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A BRIEF NOTE ON QUATERNION ANALYSIS

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Abstract. This paper presents a brief historical survey of quaternion functions of a quaternion variable.

It has long been known that holomorphic (analytic) functions of one complex variable have numerous applications in solving various important problems of natural sciences.

Because of these applications and an active mathematical interest shown in them there arose the problem whether there exists or not an analogous theory for functions depending on three and more real variables.

This problem for functions with four real variables x_0, x_1, x_2, x_3 was studied for over a decade by William Rowan Hamilton (1805-1865). In 1843, Hamilton introduced into consideration the quaternions $z = x_0 i_0 + x_1 i_1 + x_2 t_2 + x_3 i_3$ with the real unit $i_0 = 1$ and three imaginary units i_1, i_2, i_3, having the properties $i_1^2 = i_2^2 = i_3^2 = -1$, $i_1 i_2 = -i_2 i_1 = i_3$, $i_2 i_3 = -i_3 i_2 = i_1$, $i_3 i_1 = -i_1 i_3 = i_2$.

Therefore the product of quaternions depends on the order of succession of cofactors, i.e. the product does not possess the property of commutativity. That is why two equations

$$q_2 x = q_1 \tag{1}$$

and

$$x q_2 = q_1 \tag{2}$$

are considered for quaternions. The solution of equation (1) is called the left quotient of division of q_1 by q_2 and here we denote it by the symbol $\vdash\!\frac{q_1}{q_2}$ (the numerator is not allowed to move to the left). Analogously, the right quotient of division of q_1 by q_2 is called the solution of equation (2) denoted by the symbol $\frac{q_1}{q_2}\!\dashv$ (the numerator is not allowed to move to the right). For $q_1 = 1$ we see that each $q_2 \neq 0$ has an inverse quaternion $\frac{\overline{q}_2}{|q_2|^2}$, $|q| = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{1/2}$, $\overline{q} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3$. Hence we have

$$\frac{q_1}{q_2}\!\dashv = q_1 \cdot \frac{1}{q_2} \tag{3}$$

and

$$\vdash\!\frac{q_1}{q_2} = \frac{1}{q_2} \cdot q_1. \tag{4}$$

It should be said that in 1837 Janos Bolyai (1802–1860) wrote a treatise on the theory of imaginary values which he sent to Leipzig University to participate in the announced students' competition. Unfortunately the jury adopted a negative decision.

In the middle of the 19th century, Hamilton set the problem: for quaternions functions of a quaternion variable to develop a theory analogous to the theory of holomorphic functions of one complex variable.

As to this problem, we know that several not quite successful attempts were undertaken to solve it. The problem was considered for the first time by A. N. Kryloff (1863–1945) [6] who, using equalities (3) and (4), introduced into consideration the right and the left derivative at the point z for the function f by means of the equalities

$$A(z) = \lim_{h \to 0} [f(z+h) - f(z)] \cdot \frac{1}{h}, \quad B(z) = \lim_{h \to 0} \frac{1}{h} \cdot [f(z+h) - f(z)]$$

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respectively. It turned out that [7], $A(z)$ exists only for functions of the form $az+b$, $B(z)$ exists in the case of functions of the form $za+b$ and $A(z)=B(z)$ only for functions of the form $rz+b$ where the quaternions a and b are constant with respect to z, and r is a real number. Hence the derivative method is not suitable for quaternion functions. An analogous opinion exists about the polynomial method [1,5]. This perhaps led G. E. Shilov to make the following statement: "The dream of Hamilton was to create the theory of functions of a quaternion variable. However the hopes set on quaternions did not come true" – G. E. Shilov, Mathematical Analysis-Functions of Several Real Variables, Parts 1 and 2, Nauka, Moscow, 1972, p. 385.

In 1985, D. Solomentsev, the head of the mathematical analysis sector of the Mathematical Synopses Journal, advised me not to get involved in the study of Hamilton's problem because no serious reviewer could be found not in any country of the world.

Following this advice, for several years my research had been focused on problems of the existence of bihedral angular limits of partial derivatives of the spherical Poisson integral and on finding a necessary and sufficient condition for the existence of the total differential of a function of many variables (the results appeared in press).

However later I renewed the work on Hamilton's problem and finished it in a certain sense in 2009. The results obtained by me on the existence of the operation of differentiation for quaternion functions of a quaternion variable were submitted for publication in the form of a paper to an American mathematical journal. Two years after I received from the editor of that journal an enthusiastic letter with the attached reviewer's complimentary report. The editor advised me to get acquainted with the work of two Italian mathematicians and asked to inform him of my opinion. The results of the Italian colleagues turned out to be the newly obtained findings of A. N. Kryloff's results. I sent this information to the editor and soon from the new editor of the same journal I received a refusal to publish my paper.

I was surprised by this fact and told about it to Academician Hverdri Inassaridze who in his student years made a report on quaternion numbers at a session of the circle under the mechanical-and-mathematical department of Tbilisi State University and who advised me to submit my paper to Tbilisi Mathematical Journal of which he was the editor-in-chief. The paper was published [1] and presently I also have the survey paper [5] published in the Journal of Mathematical Sciences.

In these works, for quaternion functions of a quaternion variable z the notion of a H-derivative is introduced (in honor of Hamilton) and the following results are established:

1) formulas $(z^n)'=nz^{n-1}$, $(e^z)'=e^z$, $(\sin z)'=\cos z$, $(\cos z)'=-\sin z$;

2) rules of H-differentiation $(f+\varphi)'=f'+\varphi'$, $(f\cdot\varphi)'=f'\cdot\varphi+f\cdot\varphi'$, $\left(\frac{1}{\varphi}\right)'=-\frac{1}{\varphi}\cdot\varphi'\cdot\frac{1}{\varphi}$, $\left(\frac{f}{\varphi}\dashv\right)'=f'\cdot\frac{1}{\varphi}-f\cdot\frac{1}{\varphi}\cdot\varphi'\cdot\frac{1}{\varphi}$, $\left(\vdash\frac{f}{\varphi}\right)'=-\frac{1}{\varphi}\cdot\varphi'\cdot\frac{1}{\varphi}\cdot f+\frac{1}{\varphi}\cdot f'$. It should be noted that the equality for $\left(\frac{1}{\varphi}\right)'$ was established by me only after establishing the equality $\frac{1}{z_1}(z_1-z_2)\frac{1}{z_2}=\frac{1}{z_2}(z_1-z_2)\frac{1}{z_1}$ which, after simplification, yields the equality $\frac{1}{z_2}-\frac{1}{z_1}=\frac{1}{z_2}-\frac{1}{z_1}$. In addition to these results:

3) the necessary and sufficient condition for the existence of a H-derivative is obtained;

4) for a $\mathbb{C}^2$-holomorphic in the domain D quaternion function, its integral representation and its expansion into a power series with respect to two complex variables are established;

5) A. Moivre's (1667–1754) formula is obtained by using the imaginary unit I_z with the property $I_z^2=-1$ that depends on the variable z. It should be noted that F. G. Frobenius (1849–1917) established that for functions of three real variables there exists no theory analogous to the theory of holomorphic functions of one complex variable.

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