\|_{\infty,\varphi}.$$

Then T is also of the type (p,p) for all $p_0 < p < p_1$, i.e., for any $f \in L_p(\mathbb{R}_+, \varphi)$,

$$\int\limits_{\mathbb{R}_+} \big|Tf(\operatorname{ch} x)\big|^p d\nu(x) \lesssim \int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p d\varphi(x).$$

Denote by $L_{p,\omega,\lambda}(\mathbb{R}_+, G)$ the set of measurable functions on $\mathbb{R}_+$ with a finite norm

$$\big\|f\big\|_{L_{p,\omega,\lambda}(\mathbb{R}_+,G)} = \bigg(\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p\, d\omega_\lambda(x)\bigg)^{\frac{1}{p}} < \infty, \ \ 1 \le p < \infty,$$

where $d\omega_\lambda(x) = \omega(\operatorname{ch} x) d\mu_\lambda(x)$.

The following theorem is a version of the Fefferman-Stein inequality.

Theorem 3.1. *Let $1 \le p < \infty$ and g be a nonnegative function such that $g \in L^{loc}_{1,\omega,\lambda}(\mathbb{R}_+, G)$. Then for any function $f \in L_{p,\omega,\lambda}(\mathbb{R}_+, G)$, the following inequality*

$$\int\limits_{\mathbb{R}_+} (M_\mu f(\operatorname{ch} x))^p \, g(\operatorname{ch} x) d\mu_\lambda(x) \le \int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p \, M_\mu g(\operatorname{ch} x) d\mu_\lambda(x)$$

is valid.

Proof. Without loss of generality, we may assume that $M_\mu g(\operatorname{ch} x) < \infty$, a.e. $x \in \mathbb{R}_+$ and $M_\mu g(\operatorname{ch} x) > 0$. If we denote $d\nu_\lambda(x) = g(\operatorname{ch} x) d\mu_\lambda(x)$ and $d\varphi_\lambda(x) = M_\mu g(\operatorname{ch} x) d\mu_\lambda(x)$, then by the Marcinkiewicz interpolation theorem for the validity of our assertion it suffices to prove that M_μ is both of type $(L_{\infty,\varphi}, L_{\infty,\nu})$ and of weak type $(L_{1,\varphi,\lambda}, L_{1,\nu,\lambda})$.

Let us first show that M_μ is one of the type $(L_{\infty,\varphi}, L_{\infty,\nu})$. In fact, if $\|f\|_{\infty,\varphi} \le a < \infty$, then

$$\int\limits_{\{x \in \mathbb{R}_+ : |f(\operatorname{ch} x)| > a\}} M_\mu g(\operatorname{ch} x) d\mu_\lambda(x) = \left| \left\{ x \in \mathbb{R}_+ : |f(\operatorname{ch} x)| > a \right\} \right|_{\mu_\lambda} = 0.$$

Since $M_\mu g(\operatorname{ch} x) > 0$ for any $x \in \mathbb{R}_+$, we get $\left| \left\{ x \in \mathbb{R}_+ : |f(\operatorname{ch} x)| > a \right\} \right|_{\mu_\lambda} = 0$, equivalently, $|f(\operatorname{ch} x)| \le a$, a.e. $x \in \mathbb{R}_+$. Thus $M_\mu f(\operatorname{ch} x) \le a$, a.e. $x \in \mathbb{R}_+$, and thus it follows that $\|M_\mu f\|_{\infty,\nu_\lambda} \le a$.

Therefore $\|M_\mu f\|_{\infty,\nu_\lambda} \le \|f\|_{\infty,\varphi_\lambda}$.

Now we can show that M_μ has weak type $(L_{1,\varphi,\lambda}, L_{1,\nu,\lambda})$. For this we need to prove that for any $\alpha > 0$ and $f \in L_{1,\varphi,\lambda}(\mathbb{R}_+)$

$$\int\limits_{\{x \in \mathbb{R}_+ : M_\mu f(\operatorname{ch} x) > \alpha\}} g(\operatorname{ch} x) \, d\mu_\lambda(x) \lesssim \frac{1}{\alpha} \int\limits_{\mathbb{R}_+} f(\operatorname{ch} x) \, M_\mu g(\operatorname{ch} x) d\mu_\lambda(x).$$

By Theorem 2.3 (3), we have

$$\int\limits_{H_i} f(\operatorname{ch} x) M_\mu g(\operatorname{ch} x) d\mu_\lambda(x) \ge \int\limits_{H_i} f(\operatorname{ch} x) \left(\frac{1}{|H_i|_\lambda} \int\limits_{H_i} g(\operatorname{ch} t) d\mu_\lambda(t) \right) d\mu_\lambda(x)$$
$$\approx \alpha \int\limits_{H_i} g(\operatorname{ch} u) d\mu_\lambda(u).$$

Summing over i, we obtain

$$\int\limits_{\mathbb{R}_+} f(\operatorname{ch} x) M_\mu g(\operatorname{ch} x) d\mu_\lambda(x) \ge \alpha \int\limits_{\mathbb{R}_+} g(\operatorname{ch} u) d\mu_\lambda(u)$$
$$\ge \alpha \int\limits_{\{u \in \mathbb{R}_+ : M_\mu f(\operatorname{ch} u) > \alpha\}} g(\operatorname{ch} u) \, d\mu_\lambda(u).$$

Thus M_μ has weak type $(L_{1,\varphi,\lambda}, L_{1,\nu,\lambda})$ and the Fefferman-Stein inequality follows from the Marcinkiewicz interpolation theorem by $p_0 = 1$ and $p_1 = \infty$. □

Theorem 3.2. *The Chebychev type inequality*

$$\left| \left\{ x \in \mathbb{R}_+ : M_\mu f(\operatorname{ch} x) > \alpha \right\} \right|_\omega \le \frac{1}{\alpha} \int\limits_{\mathbb{R}_+} M_\mu f(\operatorname{ch} x) \, d\omega_\lambda(x)$$

is valid for all $\alpha > 0$ and $t > 0$.

Proof. Since

$$M_\mu f(\operatorname{ch} x) \ge \alpha \chi_{\{M_\mu f(\operatorname{ch} x) > \alpha\}}(\operatorname{ch} x),$$

we have

$$\int\limits_{\mathbb{R}_+} M_\mu f(\operatorname{ch} x)\, d\omega_\lambda(x) \geq \alpha \int\limits_{\mathbb{R}_+} \chi_{\{M_\mu f(\operatorname{ch} x)>\alpha\}}(\operatorname{ch} x) d\omega_\lambda(x) = \alpha \Big|\{x \in \mathbb{R}_+ : M_\mu f(\operatorname{ch} x) > \alpha\}\Big|_\omega .$$

Thus our assertion is proved. □

Definition 3.1. The weight function ω belongs to the class $A_p^\lambda(\mathbb{R}_+)$ for $1 < p < \infty$ if

$$\sup_{x\in\mathbb{R}_+, r>0} \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} u) \operatorname{sh}^{2\lambda} u\, du \right) \times \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} u)^{-\frac{1}{p-1}} \operatorname{sh}^{2\lambda} u\, du \right)^{p-1} < \infty \tag{3.1}$$

and ω belongs to $A_1^\lambda(\mathbb{R}_+)$ if there exists a positive constant C such that for any $x \in \mathbb{R}_+$ and $r > 0$

$$M_G\, \omega(\operatorname{ch} x) \leq C\omega(\operatorname{ch} x). \tag{3.2}$$

Remark 3.1. Inequality (3.2) is equivalent to the inequality

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \leq C \operatorname*{ess\,inf}_{y\in H(x,r)} \omega(\operatorname{ch} y). \tag{3.3}$$

Remark 3.2. In inequalities (3.2) and (3.3), for $C \geq 1$, by Hölder's inequalility, we have

$$\begin{aligned} 1 &= \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y)^{\frac{1}{p}} \omega(\operatorname{ch} y)^{-\frac{1}{p}} \operatorname{sh}^{2\lambda} y\, dy \\ &\leq \left\{ \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \right) \right. \\ &\quad \times \left. \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y)^{-\frac{1}{p-1}} \operatorname{sh}^{2\lambda} y\, dy \right)^{p-1} \right\}^{\frac{1}{p}} \leq C^{\frac{1}{p}}. \end{aligned}$$

We show that $\operatorname{sh}^\alpha u \in A_p^\lambda(\mathbb{R}_+)$, $1 < p < \infty$, if and only if $-(2\lambda+1) < \alpha < (2\lambda+1)(p-1)$ and $\operatorname{sh}^\alpha u \in A_1^\lambda(\mathbb{R}_+)$ if and only if $-(2\lambda+1) < \alpha \leq 0$.

By using Lemma 2.3, for $\gamma = 2\lambda - \frac{\alpha}{p-1}$ and (2.17) for $0 < x + r < 2$, we obtain

$$\begin{aligned} \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} (\operatorname{sh} u)^{2\lambda - \frac{\alpha}{p-1}} du \right)^{p-1} &\approx \left(\frac{\left(\operatorname{sh} \frac{x+r}{2}\right)^{2\lambda+1-\frac{\alpha}{p-1}}}{\left(\operatorname{sh} \frac{x+r}{2}\right)^{2\lambda+1}} \right)^{p-1} \\ &= \left(\operatorname{sh} \frac{x+r}{2} \right)^{-\alpha}, \quad \alpha < (2\lambda+1)(p-1), \end{aligned}$$

and also for $\gamma = \alpha + 2\lambda$ and (2.18),

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} (\operatorname{sh} u)^{\alpha+2\lambda} \approx \left(\operatorname{sh} \frac{x+r}{2} \right)^{-\alpha}, \quad \alpha > -2\lambda - 1.$$

Taking into account the relation in (3.1), we obtain that for

$$-(2\lambda+1) < \alpha < (2\lambda+1)(p-1)$$

$\operatorname{sh}^\alpha u \in A_p^\lambda(\mathbb{R}_+)$.

