Order theoretic correspondence for Intuitionistic mu-calculus

Sumit Sourabh
(joint work with Alessandra Palmigiano)

ILLC, Universiteit van Amsterdam

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Outline

1. A brief introduction to correspondence theory

2. Order theoretic correspondence for IML
   - Sahlqvist (and Inductive formulas in IML)
   - Ackermann lemma based algorithm

3. Extending the approach to Intuitionistic mu-calculus
   - Extend Algorithmic approach to fixed points
   - Defining Sahlqvist and Inductive formulas

Introduction to Correspondence theory

- Correspondence theory
  - gives (sufficient) syntactic conditions on modal formulas to have a first order correspondent
  - These formulas generate logics that are strongly complete w.r.t. first-order definable classes of frames.
  - Sahlqvist formulas are canonical.
- Sahlqvist formulas are classically defined as $\varphi \rightarrow \psi$, such that

$$\varphi = \bot \mid \top \mid \Box^n p \mid \eta \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \Diamond\varphi$$

where $n \geq 0$, η is a negative formula and ψ is a positive formula.

  - Eg. of sahlqvist formulas are the std. modal logic axioms T, D, S4 . . .
  - Eg. of non-sahlqvist formulas are
    $\Box\Diamond p \rightarrow \Diamond\Box p$ (McKinsey formula)
    $\Box p(\Box p \rightarrow p) \rightarrow \Box p$ ($= \Box(\neg\Box p \vee p) \rightarrow \Box p$) (Löb formula)

Motivation

- For classical logics 'no choice node in scope of a universal node'

Choice	Universal
$\vee$	
$\Diamond$	$\Box$

- Order theoretically,
  - Choice = Not a right adjoint
  - Universal = Not a left residual
- In case of IML, we discover an asymmetry in the classification. (For eg., $\rightarrow$ is neither a left residual nor a right adjoint.)

  IML is much more instructive in this respect and thus further supports the need of a positive classification, based on algebraic and order theoretic properties.

Adjunction and Residuation: Examples

- We have,

$\Diamond$ as SLA	$\Box$ as SRA
$\Diamond x \leq y$ iff $x \leq (\Diamond)^{-1} y$	$x \leq \Box y$ iff $(\Box)^{-1} x \leq \Box y$
$\Diamond x \leq y$ iff $x \leq \blacksquare y$	$x \leq \Box y$ iff $\blacklozenge\ x \leq \Box y$

- Also,

$\wedge$ as SLR	$x \wedge y \leq z$ iff $x \leq y \to z$
$\wedge$ as SRA	$x \leq y \wedge z \Leftrightarrow \Delta(x) \leq^{\times} \wedge(y, z)$

- We can classify the connectives in the IML signature

SRA	SLR	SRR
$+ \wedge$	$+ \wedge$	$+ \wedge$
	$+ \vee$	$+ \vee$
$+ \Box$	$+ \Diamond$	$+ \Box$
		$+ \to$

Order type and signed generation trees

- An order type is a vector $\epsilon \in \{1, \partial\}^n$, for some n.
  An *ϵ-critical node* in the signed generation tree of s is a leaf node $+p_i$ with $\epsilon_i = 1$ or $-p_i$ with $\epsilon_i = \partial$.
  An *ϵ-critical branch* in the tree is a branch terminating in an ϵ-critical node.

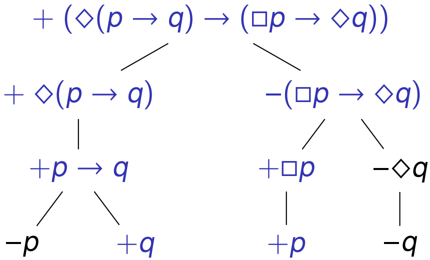

Figure: positive signed generation tree for FS axiom

ϵ-crtical brances for $(\epsilon_p = 1, \epsilon_q = 1)$

Sahlqvist formulas for IML, positive version

Sahlqvist formula

Given an order type ϵ on $s(p_1, \ldots, p_n)$ is *ϵ-sahlqvist* if

- for every ϵ-critical branch of its signed generation tree, if we traverse that branch from the leaf node to the root s, we first encounter the (possibly zero) SRA nodes, but
- as soon as we encounter one node which is not an SRA, then this node has to be SLR, and from that point on up to the root, all the nodes must be SLR.