Now, let $2 \le x+r < \infty$. Then assuming $\gamma = 2\lambda - \dfrac{\alpha}{p-1}$ in Lemma 2.3 and using (2.18), we obtain

$$\left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} (\operatorname{sh} u)^{2\lambda - \frac{\alpha}{p-1}} du \right)^{p-1} \approx \left(\operatorname{sh} \frac{x+r}{2} \right)^{-2\alpha}, \quad \alpha < (2\lambda+1)(p-1).$$

and also for $\gamma = \alpha + 2\lambda$,

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} (\operatorname{sh} u)^{\alpha+2\lambda} du \approx \left(\operatorname{sh} \frac{x+r}{2} \right)^{2\alpha}, \quad -(2\lambda+1) < \alpha \le 0.$$

That is, for $2 \le x+r < \infty$, $-(2\lambda+1) < \alpha < (2\lambda+1)(p-1)$ $(\operatorname{sh} u)^\alpha \in A_p^\lambda(\mathbb{R}_+)$ with $1 < p < \infty$.

Let $p = 1$, then for $0 < x+r < 2$ and $\gamma = \alpha + 2\lambda$, we have

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} (\operatorname{sh} u)^{\alpha+2\lambda} du \approx \left(\operatorname{sh} \frac{x+r}{2} \right)^{\alpha}, \quad -(2\lambda+1) < \alpha \le 0,$$

and for $2 \le x+r < \infty$ and $\gamma = \alpha + 2\lambda$

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} (\operatorname{sh} u)^{\alpha+2\lambda} du \approx \left(\operatorname{sh} \frac{x+r}{2} \right)^{\alpha} \lesssim \left(\operatorname{sh} \frac{x+r}{2} \right)^{\alpha}, \quad (2\lambda+1) < \alpha \le 0.$$

Thus, for any $0 < x+r < \infty$,

$$(\operatorname{sh} u)^\alpha \in A_1^\lambda(\mathbb{R}_+), \quad -(2\lambda+1) < \alpha \le 0.$$

We are going to prove some properties of $A_1^\lambda(\mathbb{R}_+)$, which we will need later. Note that in proving these properties and Theorem 3.3, we use the outline from [23].

Proposition 3.1. *If $1 \le p < q < \infty$, then $A_p^\lambda(\mathbb{R}_+) \subsetneq A_1^\lambda(\mathbb{R}_+)$. In fact, by Hölder's inequality, we have*

$$\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \le \left(\int\limits_{H(x,r)} \omega^{-\frac{k}{q-1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \right)^{\frac{1}{k}} \left(\int\limits_{H(x,r)} \operatorname{sh}^{2\lambda} y \, dy \right)^{\frac{k-1}{k}}.$$

Supposing here $k = \frac{q-1}{p-1}$, we obtain

$$\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \le \left(\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \right)^{\frac{p-1}{q-1}} \left(\int\limits_{H(x,r)} \operatorname{sh}^{2\lambda} y \, dy \right)^{\frac{q-p}{q-1}},$$

whence we have

$$\begin{aligned}
&\left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \right)^{q-1} \\
&\quad \le \frac{1}{|H(x,r)|_\lambda^{q-1}} \left(\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \right)^{p-1} \left(\int\limits_{H(x,r)} \operatorname{sh}^{2\lambda} y \, dy \right)^{q-p} \\
&\quad = \frac{|H(x,r)|_\lambda^{q-p}}{|H(x,r)|_\lambda^{q-1}} \left(\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \right)^{p-1} \\
&\quad = \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \right)^{p-1}.
\end{aligned}$$

If $p=1$, then by (3.3), we have

$$\left(\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)}\omega^{-\frac{1}{q-1}}(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy\right)^{q-1}\leq ess\sup_{y\in H(x,r)}\omega^{-1}(\operatorname{ch}y)=\left(\operatorname*{ess\,inf}_{y\in H(x,r)}\omega(\operatorname{ch}y)\right)^{-1}$$

$$\leq C\left(\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy\right)^{-1}.$$

Thus, if $\omega\in A_p^\lambda(\mathbb{R}_+)$, then $\omega\in A_q^\lambda(\mathbb{R}_+)$ for $q>p$. On the other hand, $(\operatorname{sh}u)^\alpha\in A_p^\lambda(\mathbb{R}_+)$, if and only if $-(2\lambda+1)<\alpha<(2\lambda+1)(p-1)$, therefore $A_p^\lambda(\mathbb{R}_+)\neq A_q^\lambda(\mathbb{R}_+)$.

Proposition 3.2. *If $\omega\in A_p^\lambda(\mathbb{R}_+)$ $(1\leq p<\infty)$, then for any $\alpha\in(0,1)$, there exists $\beta\in(0,1)$ such that for any measurable set $E\subset H$, $|E|_\lambda\leq\alpha|H|_\lambda$ and $\omega(E)\leq\beta\ \omega\ (H)$, where $\omega(A)=\int_A\omega(\operatorname{ch}x)\operatorname{sh}^{2\lambda}x\,dx$.*

Proof. In fact, let $S=H\backslash E$ and $f(\operatorname{ch}x)=\chi_s(\operatorname{ch}x)$. Then

$$\left(\frac{|S(x,r)|_\lambda}{|H(x,r)|_\lambda}\right)^p\omega(H)\leq C\int\limits_{S(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy$$

$$\leq C\left(\int\limits_{H(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy-\int\limits_{E(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy\right)=C\left(\omega(H)-\omega(E)\right).$$

Further,

$$\big|E(x,r)\big|_\lambda\leq\alpha\big|H(x,r)\big|_\lambda\Leftrightarrow\frac{|E(x,r)|_\lambda}{|H(x,r)|_\lambda}\leq\alpha$$

$$\Leftrightarrow-\frac{|E(x,r)|_\lambda}{|H(x,r)|_\lambda}\geq-\alpha\Leftrightarrow 1-\alpha\leq 1-\frac{|E(x,r)|_\lambda}{|H(x,r)|_\lambda}$$

$$\Leftrightarrow(1-\alpha)^p\omega(H)\leq\left(1-\frac{|E(x,r)|_\lambda}{|H(x,r)|_\lambda}\right)^p\omega(H)\leq C\big(\omega(H)-\omega(E)\big).$$

Taking into account that $C\geq1$, we obtain

$$(1-\alpha)^p\,\omega(H)\leq C\,\omega(H)-C\,\omega(E)\Leftrightarrow C\,\omega(E)\leq\big(1-(1-\alpha)^p\big)\omega(H)$$

$$\Leftrightarrow\omega(E)\leq\frac{C-(1-\alpha)^p}{C}\,\omega(H).$$

Thus, we get our assertion with $\beta=\frac{C-(1-\alpha)^p}{C}$. □

Further, we need the reverse of Hölder's inequality.

Theorem 3.3. *Let $\omega\in A_p^\lambda(\mathbb{R}_+)$, $1\leq p<\infty$. Then there exist a constant $C>0$ and $\varepsilon>0$ depending only on p such that for any interval $H(x,r)$, the inequality*

$$\left(\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)}\omega^{1+\varepsilon}(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy\right)^{\frac{1}{1\varepsilon}}\leq\frac{C}{|H(x,r)|_\lambda}\int\limits_{H(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy$$

is valid.

Proof. Fix an interval $H_0(x_0,r_0)$. By Remark 3.1, we apply inequality (3) from Theorem 2.3 with respect to H_0 for ω, and the increasing sequence $\{\beta_k\}$, $k=0,1,\dots$, we can write

$$\left\{\frac{\omega(H)}{|H|_\lambda}=\beta_0<\beta_1<\cdots<\beta_k<\cdots\right\}.$$

For each β_k, by property (1), we can get a disjoint sequence $\{H_{k,i}\}$ such that $\omega(\operatorname{ch} x) \le \beta_k$ for $x \notin \Lambda_k = \bigcup\limits_i H_{k,i}$, and by property (3),

$$\beta_k < \frac{1}{|H_{k,i}|_\lambda} \int\limits_{H_{k,i}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \; dy \le 2^{(2\lambda+1)n} \beta_k.$$

Since $\beta_{k+1} > \beta_k$, for every interval $H_{k+1,j}$, it is either equal to $H_{k,i}$ or a subinterval of $H_{k,i}$ for some i, therefore

$$|H_{k+1,j}|_\lambda < \frac{1}{\beta_{k+1}} \int\limits_{H_{k+1,j}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy = \frac{|H_{k,i}|_\lambda}{\beta_{k+1}} \cdot \frac{1}{|H_{k,i}|_\lambda} \int\limits_{H_{k+1,j}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy$$

$$\le \frac{|H_{k,i}|_\lambda}{\beta_{k+1}} \cdot \frac{1}{|H_{k,i}|_\lambda} \int\limits_{H_{k,j}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \le 2^{(2\lambda+1)n} \cdot \frac{\beta_k}{\beta_{k+1}} |H_{k,i}|_\lambda .$$

From this, we get

$$\left|H_{k,i} \cap \Lambda_{k+1}\right|_\lambda \le 2^{(2\lambda+1)n} \frac{\beta_k}{\beta_{k+1}} |H_{k,i}|_\lambda .$$

For fixed $\alpha < 1$ we choose a sequence $\{\beta_k\}$ such that

$$\frac{2^{(2\lambda+1)n}\beta_k}{\beta_{k+1}} = \alpha \Leftrightarrow \beta_k = \left(\frac{2^{(2\lambda+1)n}}{\alpha}\right)^k \beta_0,$$

where $\beta_0 = \left(\frac{\alpha}{2^{(2\lambda+1)n}}\right)^{k+1} \beta_{k+1}$. Thus,

$$\left|H_{k,i} \cap \Lambda_{k+1}\right|_\lambda \le \alpha \, |H_{k,i}|_\lambda .$$

From Property 2 of class $A_p^\lambda(\mathbb{R}_+)$, there exists $\gamma \in (0,1)$ such that

$$\omega\big(H_{k,i} \cap \Lambda_{k+1}\big) \le \gamma \omega\left(H_{k,i}\right).$$

From this, we have

$$\mathop{\cup}_i \omega\big(H_{k,i} \cap \Lambda_{k+1}\big) \le \gamma \mathop{\cup}_i \omega\left(H_{k,i}\right),$$

that equivalently

$$\omega\left(\Lambda_{k+1}\right) \le \gamma \omega\left(\Lambda_k\right),$$

from which it follows that

$$\omega\left(\Lambda_{k+1}\right) \le \gamma^k \omega\left(\Lambda_0\right).$$

Analogously, we have $|\Lambda_{k+1}|_\lambda \le \alpha \, |\Lambda_k|_\lambda$ and $|\Lambda_{k+1}|_\lambda \le \alpha^k \, |\Lambda_0|_\lambda$. Consequently,

$$\big| \mathop{\cap}_{k=0}^{\infty} \Lambda_k \big|_\lambda = \lim_{k\to\infty} |\Lambda_k| = 0.$$

Thus,

$$\int\limits_H \omega^{1+\varepsilon}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy = \int\limits_{H\backslash\Lambda_0} \omega^{1+\varepsilon}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy + \sum_{k=0}^{\infty} \int\limits_{\Lambda_k\backslash\Lambda_{k+1}} \omega^{1+\varepsilon}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy$$

$$\le \beta_0^\varepsilon \omega\left(H\backslash\Lambda_0\right) + \sum_{k=0}^{\infty} \beta_{k+1}^\varepsilon \omega\left(\Lambda_k\backslash\Lambda_{k+1}\right)$$

$$\le \beta_0^\varepsilon \left(\omega\left(H\backslash\Lambda_0\right) + \sum_{k=0}^{\infty} \left(\frac{2^{(2\lambda+1)n}}{\alpha}\right)^{(k+1)\varepsilon} \gamma^k \omega\left(\Lambda_0\right) \right)$$

$$\le \beta_0^\varepsilon \left(\left[\omega\left(H\backslash\Lambda_0\right) + \left(\frac{2^{(2\lambda+1)n}}{\alpha}\right)^\varepsilon \sum_{k=0}^{\infty} \left(\frac{2^{(2\lambda+1)n}}{\alpha}\right)^\varepsilon \gamma \right]^k \omega\left(\Lambda_0\right) \right).$$

Let $\varepsilon > 0$ be small enough such that $\left(\frac{2^{(2\lambda+1)n}}{\alpha}\right)^{\varepsilon}\gamma < 1$. Then the series converges. Therefore we have

$$\int_H \omega^{1+\varepsilon}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy \le C\beta_0^{\varepsilon}\left(\omega\left(H\backslash\Lambda_0\right)+\omega\left(\Lambda_0\right)\right) = C\beta_0^{\varepsilon}\omega(H)$$

$$= C\frac{\omega^{\varepsilon}(H)}{|H|_{\lambda}^{\varepsilon}}\cdot\omega(H) = C\frac{\omega^{1+\varepsilon}(H)}{|H|_{\lambda}^{1+\varepsilon}}|H|_{\lambda} = C\left(\frac{1}{|H|_{\lambda}}\int_H \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy\right)^{1+\varepsilon}|H|_{\lambda},$$

thus there follows the assertion of theorem. □

Proposition 3.3. *Let $\omega \in A_p^{\lambda}(\mathbb{R}_+)$, $1 < p < \infty$. Then there exists an $\varepsilon > 0$ such that $p-\varepsilon > 1$ and $\omega \in A_{p-\varepsilon}^{\lambda}(\mathbb{R}_+)$.*

Proof. If $\omega \in A_p^{\lambda}(\mathbb{R}_+)$, then by Property 2, $\omega^{-\frac{1}{p-1}} \in A_{1+\frac{1}{p-1}}^{\lambda}(\mathbb{R}_+)$. Applying Theorem 3.3, we obtain

$$\left(\frac{1}{|H(x,r)|_{\lambda}}\int_{H(x,r)} \omega(\operatorname{ch} y)^{\frac{1+\theta}{1-p}}\operatorname{sh}^{2\lambda} y\,dy\right)^{\frac{p-1}{1+\theta}} \le C^{p-1}\left(\frac{1}{|H(x,r)|_{\lambda}}\int_{H(x,r)} \omega(\operatorname{ch} y)^{-\frac{1}{1-p}}\operatorname{sh}^{2\lambda} y\,dy\right)^{p-1},$$

where $\theta > 0$. Multiplying both sides of the inequality by

$$\frac{1}{|H(x,r)|_{\lambda}}\int_{H(x,r)} \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy,$$

we have

$$\left(\frac{1}{|H(x,r)|_{\lambda}}\int_{H(x,r)} \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy\right)\times\left(\frac{1}{|H(x,r)|_{\lambda}}\int_{H(x,r)} \omega(\operatorname{ch} y)^{-\frac{1+\theta}{1-p}}\operatorname{sh}^{2\lambda} y\,dy\right)^{\frac{p-1}{1+\theta}}$$

$$\le C^{p-1}\left(\frac{1}{|H(x,r)|_{\lambda}}\int_{H(x,r)} \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy\right)$$

$$\times\left(\frac{1}{|H(x,r)|_{\lambda}}\int_{H(x,r)} \omega(\operatorname{ch} y)^{-\frac{1}{1-p}}\operatorname{sh}^{2\lambda} y\,dy\right)^{p-1} \le C_1.$$

Suppose $\frac{1+\theta}{p-1} = \frac{1}{q-1} \Leftrightarrow (q-1)(1+\theta) = p-1 \Leftrightarrow p-q = \theta(q-1) > 0 \Leftrightarrow p > q$, then $p > q > 1$ and $\omega \in A_q^{\lambda}(\mathbb{R}_+)$. Thus we get Property 3 with $\varepsilon = p-q$. □

The following theorems are the analogues of the corresponding Theorems 2 and 3 from [29].

Theorem 3.4. *Let $0 \le \alpha < 2\lambda+1$, $1 \le p < \frac{2\lambda+1}{\alpha}$, $\frac{1}{p}-\frac{1}{q} = \frac{\alpha}{2\lambda+1}$, $\beta > 0$, $E_{\beta} = \{x \in \mathbb{R}_+ : M_G^{\alpha}f(\operatorname{ch} x) > \beta\}$ and $V(\operatorname{ch} x)$ is a nonnegative function on $\mathbb{R}_+$ such that for every interval $H \subset \mathbb{R}_+$, the inequality*

$$\left(\frac{1}{|H|_{\lambda}}\int_H V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}}\left(\frac{1}{|H|_{\lambda}}\int_H V(\operatorname{ch} x)^{-p'}\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{p'}} \le K \tag{3.4}$$

holds with K, independent of H, then there is a C, independent of f such that

$$\left(\int_{E_{\beta}} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}} \le \frac{C}{\beta}\left(\int_{\mathbb{R}_+}|f(\operatorname{ch} x)V(\operatorname{ch} x)|^p\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{p}}. \tag{3.5}$$

Proof. Fix $M>0$ and let $E_{\beta,M}$, an interval of radius M, be the intersection of the set E_β. For each $x\in E_{\beta,M}$, there is an interval H centered at x such that

$$|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H |f(\operatorname{ch}x)|\operatorname{sh}^{2\lambda}x\,dx>\beta. \tag{3.6}$$

By the Lindelof covering theorem (see [24]), there is a sequence $\{H_k\}$ such that $E_{\beta,M}\subset\cup H_k$, then we can write

$$\begin{aligned}\Bigg(\int\limits_{E_{\beta,M}} V(\operatorname{ch}x)^q\operatorname{sh}^{2\lambda}x\,dx\Bigg)^{\frac{p}{q}} &\le \Bigg(\sum_k\int\limits_{H_k} V(\operatorname{ch}x)^q\operatorname{sh}^{2\lambda}x\,dx\Bigg)^{\frac{p}{q}}\\ &\le \sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch}x)^q\operatorname{sh}^{2\lambda}x\,dx\Bigg)^{\frac{p}{q}}, \end{aligned}\tag{3.7}$$

so, $\frac{p}{q}\le 1$.

Since interval H_k satisfies (3.6), from (3.7), we have

$$\begin{aligned}\sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch}x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}} &\le \sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch}x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}}\\ &\times\Bigg(\frac{1}{\beta}|H_k|^{\frac{\alpha}{2\lambda+1}-1}\int\limits_{H_k}\big|f(\operatorname{ch}x)\big|\operatorname{sh}^{2\lambda}x,dx\Bigg)^p\\ &=\sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch}x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}}\frac{1}{\beta^p}|H_k|^{1-p-\frac{p}{q}}\\ &\times\Bigg(\int\limits_{H_k}|f(\operatorname{ch}x)|\,V(\operatorname{ch}x)V(\operatorname{ch}x)^{-1}\operatorname{sh}^{2\lambda}x,dx\Bigg)^p.\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}&\Bigg(\int\limits_{H_k}\big|f(\operatorname{ch}x)\big|\,V(\operatorname{ch}x)\,V(\operatorname{ch}x)^{-1}\operatorname{sh}^{2\lambda}x,dx\Bigg)^p\\ &\le\Bigg(\int\limits_{H_k}\big|f(\operatorname{ch}x)\,V(\operatorname{ch}x)\big|^p\operatorname{sh}^{2\lambda}x,dx\Bigg)\Bigg(\int\limits_{H_k}V(\operatorname{ch}x)^{-p'}\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{p'}}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch}x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}} &\le\sum_k\Bigg(\frac{1}{|H_k|_\lambda}\int\limits_{H_k} V(\operatorname{ch}x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}}\\ \times\Bigg(\frac{1}{\beta^p}\int\limits_{H_k}\big|f(\operatorname{ch}x)V(\operatorname{ch}x)\big|^p\operatorname{sh}^{2\lambda}x,dx\Bigg)&\Bigg(\frac{1}{|H_k|_\lambda}\int\limits_{H_k}V(\operatorname{ch}x)^{-p'}\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{p'}}.\end{aligned}$$

Taking into account (3.4), we have

$$\sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch}x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}}\le C\beta^{-p}\Bigg(\int\limits_{H_k}\big|f(\operatorname{ch}x)V(\operatorname{ch}x)\big|^p\operatorname{sh}^{2\lambda}x,dx\Bigg).$$

From this and (3.7), it follows that

$$\left(\int\limits_{E_{\beta,M}} V(\operatorname{ch} x)^q \operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}} \le \frac{C}{\beta}\left(\int\limits_{H_k} |f(\operatorname{ch} x)V(\operatorname{ch} x)|^p \operatorname{sh}^{2\lambda} x dx\right)^{\frac{1}{p}}$$
$$\le \frac{C}{\beta}\left(\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)V(\operatorname{ch} x)|^p \operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{p}}.$$