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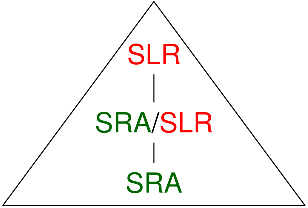

Figure: ϵ-critical branch

Sahlqvist formulas

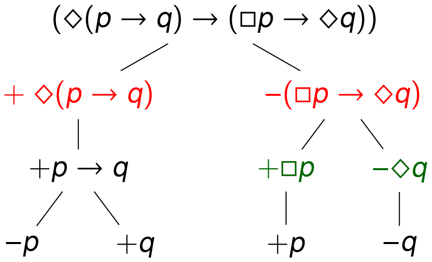

Figure: $\epsilon-$ sahlqvist for $(\epsilon_p = 1, \epsilon_q = \partial)$

McKinsey axiom

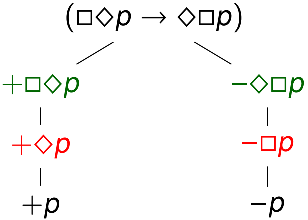

Figure: McKinsey axiom is not Sahlqvist for any ϵ

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  Use the approximation and residuation rules (due to order theoretic properties of connectives) to reduce the inequality to an Ackermann form.

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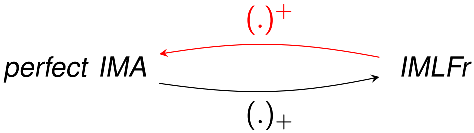
- Step 3: *Variable elimination*
  Use the Ackermann lemma to eliminate the propositional variables.

Step 1 : Preprocessing and first approximation

Theorem

For every IML frame $\mathcal{F}$, $\mathcal{F}^+$ is a perfect IMA.



- In setting of perfect IMA, we introduce nominals and co-nominals which range over $J^\infty(\mathbb{C})$ and $M^\infty(\mathbb{C})$ resp.
- We use the following equivalence for the first approximation

$$\mathcal{F}, V, w \models \varphi \ \text{ iff } \ \mathcal{F}^+, V'^+ \models \mathbf{i} \leq \varphi \ \text{ iff } \ \mathcal{F}^+, V'^+ \models \varphi \leq \mathbf{m}$$

$$\frac{\varphi_i \to \psi_i}{\mathbf{i}_0 \leq \varphi_i \quad \psi_i \leq \mathbf{m}_0}$$

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  $\Leftrightarrow \quad \mathbf{i} \leq \Diamond \bigvee\{a \in J^{\infty} | a \leq \alpha\}$ (perfect lattice env.)

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$$\begin{array}{lll} \text{Suppose} & \mathbf{i} \leq \Diamond\alpha & \\ \Leftrightarrow & \mathbf{i} \leq \Diamond \bigvee\{a \in J^{\infty} | a \leq \alpha\} & \text{(perfect lattice env.)} \\ \Leftrightarrow & \mathbf{i} \leq \bigvee\{\Diamond a \in J^{\infty} | a \leq \alpha\} & (\Diamond \text{ is join preserving}) \end{array}$$

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\Leftrightarrow & \mathbf{j} \leq \alpha \quad \mathbf{i} \leq \Diamond\mathbf{j} & (\mathbf{i} \text{ is join-prime})
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- $\dfrac{\mathbf{i} \leq \Diamond\alpha}{\mathbf{j} \leq \alpha \quad \mathbf{i} \leq \Diamond \mathbf{j}}$
- $\dfrac{\Box\alpha \leq \mathbf{m}}{\alpha \leq \mathbf{n} \quad \Box\mathbf{n} \leq \mathbf{m}}$
- $\dfrac{\alpha \rightarrow \beta \leq \mathbf{m}}{\mathbf{j} \leq \alpha \quad \mathbf{j} \rightarrow \beta \leq \mathbf{m}}$
- $\dfrac{\alpha \rightarrow \beta \leq \mathbf{m}}{\beta \leq \mathbf{n} \quad \alpha \rightarrow \mathbf{n} \leq \mathbf{m}}$

Approximation and Residuation

- The operations being left or right adjoints or residuals provides us with invertible rules

$\wedge$	$\vee$	$\Diamond$	$\Box$
$\rightarrow$	$-$	$\blacksquare$	$\blacklozenge$

- $\dfrac{\alpha \wedge \beta \leq \gamma}{\alpha \leq \beta \rightarrow \gamma}$
- $\dfrac{\Diamond\alpha \leq \beta}{\alpha \leq \blacksquare\beta}$
- $\dfrac{\alpha \leq \beta \vee \gamma}{\alpha - \beta \leq \gamma}$
- $\dfrac{\alpha \leq \Box\beta}{\blacklozenge\ \alpha \leq \beta}$

Step 3: Elimination of Propositional variables

- In the algebraic setting, ackermann lemma generalizes minimal valuation

Right Ackermann Lemma

Let $p \notin \mathrm{Prop}(\alpha)$, β be positive in p and γ be negative in p. Tfae