So, (3.5) follows from the monotone convergence theorem. □

Theorem 3.5. *Let $0 < \alpha < 2\lambda + 1$, $1 < p < \frac{2\lambda+1}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$ and $V(\operatorname{ch} x)$ be a nonnegative function on $\mathbb{R}_+$ such that for every interval H, (3.4) holds with K, independent of H. Then there is a constant C, independent of φ such that*

$$\left(\int\limits_{\mathbb{R}_+} \left[M_G^\alpha \varphi(\operatorname{ch} x)V(\operatorname{ch} x)\right]^q \operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}} \le C\left(\int\limits_{\mathbb{R}_+} \left|\varphi(\operatorname{ch} x)V(\operatorname{ch} x)\right|^p \operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{p}}. \tag{3.8}$$

Proof. Suppose $W(\operatorname{ch} x) = V(\operatorname{ch} x)^q$ and note that the condition (3.4) is equivalent to

$$\left(\frac{1}{|H|_\lambda}\int\limits_H W(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x, dx\right)\left(\frac{1}{|H|_\lambda}\int\limits_H W(\operatorname{ch} x)^{-\frac{1}{r-1}} \operatorname{sh}^{2\lambda} x, dx\right)^{r-1} \le C,$$

where $r = 1 + \frac{q}{p'}$. That is, $W(\operatorname{ch} x)$ belongs to class $A_r^\lambda(\mathbb{R}_+)$. Then by Properties 1 and 3, there exists an $\varepsilon > 0$ such that $r_2(\varepsilon) < r < r_1(\varepsilon)$ and simultaneously, $W \in A_{r_1(\varepsilon)}^\lambda(\mathbb{R}_+)$ and $W \in A_{r_2(\varepsilon)}^\lambda(\mathbb{R}_+)$. Let $0 < \varepsilon < \min(\alpha, 2\lambda + 1 - \alpha)$.

Suppose

$$\begin{aligned}
\frac{1}{p_1} &= \frac{1}{p} - \frac{\varepsilon}{2\lambda+1} < \frac{1}{p} < \frac{1}{p} + \frac{\varepsilon}{2\lambda+1} = \frac{1}{p_2} \Longrightarrow p_2 < p < p_1,\\
\frac{1}{q_1} &= \frac{1}{p} - \frac{\alpha+\varepsilon}{2\lambda+1} = \frac{}{p} - \frac{\varepsilon}{2\lambda+1} - \frac{\alpha}{2\lambda+1} = \frac{1}{p_1} - \frac{\alpha}{2\lambda+1},\\
\frac{1}{q_2} &= \frac{1}{p} - \frac{\alpha-\varepsilon}{2\lambda+1} = \frac{1}{p} + \frac{\varepsilon}{2\lambda+1} - \frac{\alpha}{2\lambda+1} = \frac{1}{p_2} - \frac{\alpha}{2\lambda+1}.
\end{aligned}$$

From this it follows that simultaneously $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{2\lambda+1}$ and $\frac{1}{q_2} = \frac{1}{p_2} - \frac{\alpha}{2\lambda+1}$, and then suppose

$$\begin{aligned}
r_1(\varepsilon) &= 1 + \frac{p_1(2\lambda+1)}{p_1'(2\lambda+1-(\alpha+\varepsilon)p_1)}\\
&= 1 + \frac{p_1(2\lambda+1)}{p_1'(2\lambda+1-\alpha p_1)} = r_1 = 1 + \frac{q_1}{p_1'}, \quad p_1p_1' = p_1 + p_1',\\
r_2(\varepsilon) &= 1 + \frac{p_2(2\lambda+1)}{p_2'(2\lambda+1-(\alpha+\varepsilon)p_2)}\\
&= 1 + \frac{p_2(2\lambda+1)}{p_2'(2\lambda+1-\alpha p_2)} = r_2 = 1 + \frac{q_2}{p_2'}, \quad p_2p_2' = p_2 + p_2'.
\end{aligned}$$

We obtain for $r_2 < r < r_1$, from $p_2 < p < p_1$ it follows that $p_1' < p' < p_2'$, but then simultaneously $W \in A^\alpha_{1+\frac{q_1}{p_1'}}(\mathbb{R}_+)$ and $W \in A^\alpha_{1+\frac{q_2}{p_2'}}(\mathbb{R}_+)$.

By Theorem 3.4, there exists a constant C such that

$$\left(\int\limits_{E_\beta} W(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x, dx\right)^{\frac{p_i}{q_i}} \le C\beta^{-p_i}\int\limits_{\mathbb{R}_+} \left|\varphi(\operatorname{ch} x)\right|^{p_i} W(\operatorname{ch} x)^{\frac{p_i}{q_i}} \operatorname{sh}^{2\lambda} x, dx, \quad i = 1, 2. \tag{3.9}$$

Now define a sublinear operator T by

$$Tg(\operatorname{ch} x) = M_G^{\alpha}\left[g(\operatorname{ch} x)W(\operatorname{ch} x)^{\frac{\alpha}{2\lambda+1}}\right].$$

Then with $\varphi(\operatorname{ch} x) = g(\operatorname{ch} x)W(\operatorname{ch} x)^{\frac{\alpha}{2\lambda+1}}$, (3.9) can be written in the form

$$\int\limits_{\{x\in E_\beta : Tg(\operatorname{ch} x)>\beta\}} W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx \le C\beta^{-q_i}\left(\int\limits_{\mathbb{R}_+} |g(\operatorname{ch} x)|^{p_i} W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{q_i}{p_i}}, \quad i=1,2.$$

From this it follows that the operator T has simultaneously weak type (p_1, q_1) and (p_2, q_2).

$$\left(\int\limits_{\mathbb{R}_+} [Tg(\operatorname{ch} x)]^q W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}} \le C\left(\int\limits_{\mathbb{R}_+} [g(\operatorname{ch} x)]^p W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{p}}.$$

Supposing here $g(\operatorname{ch} x) = \varphi(\operatorname{ch} x)W(\operatorname{ch} x)^{-\frac{\alpha}{2\lambda+1}}$ and $W(\operatorname{ch} x) = V(\operatorname{ch} x)^q$, we obtain the assertion of the theorem. □

4. Main Results

4.1. Weighted $(L_{p,\omega,\lambda},\ L_{q,\omega,\lambda})$ Boundedness Gegenbauer Fractional Maximal Operator. Next two theorems are analogues of works [12] and [30].

Theorem 4.1. *Let* $1 < p < \frac{2\lambda+1}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$. *Then the next two conditions are equivalent:*

(i) $\exists C > 0$ *such that* $\forall f \in L_{p,\omega,\lambda}(\mathbb{R}_+, G)$ *the following inequality*

$$\left\{\int\limits_{\mathbb{R}_+} \left[M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)\right]^q \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right\}^{\frac{1}{q}} \le C\left(\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{p}} \text{ is valid,} \tag{4.1}$$

(ii) $\omega \in A_{1+\frac{q}{p'}}(\mathbb{R}_+)$, $p\,p' = p + p'$,

$$\sup_H \left(\frac{1}{|H|_\lambda}\int\limits_H \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\, dy\right)\left(\frac{1}{|H|_\lambda}\int\limits_H \omega(\operatorname{ch} y)^{-\frac{p'}{q}}\operatorname{sh}^{2\lambda} y\, dy\right)^{\frac{q}{p'}} < \infty. \tag{4.2}$$

Proof. We show that from (4.1), (4.2) we have the following. For every fixing interval $H \subset [0,\infty)$, we can write

$$\begin{aligned} M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x) &= \sup_H\left(|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \left|\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)\right|\operatorname{sh}^{2\lambda} y\, dy\right) \\ &\ge \left(|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \left|\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)\right|\operatorname{sh}^{2\lambda} y\, dy\right)\chi_H(\operatorname{ch} x). \end{aligned}$$

Taking into account (4.1), we obtain

$$\begin{aligned} &\left(\int\limits_H \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\, dx\right)^{\frac{1}{q}}\left(\left|H\right|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \left|\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)\right|\operatorname{sh}^{2\lambda} y\, dy\right) \\ &\qquad \le C\left(\int\limits_H |f(\operatorname{ch} x)|^p \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\, dx\right)^{\frac{1}{p}}. \end{aligned}$$

Thus,

$$|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \left|\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)\right|\operatorname{sh}^{2\lambda} y\,dy$$

$$\le C\left(\int\limits_H \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\right)^{-\frac{1}{q}}\left(\int\limits_H |f(\operatorname{ch} x)|^p\,\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{p}}.$$

Supposing $f(\operatorname{ch} x)=\omega(\operatorname{ch} x)^{-\frac{1}{p}\left(1+\frac{p'}{q}\right)}$, we obtain

$$|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \omega(\operatorname{ch} x)^{-\frac{p'}{q}}\operatorname{sh}^{2\lambda} x\,dx\le C\left(\int\limits_H \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\right)^{-\frac{1}{q}}\left(\int\limits_H \omega(\operatorname{ch} x)^{-\frac{p'}{q}}\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{p}}.$$

From this it follows that

$$|H|_\lambda^{\left(\frac{\alpha}{2\lambda+1}-1\right)q}\left(\int\limits_H \omega(\operatorname{ch} x)^{-\frac{p'}{q}}\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{q}{p'}}\le C^q\left(\int\limits_H \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\right)^{-1}.$$

So, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}\Leftrightarrow\frac{1}{q}-\frac{1}{p}+1=1-\frac{\alpha}{2\lambda+1}\Leftrightarrow\frac{1}{q}+\frac{1}{p'}=1-\frac{\alpha}{2\lambda+1}\Leftrightarrow 1+\frac{q}{p'}=\left(1-\frac{\alpha}{2\lambda+1}\right)q$, then (4.2) is provided. We show that from inequality (4.2) follows inequality (4.1). Suppose in (3.8) $\varphi(\operatorname{ch} x)=f(\operatorname{ch} x)\;\omega(\operatorname{ch} x)^{\frac{\alpha}{2\lambda+1}}$ and $V(\operatorname{ch} x)^q=\omega(\operatorname{ch} x)$, we obtain

$$\left(\int\limits_{\mathbb{R}_+}\left[M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)\right]^q\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{q}}\le C\left(\int\limits_{\mathbb{R}_+}[f(\operatorname{ch} x)]^p\,[\omega(\operatorname{ch} x)]^{\frac{p\alpha}{2\lambda+1}+\frac{p}{q}}\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{p}}$$

$$=C\left(\int\limits_{\mathbb{R}_+}[f(\operatorname{ch} x)]^p\,\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{p}},$$

since $p\left(\frac{\alpha}{2\lambda+1}+\frac{1}{q}\right)=1\Leftrightarrow\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$. □

Theorem 4.2. *Let $q=\frac{2\lambda+1}{2\lambda+1-\alpha}$. Then the next two conditions are equivalent:*

(i) $$\int\limits_{\left\{x\in\mathbb{R}_+:M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)>\beta\right\}}\le C\left(\frac{1}{\beta}\int\limits_{\mathbb{R}_+}|f(\operatorname{ch} x)|\,\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\right)^q,$$

where the constant C does not depend on f and $\beta>0$.

(ii) $\omega\in A_1^\lambda(\mathbb{R}_+)$, *i.e.*, $M\omega(\operatorname{ch} x)\le C\omega(\operatorname{ch} x)$.

Proof. Let $H_1\subset H$. Suppose $f\omega^{\frac{\alpha}{2\lambda+1}}=|H|_\lambda^{\frac{\alpha}{2\lambda+1}}\chi_{H_1}$, where χ_{H_1} is the characteristic function of H_1. From this we have

$$M_\mu^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)=|H|_\lambda^{\frac{\alpha}{2\lambda+1}}\,M_\mu\chi_{H_1}(\operatorname{ch} x). \tag{4.3}$$

But for any $x\in H$,

$$M_G\chi_{H_1}(\operatorname{ch} x)=\sup_{r>0}\frac{|H_1\cap H|_\lambda}{|H|_\lambda}\ge\frac{|H_1|_\lambda}{|H|_\lambda}. \tag{4.4}$$

From (4.3) and (4.4), for any $x\in H$, we have

$$M\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)\ge|H|_\lambda^{\frac{\alpha}{2\lambda+1}}\frac{|H_1|_\lambda}{|H|_\lambda}>\beta>0,$$

from this it follows that

$$H\subset\left\{x\in\mathbb{R}:M\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)>\beta\right\}$$

for every $0<\beta<|H|_\lambda^{\frac{\alpha}{2\lambda+1}}\frac{|H_1|_\lambda}{|H|_\lambda}$.

By (i) and Hölder's inequality, we obtain

$$\beta^q \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le \beta^q \int\limits_{\left\{y\in\mathbb{R}_+ : M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)>\beta\right\}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy$$

$$\le \Bigg(\int\limits_{H_1(x,r_1)} \omega^{1-\frac{\alpha}{2\lambda+1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy\Bigg)^q = C\Bigg(\int\limits_{H_1(x,r_1)} \left[\omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\right]^{\frac{1}{q}} (\operatorname{sh}^{2\lambda} y)^{1-\frac{1}{q}} dy\Bigg)^q$$

$$\le C\Bigg(\int\limits_{H_1(x,r_1)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy\Bigg)\Bigg(\int\limits_{H_1(x,r_1)} \operatorname{sh}^{2\lambda} y\, dy\Bigg)^{q-1}$$

$$= C\,|H_1(x,r_1)|_\lambda^q \Bigg(\frac{1}{|H_1(x,r_1)|_\lambda} \int\limits_{H_1(x,r_1)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy\Bigg).$$

From this it follows that

$$\frac{|H_1(x,r_1)|_\lambda^q}{|H(x,r)|_\lambda^q}\, |H(x,r)|_\lambda^{q-1} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy$$

$$\le C\,|H_1(x,r_1)|_\lambda^q \Bigg(\frac{1}{|H_1(x,r_1)|_\lambda} \int\limits_{H_1(x,r_1)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy\Bigg),$$

which is equivalent to

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le C \frac{1}{|H_1(x,r_1)|_\lambda} \int\limits_{H_1(x,r_1)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy.$$

Applying the Lebesgue differentiation theorem, we obtain

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le C\omega(\operatorname{ch} x)$$

for a.e. $x \in \mathbb{R}_+$.

Thus, $\omega \in A_1^\lambda(\mathbb{R}_+)$.

Now we show that from (ii) (i) we have the following. Applying Hölder's inequality, we obtain

$$M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x) = \sup_{r>0} \frac{1}{|H(x,r)|_\lambda^{1-\frac{\alpha}{2\lambda+1}}} \int\limits_{H(x,r)} |f(\operatorname{ch} t)|\, \omega(\operatorname{ch} t)^{\frac{\alpha}{2\lambda+1}} \operatorname{sh}^{2\lambda} t\, dt$$

$$= \sup_{r>0} \frac{1}{|H(x,r)|_\lambda^{1-\frac{\alpha}{2\lambda+1}}} \int\limits_{H(x,r)} \left[(f\omega)(\operatorname{ch} t)\operatorname{sh}^{2\lambda} t\right]^{\frac{\alpha}{2\lambda+1}} \left[f(\operatorname{ch} t)\operatorname{sh}^{2\lambda} t\right]^{1-\frac{\alpha}{2\lambda+1}} dt$$

$$\le \sup_{r>0} \frac{1}{|H(x,r)|_\lambda^{1-\frac{\alpha}{2\lambda+1}}} \Bigg(\int\limits_{H(x,r)} f(\operatorname{ch} t)\omega(\operatorname{ch} t)\operatorname{sh}^{2\lambda} t\, dt\Bigg)^{\frac{\alpha}{2\lambda+1}} \Bigg(\int\limits_{H(x,r)} f(\operatorname{ch} t)\operatorname{sh}^{2\lambda} t\, dt\Bigg)^{1-\frac{\alpha}{2\lambda+1}}$$

$$\le \left(M_\mu f(\operatorname{ch} x)\right)^{\frac{1}{q}} \left(\|f\|_{L^\lambda_{1,\omega,\lambda}}\right)^{1-\frac{1}{q}}.$$

From this it follows that

$$M_\mu f(\operatorname{ch} x) \ge M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)^q \|f\|_{L_{1,\omega,\lambda}}^{1-q}.$$

Then taking into account Theorem 3.2, we obtain

$$\int\limits_{\left\{y\in H(x,r):M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)>\beta\right\}} \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy \le \int\limits_{\left\{y\in\mathbb{R}_+:M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)>\beta\right\}} \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy$$
$$\le \int\limits_{\left\{y\in\mathbb{R}_+:M_\mu f(\operatorname{ch} y)>\beta^q\|f\|_{L_{1,\omega,\lambda}}^{1-q}\right\}} \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy \le C\beta^{-q}\|f\|_{L_{1,\omega,\lambda}}^{1-q}\int\limits_{\mathbb{R}_+} M_\mu f(\operatorname{ch} y)\omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy.$$

Using Theorem 3.1, for $p=1$, and also condition (ii), we get

$$\int\limits_{\left\{y\in\mathbb{R}_+:M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)>\beta\right\}} \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy \le C\beta^{-q}\|f\|_{L_{1,\omega,\lambda}}^{q-1}\int\limits_{\mathbb{R}_+} f(\operatorname{ch} y)M_\mu\omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy$$
$$\le C\beta^{-q}\|f\|_{L_{1,\omega,\lambda}}^{q-1}\int\limits_{\mathbb{R}_+} f(\operatorname{ch} y)\omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy = C\left(\frac{1}{\beta}\|f\|_{L_{1,\omega,\lambda}}\right)^q. \qquad \square$$

4.2. Weighted $(L_{p,\omega,\lambda},\ L_{q,\omega,\lambda})$ Boundedness of G-Riesz Potential. In this section we obtain some results for the G-Riesz potential (1.1), which are analogous to the corresponding results obtained in [12] for the B-Riesz potential.

Lemma 4.1. *Let $0<\alpha<2\lambda+1,\ 1\le p<\frac{\beta}{\alpha}$. Then there is a positive constant C such that for any $r>0$ and $x\in\mathbb{R}_+$, we have*

$$|I_G^{\alpha} f(\operatorname{ch} x)| \le C\Big((\operatorname{sh} r)^{\alpha} M_G f(\operatorname{ch} x) + (\operatorname{sh} r)^{\alpha-\frac{\beta}{p}} M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\Big). \tag{4.5}$$

Proof. From (1.1), we have

$$I_G^{\alpha} f(\operatorname{ch} x) = \left(\int\limits_0^r + \int\limits_r^{\infty}\right)\left(\int\limits_0^{\infty} r^{\frac{\alpha}{2}-1} h_r(\operatorname{ch} t)dr\right) A_{\operatorname{ch}\ t}^{\lambda} f(\operatorname{ch} x)\operatorname{sh}^{2\lambda} t\,dt = A_1(x,r) + A_2(x,r). \tag{4.6}$$

We consider $A_1(x,r)$. Let $0<r<2$. Then from Lemma 3.2 and Corollary 3.1 [15], we have

$$|A_1(x,r)| \le \int\limits_0^r A_{\operatorname{ch} t}^{\lambda}|f(\operatorname{ch} x)|(\operatorname{sh} t)^{2\lambda}(\operatorname{sh} t)^{\alpha-2\lambda-1}dt \le \sum_{k=0}^{\infty}\int\limits_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{\operatorname{ch} t}^{\lambda}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t\,dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}$$
$$\le \sum_{k=0}^{\infty}\left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\alpha}\left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{-2\lambda-1}\int\limits_0^{\frac{r}{2^k}} A_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t\,dt$$
$$\lesssim M_G f(\operatorname{ch} x)\sum_{k=0}^{\infty}\left(\frac{1}{2^{k+1}}\operatorname{sh} r\right)^{\alpha} \lesssim (\operatorname{sh} r)^{\alpha} M_G f(\operatorname{ch} x)\sum_{k=0}^{\infty}\frac{1}{2^{(k+1)^{\alpha}}}$$
$$\lesssim (\operatorname{sh} r)^{\alpha} M_G f(\operatorname{ch} x), \tag{4.7}$$

since $\operatorname{sh}\frac{t}{a} \le \frac{1}{a}\operatorname{sh} t$ for $a\ge 1$.