1. $\mathcal{F}, \Vdash (\alpha \leq p \wedge \ \beta(p) \leq \gamma(p))$,
2. $\mathcal{F} \Vdash \beta(\alpha/p) \leq \gamma(\alpha/p)$

- The requirement $p \neq Prop(\alpha)$ can be relaxed for the Recursive ackermann lemma.
  We replace $\alpha \leq p$ and $\beta(p) \leq \gamma(p)$ by $\beta(\mu p.\alpha(p)/p) \leq \gamma(\mu p.\alpha(p)/p)$

Transitivity

Consider $\forall p[\Diamond\Diamond p \leq \Diamond p]$

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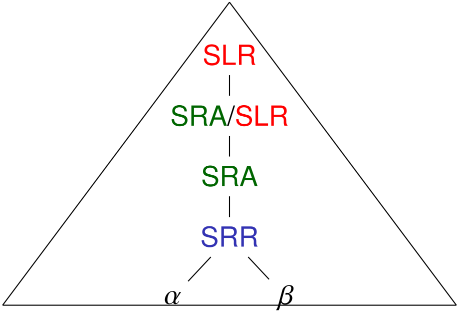
$$\forall \mathbf{m}[\Diamond\Diamond\blacksquare \mathbf{m} \leq \mathbf{m}]$$

$$\forall w[R_\Diamond[R_\Diamond[w]] \subseteq R_\Diamond[w]]. \quad \textit{(Valuation for } \boldsymbol{m}\textit{)}$$

Inductive formulas

Expand the definition of ϵ-sahlqvist formulas by

- introducing an irreflexive, transitive ordering Ω on prop. variables occurring in the formula
- allow for binary SRR in scope of SRA nodes on ϵ-critical branch under certain conditions .



For every p_j occurring in α we have $p_j \leq_\Omega p_i$, where β is the ϵ-critical branch.

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Theorem (Correctness of ALBA)

If ALBA succeeds in eliminating all the propositional variables from a formula φ, the first order formula returned is locally equivalent on IML frames to φ.

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Theorem (Scope of ALBA)

ALBA successfully reduces all IML inductive (and hence Sahlqvist) formulas.

Intuitionisitic mu-calculus

- The approach for IML can be extended to IML + fixed points as well.
  - We need to define rules (approximation-type & residuation/adjunction-type) to reach Ackermann shape.
  - Define sahlqvist and inductive formulas for Intuitionisitic mu-calculus.
- The semantic interpretation of fixpoint binders does not have any intrinsic order theoretic properties; Therefore, we need to make the rules contextual.

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$\{\mu X. \blacklozenge\, (X \vee \mathbf{i}) \leq \mathbf{m}\}$ (recursive Ackermann lemma)

Sound approximation rules for fixpoint binders

Consider the rules:

$$\frac{\mathbf{i} \leq \mu X.\varphi(X, \psi/!x)}{\exists \mathbf{j}[\mathbf{i} \leq \mu X.\varphi(X, \mathbf{j}/!x)\ \&\ \mathbf{j} \leq \psi]}\ (\mu\text{-A})$$

$$\frac{\nu X.\varphi(X, \psi/!x) \leq \mathbf{m}}{\exists \mathbf{n}[\nu X.\varphi(X, \mathbf{n}/!x) \leq \mathbf{m}\ \&\ \psi \leq \mathbf{n}]}\ (\nu\text{-A})$$

On the left, $[\varphi]$ is completely $\bigvee$-preserving in x (resp. completely $\bigwedge$-preserving in x on the right). In both rules the variable $x \in Var$ is assumed to not occur in ψ.

Adjucntion rules for fixed points

$$\frac{\mu X.(A(X) \vee B(p)) \leq \chi}{p \leq \nu X.(E(X) \wedge D(\chi/p))} \ (\mu\text{-Adj}) \qquad \frac{\chi \leq \nu X.(E(X) \wedge D(p))}{\mu X.(A(X) \vee B(\chi/p)) \leq p} \ (\nu\text{-Adj})$$

where, in each rule,

$$\begin{array}{ll} A(X) = \bigvee_{i \in I} \delta_i(X) & B(p) = \bigvee_{j \in J} \delta'_j(p) \\ E(X) = \bigwedge_{i \in I} \beta_i(X) & D(p) = \bigwedge_{j \in J} \beta'_j(p) \end{array}$$

- δ_i, δ'_j are unary left adjoints
- β_i, β'_j are unary right adjoints
- $\delta_i \dashv \beta_i$ and $\delta'_j \dashv \beta'_j$.