Now let $2 \le r < \infty$ and $0 < \alpha < 4\lambda$. Then from the proof of Corollary 3.1 in [15], we have

$$\begin{aligned}
|A_1(x,r)| &\le \int_0^r \frac{A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t dt}{(\mathrm{ch}\, t)^{2\lambda+1-\alpha}} \le \int_0^r \frac{A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t\, dt}{(\mathrm{ch}\, t)^{4\lambda-\alpha}} \\
&\le \int_0^r \frac{A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t dt}{(\mathrm{sh}\, t)^{4\lambda-\alpha}} \le \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t dt}{(\mathrm{sh}\, t)^{4\lambda-\alpha}} \\
&\le \sum_{k=0}^\infty \Big(\mathrm{sh}\, \frac{r}{2^{k+1}}\Big)^\alpha \Big(\mathrm{sh}\, \frac{r}{2^{k+1}}\Big)^{-4\lambda} \int_0^{\frac{r}{2^k}} A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t dt \\
&\le M_G f(\mathrm{ch}\, x) \sum_{k=0}^\infty \Big(\mathrm{sh}\, \frac{r}{2^{k+1}}\Big)^\alpha \le (\mathrm{sh}\, r)^\alpha M_G f(\mathrm{ch}\, x), \quad 0 < \alpha < 4\lambda.
\end{aligned} \tag{4.8}$$

Now let $4\lambda \le \alpha < 2\lambda+1$. From the proof of Corollary 3.1 in [15], it follows that $\int_0^\infty r^{\frac{\alpha}{2}-1} h_2(\mathrm{ch}\, t) dr \lesssim 1$, then

$$\begin{aligned}
|A_1(x,r)| &\le \int_0^r \frac{A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t}{(\mathrm{ch}\, t)^{2\lambda+1-\alpha}} dt \\
&\le \int_0^r A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t dt = \frac{(\mathrm{sh}\, \frac{r}{2})^{4\lambda}}{(\mathrm{sh}\, \frac{r}{2})^{4\lambda}} \int_0^r A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t\, dt \\
&\le \Big(\mathrm{sh}\, \frac{r}{2}\Big)^{4\lambda} M_G\, f(\mathrm{ch}\, x) \le (\mathrm{sh}\, r)^\alpha M_G f(\mathrm{ch}\, x), \quad 4\lambda < \alpha < 2\lambda+1.
\end{aligned} \tag{4.9}$$

Thus from (4.7)–(4.9), it follows that for every $0 < r < \infty$ and $0 < \alpha < 2\lambda + 1$,

$$|A_1(x,r)| \lesssim (\mathrm{sh}\, r)^\alpha M_G f(\mathrm{ch}\, x). \tag{4.10}$$

We estimate $A_2(x,r)$. Let $0 < r < 2$. Then

$$\begin{aligned}
|A_2(x,r)| &\le \int_r^\infty \frac{A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t}{(\mathrm{sh}\, t)^{2\lambda+1-\alpha}} dt = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1} r} \frac{A^\lambda_{\mathrm{ch} t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t}{(\mathrm{sh}\, t)^{2\lambda+1-\alpha}} dt \\
&\le \sum_{k=0}^\infty \big(\mathrm{sh}\, 2^k r\big)^{\alpha-2\lambda-1} \int_0^{2^{k+1} r} A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t dt \\
&= \sum_{k=0}^\infty \big(\mathrm{sh}\, 2^k r\big)^{\alpha-\frac{\beta}{p}} \big(\mathrm{sh}\, 2^k r\big)^{\frac{\beta}{p}-2\lambda-1} \int_0^{2^{k+1} r} A^\lambda_{\mathrm{ch}\, t}|f(\mathrm{ch}\, x)|\,\mathrm{sh}^{2\lambda} t\, dt \\
&\le M_G^{\frac{\beta}{p}} f(\mathrm{ch}\, x) \sum_{k=0}^\infty \big(\mathrm{sh}\, 2^k r\big)^{\alpha-\frac{\beta}{p}} \le (\mathrm{sh}\, r)^{\alpha-\frac{\beta}{p}} M_G^{\frac{\beta}{p}} f(\mathrm{ch}\, x) \sum_{k=0}^\infty 2^{k(\alpha-\frac{\beta}{p})} \\
&\lesssim (\mathrm{sh}\, r)^{\alpha-\frac{\beta}{p}} M_G^{\frac{\beta}{p}} f(\mathrm{ch}\, x),
\end{aligned} \tag{4.11}$$

by the condition $\alpha - \frac{\beta}{p} < 0$.

Now let $2 \le r < \infty$. Then for $0 < \alpha < 4\lambda$, we have

$$A_2(x,r) \le \int_r^\infty \frac{A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{4\lambda-\alpha}}dt \le \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \frac{A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{4\lambda-\alpha}}dt$$
$$\le \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-4\lambda} \int_0^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt$$
$$= \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-\frac{\beta}{p}} (\operatorname{sh} 2^k r)^{\frac{\beta}{p}-4\lambda} \int_0^{2^k r} A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh} 2\lambda t dt$$
$$\le M_G^{\frac{\beta}{p}} f(\operatorname{ch} x) \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-\frac{\beta}{p}} \le (\operatorname{sh} r)^{\alpha-\frac{\beta}{p}} M_G^{\frac{\beta}{p}} f(\operatorname{ch} x). \tag{4.12}$$

We consider the case $4\lambda < \alpha < 2\lambda+1$. Then

$$|A_2(x,r)| \lesssim \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \frac{A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}dt \le \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-2\lambda-1} \int_{2^k r}^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t\, dt$$
$$= \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-\frac{\beta}{p}} (\operatorname{sh}\ 2^k r)^{\frac{\beta}{p}-4\lambda} (\operatorname{sh}\ 2^k r)^{2\lambda-1} \int_{2^k r}^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt$$
$$\lesssim \sum_{k=0}^\infty (\operatorname{sh}\ 2^k r)^{\alpha-\frac{\beta}{p}} (\operatorname{sh}\ 2^k r)^{\frac{\beta}{p}-4\lambda} \int_{2^k r}^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt$$
$$\lesssim (\operatorname{sh}\ r)^{\alpha-\frac{\beta}{p}} M_G^{\frac{\beta}{p}} f(\operatorname{ch} x). \tag{4.13}$$

From (4.11)–(4.13) it follows that for any $0 < r < \infty$ and $0 < \alpha < 2\lambda+1$, the inequality

$$A_2(x,r) \lesssim (\operatorname{sh}\ r)^{\alpha-\frac{\beta}{p}} M_G^{\frac{\beta}{p}} f(\operatorname{ch} x) \tag{4.14}$$

is valid. Taking into account (4.10) and (4.14) in (4.6), we obtain the statement of Lemma 4.1. □

Theorem 4.3. *Let $0 < \alpha < \beta \le 2\lambda+1$, $1 < p < \frac{\beta}{\alpha}$, $1 \le r \le \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta} + \frac{\alpha p}{\beta r}$. Then for any function $f \in L_{p,\lambda}(\mathbb{R}_+)$ and $M_G^{\frac{\beta}{p}} f \in L_{r,\lambda}(\mathbb{R}_+)$ the estimate*

$$\left\|I_G^\alpha f\right\|_{L_{q,\lambda}(\mathbb{R}_+)} \lesssim \left\|M_G^{\frac{\beta}{p}} f\right\|_{L_{r,\lambda}(\mathbb{R}_+)}^{\frac{\alpha\beta}{\beta}} \cdot \left\|f\right\|_{L_{p,\lambda}(\mathbb{R}_+)}^{1-\frac{\alpha\beta}{\beta}}$$

is valid.

Proof. From (4.5), for

$$\operatorname{sh} r = \operatorname{sh} r(\operatorname{ch} x) = \left(\frac{M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)}{M_G f(\operatorname{ch} x)}\right)^{\frac{p}{\beta}},$$

we obtain

$$\left|I_G^\alpha f(\operatorname{ch} x)\right| \lesssim \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\right)^{\frac{\alpha p}{\beta}} \left(M_G f(\operatorname{ch} x)\right)^{1-\frac{\alpha p}{\beta}}$$

for each $x \in \mathbb{R}_+$.

Considering both sides of inequality (4.1) to the power of q, integrate by x and using Hölder's inequality, we have

$$\int\limits_{\mathbb{R}_+} \left|I_G^\alpha f(\operatorname{ch} x)\right|^q \operatorname{sh}^{2\lambda} x\,dx \lesssim \int\limits_{\mathbb{R}_+} \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\right)^{\frac{\alpha pq}{\beta}} \left(M_G f(\operatorname{ch} x)\right)^{q-\frac{\alpha pq}{\beta}} \operatorname{sh}^{2\lambda} x\,dx$$

$$\lesssim \left(\int\limits_{\mathbb{R}_+} \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\right)^{\frac{\alpha pq}{\beta}s'} \operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{s'}} \left(\int\limits_{\mathbb{R}_+} \left(M_G f(\operatorname{ch} x)\right)^{\left(q-\frac{\alpha pq}{\beta}\right)s} \operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{s}},$$

where

$$\left(q-\frac{\alpha pq}{\beta}\right)s = p, \quad s' = \frac{s}{s-1} = \frac{\beta r}{\alpha pq}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta} + \frac{\alpha p}{\beta r}.$$

Therefore,

$$\left(\int\limits_{\mathbb{R}_+} |I_G^\alpha f(\operatorname{ch} x)|^q\,d\mu_\lambda(x)\right)^{\frac{1}{q}} \lesssim \left(\int\limits_{\mathbb{R}_+} \left(M_G f(\operatorname{ch} x)\right)^p d\mu_\lambda(x)\right)^{\frac{1}{sq}} \left(\int\limits_{\mathbb{R}_+} \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\right)^r d\mu_\lambda(x)\right)^{\frac{\alpha p}{\beta r}}$$

$$\lesssim \left(\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p\,d\mu_\lambda(x)\right)^{\frac{1}{sq}} \left(\int\limits_{\mathbb{R}_+} \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\right)^r d\mu_\lambda(x)\right)^{\frac{\alpha p}{\beta r}},$$

which is equivalent to

$$\left\|I_G^\alpha f\right\|_{L_{q,\lambda}(\mathbb{R}_+)} \lesssim \left\|f\right\|_{L_{p,\lambda}(\mathbb{R}_+)}^{1-\frac{\alpha p}{\beta}} \cdot \left\|M_G^{\frac{\beta}{p}} f\right\|_{L_{r,\lambda}(\mathbb{R}_+)}^{\frac{\alpha p}{\beta}}.$$

The theorem is proved. □

Lemma 4.2. *Let $0 < \varepsilon < \min(\alpha,\ 2\lambda+1-\alpha)$. Then there is the constant $C_\varepsilon > 0$ such that for any nonnegative function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and for every point $x \in \mathbb{R}_+$, the inequality*

$$I_G^\alpha \varphi(\operatorname{ch} x) \le C_\varepsilon \sqrt{M_G^{\alpha-\varepsilon}\varphi(\operatorname{ch} x) M_G^{\alpha+\varepsilon}\varphi(\operatorname{ch} x)} \tag{4.15}$$

is valid.

Proof. Let r be an arbitrary positive number. Using the scheme of the proof of Lemma 4.1, we have

$$I_G^\alpha \varphi(\operatorname{ch} x) \lesssim \left(\int\limits_0^r + \int\limits_r^\infty\right) A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ (\operatorname{sh} t)^{\alpha-2\lambda-1} \operatorname{sh}^{2\lambda} t\,dt = J_1 + J_2. \tag{4.16}$$

Let $0 < \varepsilon < \alpha$, then

$$J_1 = \int\limits_0^r \frac{A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}dt = \sum_{k=0}^\infty \int\limits_{2^{-k-1}r}^{2^{-k}r} \frac{A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}dt$$

$$\le \sum_{k=0}^\infty \left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\alpha-2\lambda-1} \int\limits_0^{2^{-k}r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\,dt$$

$$\le \sum_{k=0}^\infty \left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\varepsilon} \left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\alpha-2\lambda-1-\varepsilon} \int\limits_0^{2^{-k}r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\,dt$$

$$\le (\operatorname{sh} r)^\varepsilon \sum_{k=0}^\infty 2^{-(k+1)\varepsilon} \left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\alpha-2\lambda-1-\varepsilon} \int\limits_0^{2^{-k}r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\,dt$$

$$\le C_\varepsilon (\operatorname{sh} r)^\varepsilon M_G^{\alpha-\varepsilon}\varphi(\operatorname{ch} x). \tag{4.17}$$

Now let $0 < \varepsilon < 2\lambda + 1 - \alpha$. Then

$$\begin{aligned} J_2 &= \int\limits_r^\infty \frac{A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} dt = \sum_{k=0}^\infty \int\limits_{2^k r}^{2^{k+1} r} \frac{A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} dt \\ &\le \sum_{k=0}^\infty \left(\operatorname{sh}\ 2^k r\right)^{\alpha-2\lambda-1} \int\limits_0^{2^{k+1} r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t\, dt \\ &\le \sum_{k=0}^\infty \left(\operatorname{sh}\ 2^k r\right)^{-\varepsilon} \left(\operatorname{sh}\ 2^k r\right)^{\alpha+\varepsilon-2\lambda-1} \int\limits_0^{2^{k+1} r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t\, dt \\ &\le (\operatorname{sh} r)^{-\varepsilon} \sum_{k=0}^\infty \left(2^{-k\varepsilon}\right) \left(\operatorname{sh}\ 2^k r\right)^{\alpha+\varepsilon-2\lambda-1} \int\limits_0^{2^{k+1} r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t\, dt \\ &\le C_\varepsilon (\operatorname{sh} r)^{-\varepsilon} M_G^{\alpha+\varepsilon}\varphi(\operatorname{ch} x). \end{aligned} \tag{4.18}$$

Taking into account (4.17) and (4.18) in (4.16), we get that for any $\varepsilon > 0$ with $0 < \varepsilon < \min(\alpha, 2\lambda + 1 - \alpha)$, there exists $C_\varepsilon > 0$ such that for every nonnegative function φ, for any point $x \in \mathbb{R}_+$ and $r > 0$, the following inequality

$$I_G^\alpha \varphi(\operatorname{ch} x) \le C_\varepsilon \left((\operatorname{sh} r)^\varepsilon M_G^{\alpha-\varepsilon}\varphi(\operatorname{ch} x) + (\operatorname{sh} r)^{-\varepsilon} M_G^{\alpha+\varepsilon}\varphi(\operatorname{ch} x)\right) \tag{4.19}$$

holds.

Assuming in (4.19)

$$(\operatorname{sh} r)^\varepsilon = \left(\frac{M_G^{\alpha+\varepsilon}\varphi(\operatorname{ch} x)}{M_G^{\alpha-\varepsilon}\varphi(\operatorname{ch} x)} \right)^{\frac{1}{q}},$$

we obtain inequality (4.15). □

Theorem 4.4. *Let $1 < p < \frac{2\lambda+1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2\lambda+1}$. Then for ensuring the inequality*

$$\left(\int\limits_{\mathbb{R}_+} \left| I_G^\alpha \left(f\omega^\alpha\right)(\operatorname{ch} x)\right|^q \omega(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx \right)^{\frac{1}{q}} \lesssim \left(\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p \omega(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx \right)^{\frac{1}{p}}$$

the necessary and sufficient condition is

$$\omega \in A_\beta^\lambda(\mathbb{R}_+), \quad \beta = 1 + \frac{q}{p'}, \quad pp' = p + p'$$

for any $f \in L_{p,\omega,\lambda}(\mathbb{R}_+)$.

Proof. Sufficiency. Let $\omega \in A_\beta^\lambda(\mathbb{R}_+)$, then $\omega \in A_{\beta-\mu}^\lambda(\mathbb{R}_+)$ for any $\mu > 0$ sufficiently small. Therefore, for $0 < \varepsilon < \min(\alpha,\ 2\lambda + 1 - \alpha)$, we have $\omega \in A_{\beta_1}^\lambda(\mathbb{R}_+)$ with $\beta_1 = 1 + \frac{p(2\lambda+1)}{p'(2\lambda+1-(\alpha+\varepsilon)p)}$ and $\omega \in A_{\beta_2}^\lambda(\mathbb{R}_+)$ with $\beta_2 = 1 + \frac{p(2\lambda+1)}{p'(2\lambda+1-(\alpha-\varepsilon)p)}$. Now, if we take

$$\frac{1}{q_\varepsilon} = \frac{1}{p} - \frac{\alpha+\varepsilon}{2\lambda+1}, \quad \frac{1}{\overline{q}_\varepsilon} = \frac{1}{p} - \frac{\alpha-\varepsilon}{2\lambda+1},$$

then we find that $\omega \in A_{1+\frac{q_\varepsilon}{p'}}^\lambda(\mathbb{R}_+)$ and $\omega \in A_{1+\frac{\overline{q}_\varepsilon}{p'}}^\lambda(\mathbb{R}_+)$.

In view of $p_1 = \frac{2q_\varepsilon}{q}$ and $p_2 = \frac{2\overline{q}_\varepsilon}{q}$, we will have

$$\begin{aligned} \frac{1}{p_1} + \frac{1}{p_2} &= \frac{q}{2}\left(\frac{1}{q_\varepsilon} + \frac{1}{\overline{q}_\varepsilon}\right) = \frac{q}{2}\left(\frac{1}{p} - \frac{\alpha+\varepsilon}{2\lambda+1} + \frac{1}{p} - \frac{\alpha-\varepsilon}{2\lambda+1}\right) \\ &= q\left(\frac{1}{p} - \frac{\alpha}{2\lambda+1}\right) = 1 \Longleftrightarrow \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2\lambda+1}. \end{aligned}$$

Suppose

$$F_1(\operatorname{ch} x) = \left(M_G^{\alpha+\varepsilon}\left(f\omega^\alpha\right)(\operatorname{ch} x)\right)^{\frac{q}{2}} \omega(\operatorname{ch} x)^{\frac{1}{p_1}}$$

and

$$F_2(\operatorname{ch} x) = \left(M_G^{\alpha-\varepsilon}\left(f\omega^\alpha\right)(\operatorname{ch} x)\right)^{\frac{q}{2}} \omega(\operatorname{ch} x)^{\frac{1}{p_2}}.$$

From (4.12), by Ḧolder's inequality, we have

$$\begin{aligned}
\int\limits_{\mathbb{R}_+} \left|I_G^\alpha\left(f\omega^\alpha\right)(\operatorname{ch} x)\right|^q \omega(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx &\le C_\varepsilon \int\limits_{\mathbb{R}_+} F_1(\operatorname{ch} x) F_2(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx \\
&\le C_\varepsilon \Bigg(\int\limits_{\mathbb{R}_+} \left(M_G^{\alpha+\varepsilon}(f\omega^\alpha)(\operatorname{ch} x)\right)^{\frac{qp_1}{2}} \omega(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx \Bigg)^{\frac{1}{p_1}} \\
&\quad \times \Bigg(\int\limits_{\mathbb{R}_+} \left(M_G^{\alpha-\varepsilon}(f\omega^\alpha)(\operatorname{ch} x)\right)^{\frac{qp_2}{2}} \omega(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx \Bigg)^{\frac{1}{p_2}} \\
&= C_\varepsilon \Bigg(\int\limits_{\mathbb{R}_+} \left(M_G^{\alpha+\varepsilon}(f\omega^\alpha)(\operatorname{ch} x)\right)^{q_\varepsilon} \omega(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx \Bigg)^{\frac{1}{p_1}} \\
&\quad \times \Bigg(\int\limits_{\mathbb{R}_+} \left(M_G^{\alpha-\varepsilon}(f\omega^\alpha)(\operatorname{ch} x)\right)^{\overline{q}_\varepsilon} \omega(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx \Bigg)^{\frac{1}{p_2}}.
\end{aligned}$$

Finally, using Theorem 4.1, we obtain

$$\left\|I_G^\alpha\left(f\omega^\alpha\right)\right\|_{L_{q,\omega,\lambda}(\mathbb{R}_+)} \lesssim \|f\|_{L_{p,\omega,\lambda}(\mathbb{R}_+)}.$$

Necessity. We show that

$$M_G^\alpha\left(f\omega^\alpha\right)(\operatorname{ch} x) \lesssim I_G^\alpha\left(|f|\,\omega^\alpha\right)(\operatorname{ch} x). \tag{4.20}$$