Nested fixed points

Lemma

In case of nested fixed points,

$$\mu X.\varphi(A_1(X) \vee B_1(\mu Y.[A_2(Y) \vee B_2(p) \vee B_3(X)])/!x, \overrightarrow{z}) = \bigvee_{\lambda \in Ord} A'^{\lambda}(B'(p))$$

for some left residuals $A'(x, \overrightarrow{z})$ *and* $B'(p, \overrightarrow{z})$

Proof.

Use

$$\mu X.(A(X) \vee B) = \bigvee_{k \in Ord} A^k(B)$$

□

Sahlqvist formulas for modal mu-calculus

- In a recent paper [vBBH12] the authors define the Sahlvist formulas for modal mu calculus using PIA and Skeleton formulas.
- *Positive implies atomic* (PIA) formulas $\iff$ minimal valuation is guaranteed.

$$\forall y(\psi(P, x, y) \rightarrow Py)$$

- Skeleton supports PIA and negative formulas
- We can generalize the definition to inductive formulas using our approach.

SLR, SRA, Skeleton & PIA nodes

SLR	SRA	Skeleton	PIA
			$+ \quad \land$
$+ \quad \lor$		$+ \quad \lor$	$+ \quad \Box$
$+ \quad \land$	$+ \quad \land$	$+ \quad \land$	$+ \quad \lor$
$+ \quad \Diamond$	$+ \quad \Box$	$+ \quad \Diamond$	$+ \quad \rightarrow$
		$+ \quad \mu X$	$+ \quad \nu X$
—	—	—	—
$- \quad \lor$	$- \quad \lor$	$- \quad \lor$	$- \quad \lor$
$- \quad \land$	$- \quad \Diamond$	$- \quad \land$	$- \quad \Diamond$
$- \quad \Box$		$- \quad \Box$	$- \quad \land$
$- \quad \rightarrow$		$- \quad \rightarrow$	$- \quad \mu X$
		$- \quad \nu X$	

Table: SLR, SRA, Skeleton and PIA nodes.

Recursive inequalities and the enhanced ALBA

The intuitive idea of the ϵ-recursive shape $\varphi \leq \psi$ is that, on either side of the inequality, it consists of *three types of ingredients*.

- The first ingredient is an outer, approximation-friendly (exo)*skeleton* $\varphi'(!x_1, \ldots, !x_n) \leq \psi'(!y_1, \ldots, !y_m)$;
- ϵ^∂-*formulas* γ (i.e. formulas such that either $\epsilon^\partial(\gamma) \prec +\varphi$ or $\epsilon^\partial(\gamma) \prec -\psi$),
- *PIA-formulas* β.

Claim:

- the skeleton shape guarantees that the approximation rules are applicable
- the PIA shape guarantees that the adjunction/residuation rules and rewriting procedure are applicable

Reduction

Consider the initial clause

$$\forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \varphi'(\overrightarrow{\gamma}, \overrightarrow{\beta}) \ \& \ \psi'(\overrightarrow{\gamma}, \overrightarrow{\beta}) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}];$$

Reduction

Consider the initial clause

$$\forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \varphi'(\overrightarrow{\gamma}, \overrightarrow{\beta}) \ \& \ \psi'(\overrightarrow{\gamma}, \overrightarrow{\beta}) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}];$$

the skeleton shape guarantees that the approximation rules are applicable, using which, the initial clause can be equivalently rewritten as

$$\forall \mathbf{i} \forall \mathbf{m} \forall \overrightarrow{\mathbf{j}}[(\overrightarrow{\mathbf{j}} \leq \overrightarrow{\beta} \ \& \ \mathbf{i} \leq \varphi'(\overrightarrow{\gamma}, \overrightarrow{\mathbf{j}}) \ \& \ \psi'(\ldots) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}];$$

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Using the adjunction rules, each inequality $\mathbf{j} \leq \beta_i$ can be equivalently rewritten into $\alpha_i \leq p$;

$$\forall \mathbf{i} \forall \mathbf{m} \forall \overrightarrow{\mathbf{j}}[(\bigvee_i \alpha_i \leq p \;\&\; \mathbf{i} \leq \varphi'(\overrightarrow{\gamma}, \overrightarrow{\mathbf{j}}) \;\&\; \psi'(\ldots) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}].$$

The fact that the γ's are ϵ^∂ guarantees that the Ackermann's rule can be applied, which eliminates p.

Future work

- In particular, the adjunction/residuation rules we have need to be extended so as to be able to solve for all the critical variables at once;
- Characterising syntactic shape of inequalities which guarantee the solvability through applications of the non-recursive Ackermann.
- Extend these results to fixpoint expansions of logics with a nondistributive lattice base.

Thank you

Questions?