In fact,

$$\begin{aligned}
\int\limits_{H(0,r)} A_{\operatorname{ch} t}\left(f\omega^\alpha\right)(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\, dt &= \int\limits_0^r A_{\operatorname{ch} t}\left(f\omega^\alpha\right)(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\, dt \\
&= \int\limits_0^r \frac{A_{\operatorname{ch} t}\left(f\omega^\alpha\right)(\operatorname{ch} x)(\operatorname{sh} t)^{2\lambda+1-\alpha} \operatorname{sh}^{2\lambda} t\, dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} \\
&= \sum_{k=0}^\infty \int\limits_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{\operatorname{ch} t}\left(f\omega^\alpha\right)(\operatorname{ch} x)(\operatorname{sh} t)^{2\lambda+1-\alpha} \operatorname{sh}^{2\lambda} t\, dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} \\
&\le \sum_{k=0}^\infty \left(\operatorname{sh}\frac{r}{2^k}\right)^{2\lambda+1-\alpha} \int\limits_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{\operatorname{ch} t}\left(|f|\,\omega^\alpha\right)(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\, dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} \\
&\le \left(\operatorname{sh}\frac{r}{2}\right)^{2\lambda+1-\alpha} \sum_{k=0}^\infty \frac{1}{2^{(k-1)(2\lambda+1-\alpha)}} \times \int\limits_0^\infty \frac{A_{\operatorname{ch} t}\left(|f|\,\omega^\alpha\right)(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\, dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} \\
&\lesssim \left(\operatorname{sh}\frac{r}{2}\right)^{2\lambda+1-\alpha} I_G^\alpha\left(|f|\,\omega^\alpha\right)(\operatorname{ch} x).
\end{aligned} \tag{4.21}$$

Taking into account Lemma 2.1, by $0 < r < 2$ and (4.21), we have

$$M_G^\alpha (f\omega^\alpha)(\operatorname{ch} x) = \sup_{r>0} |H(0,r)|^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(0,r)} A_{\operatorname{ch} t}(f\omega^\alpha)(\operatorname{ch} x)\operatorname{sh}^{2\lambda} t\, dt$$
$$\lesssim \sup_{r>0} \left(\operatorname{sh}\frac{r}{2}\right)^{(2\lambda+1)\left(\frac{\alpha}{2\lambda+1}-1\right)} \left(\operatorname{sh}\frac{r}{2}\right)^{2\lambda+1-\alpha} I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x) \lesssim I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x). \tag{4.22}$$

On the other hand,

$$\int\limits_0^r A_{\operatorname{ch} t}(f\omega^\alpha)(\operatorname{ch} x)\operatorname{sh}^{2\lambda} t dt \le \sum_{k=0}^\infty \int\limits_{\left(\frac{r}{2^{k+1}}\right)^{\frac{4\lambda}{2\lambda+1}}}^{\left(\frac{r}{2^k}\right)^{\frac{4\lambda}{2\lambda+1}}} \frac{A_{\operatorname{ch} t}(f\omega^\alpha)(\operatorname{ch} x)(\operatorname{sh} t)^{2\lambda+1-\alpha}\operatorname{sh}^{2\lambda} t\, dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}$$
$$\le \sum_{k=0}^\infty \left(\operatorname{sh}\frac{r}{2^k}\right)^{\frac{4\lambda}{2\lambda+1}(2\lambda+1-\alpha)} \int\limits_0^\infty \frac{A_{\operatorname{ch} t}(|f|\,\omega^\alpha)(\operatorname{ch} x)\operatorname{sh}^{2\lambda} t\, dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}$$
$$\le \left(\operatorname{sh}\frac{r}{2}\right)^{\frac{4\lambda}{2\lambda+1}(2\lambda+1-\alpha)} I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x) \sum_{k=0}^\infty 2^{(1-k)\frac{4\lambda}{2\lambda+1}(2\lambda+1-\alpha)}$$
$$\lesssim \left(\operatorname{sh}\frac{r}{2}\right)^{\frac{4\lambda}{2\lambda+1}(2\lambda+1-\alpha)} I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x). \tag{4.23}$$

Applying Lemma 2.1, for $2 \le r < \infty$ and (4.23), we obtain

$$M_G^\alpha (f\omega^\alpha)(\operatorname{ch} x) \lesssim \sup_{r>0} \left(\operatorname{sh}\frac{r}{2}\right)^{4\lambda\left(\frac{\alpha}{2\lambda+1}-1\right)} \left(\operatorname{sh}\frac{r}{2}\right)^{4\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)} I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x)$$
$$\lesssim I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x). \tag{4.24}$$

Inequality (4.20) follows from inequalities (4.22) and (4.24). □

Theorem 4.5. *Let $q = \frac{2\lambda+1}{2\lambda+1-\alpha}$. Then the following two conditions are equivalent:*

(i) $$\int\limits_{\left\{x\in\mathbb{R}:\ I_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)>\beta\right\}} \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\, dx \le C\left(\frac{1}{\beta}\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|\,\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\, dx\right)^q$$

with a constant C, independent of f and $\lambda > 0$,

(ii) $\omega \in A_1^\lambda(\mathbb{R}_+)$.

The assertion of the Theorem follows from inequality (4.20) and Theorem 4.2.

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References

1. D. Adams, A note on Riesz potentials. *Duke Math. J.* **42** (1975), no. 4, 765–778.
2. I. A. Aliev, A. D. Gadjiev, Weighted estimates for multidimensional singular integrals generated by a generalized shift operator. (Russian) *Russian Acad. Sci. Sb. Math.* **77** (1994), no. 1, 37–55.
3. O. V. Besov, V. P. Il'in, S. M. Nikol'skiĭ, *Integral Representations of Functions, and Embedding Theorems*. Second edition. (Russian) Fizmatlit Nauka, Moscow, 1996, 480 pp.
4. A. P. Calderon, Inequalities for the maximal function relative to a metric. *Studia Math.* **57** (1976), no. 3, 297–306.
5. R. R. Coifman, C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* **51** (1974), 241–250.

6. Y. Ding, S. Z. Lu, Weighted norm inequalities for fractional integral operator with rough kernel. Canad. J. Math. 50 (1998), 29–39.
7. L. Durand, P. M. Fishbane, L. M. Simmons, Expansion formulas and addition theorems for Gegenbauer functions. *J. Mathematical Phys.* **17** (1976), no. 11, 1933–1948.
8. C. Fefferman, E. Stein, Some maximal inequalities. *Amer. J. Math.* **93** (1971), 107–115.
9. A. I. Gadziyev, On fractional maximal functions and fractional integrals, generated by Bessel differential operators. *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **20** (2000), no. 4, Math. Mech., 52–63, 265.
10. I. S. Gradshteyn, I. M. Rijik, *Tables of Integrals, Sums, Series and Products.* Nanka, Moscow, 1971.
11. V. S. Guliev, On maximal function and fractional integral, associated with the Bessel differential operator. *Math. Inequal. Appl.* **6** (2003), no. 2, 317–330.
12. E. V. Guliyev, Weighted inequality for fractional maximal functions and fractional integrals, associated with the Laplace-Bessel differential operator. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **26** (2006), no. 1, Math. Mech., 71–80.
13. V. S. Guliyev, E. J. Ibrahimov, S. Ar. Jafarova, Gegenbauer harmonic analysis and approximation of functions on the half line. *Advances in Analysis* **2** (2017), no. 3, 167–195.
14. G. H. Hardy, J. E. Littlewood, Some properties of fractional integrals. I. *Math. Z.* **27** (1928), no. 1, 565–606.
15. E. J. Ibrahimov, A. Akbulut, The Hardy-Littlewood-Sobolev theorem for Riesz potential generated by Gegenbauer operator. *Trans. A. Razmadze Math. Inst.* **170** (2016), no. 2, 166–199.
16. I. A. Kipriyanov, M. N. Ključancev, Estimates of a surface potential which is generated by a generalized shift operator. (Russian) *Dokl. Akad. Nauk SSSR* **188** (1969), 997–1000.
17. I. A. Kipriyanov, M. N. Ključancev, The boundedness of a certain class of singular integral operators. (Russian) *Dokl. Akad. Nauk SSSR* **186** (1969), 1253–1255.
18. I. A. Kipriyanov, M. I. Ključancev, Singular integrals that are generated by a generalized shift operator. II. (Russian) *Sibirsk. Mat. Ž.* **11** (1970) 1060–1083, 1196–1197.
19. M. N. Ključancev, ingular integrals that are generated by a generalized shift operator. I. (Russian) *Sibirsk. Mat. Ž* **11** (1970)m 810–821, 956–957.
20. V. M. Kokilashvili, A. Kufner, Fractional integrals on spaces of homogeneous type. *Comment. Math. Univ. Carolin.* **30** (1989), no. 3, 511–523.
21. V. M. Kokilashvili, S. Samko, Singular integrals in weighted Lebesgue spaces with variable exponent. *Georgian Math. J.* **10** (2003), no. 1, 145–156.
22. B. M. Levitan, Expansion in Fourier series and integrals with Bessel functions. (Russian) *Uspehi Matem. Nauk (N.S.)* **6** (1951). no. 2(42), 102–143.
23. B. M. Levitan, *Theory of Generalized Shift Operators.* (Russian) Izdat. Nauka, Moscow, 1973, 312 pp.
24. L. Lindelöf, *Le Calcul Des Résidus Et Ses Applications à la Théorie des Fonctions.* vol. 8. Gauthier-Villars, 1905.
25. L. N. Lyakhov, Inversion of the B-Riesz potentials. (Russian) *Dokl. Akad. Nauk SSSR* **321** (1991), no. 3. 466–469.
26. L. N. Lyakhov, Spaces of Riesz B-potentials. (Russian) *translated from Dokl. Akad. Nauk* **334** (1994), no. 3, 278–280 *Russian Acad. Sci. Dokl. Math.* **49** (1994), no. 1, 83–87
27. R. A. Macias, C. Segovia, *A Well Behaved Quasi-Distance for Spaces of Homogeneous Type.* vol. 32 of Trabajos de Matemática. Inst. Argentino Mat. 1981.
28. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165** (1972), 207–226.
29. B. Muckenhoupt, R. L. Wheeden, Weighted norm inequalities for fractional integrals. *Trans. Amer. Math. Soc.* **192** (1974), 261–274.
30. G. Welland, Weighted norm inequalities for fractional integrals. *Proc. Amer. Math. Soc.* **51** (1975), 143–148.
31. S. Yu, Y. Ding, D. Yan, *Singular Integrals and Related Topics.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
32. A. Zygmund, *Trigonometric Series.* (Russian) Izdat. Mir, Moscow, 1965 vol. I: 615 pp.; vol. II: 537 pp.

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