A Brief Introduction to Topological Semantics for Modal Logic

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Why Modal Logic

Simplicity

Simple extension of classical propositional logic by

$$\Box \text{ and } \Diamond$$

Expressivity: many interpretations of $\Box/\Diamond$

1. Necessity/Possibility
2. Obligation/Permission
3. Kripke or relational frames
4. Descriptive frames
5. Algebraic
6. Topological

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Modal Logic

Language and Well Formed Formulas

Symbols

1. Propositional variables/letters: $\mathfrak{Var} = \{p_0,\ p_1,\ p_2,\ \dots\}$
2. Logical connectives: $\top,\ \bot,\ \neg,\ \Diamond,\ \Box,\ \wedge,\ \vee,\ \rightarrow$
3. Punctuation: (and)

WFFs

1. Propositional letters and $\top$ and $\bot$
2. If φ, ψ are WFF then so are
   $(\neg\varphi),\ (\Diamond\varphi),\ (\Box\varphi)$
   $(\varphi \wedge \psi),\ (\varphi \vee \psi),\ (\varphi \rightarrow \psi)$

Note: Drop parenthesis–Unary connectives bind closer than binary; e.g. write $\Box p \rightarrow p$ for $((\Box p) \rightarrow p)$

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Modal Logic

Basic Axioms and Inference Rules

Definition

L is a (normal) modal logic if **L** contains:

1. Classical tautologies: e.g. $p \vee \neg p$ and $p \rightarrow (q \rightarrow p)$
2. $\mathbf{K} = \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
3. $\Box p \leftrightarrow \neg\Diamond\neg p$

and **L** is closed under the inference rules:

1. Modus Ponens (MP)
2. Substitution (Sub)
3. $\Box$-necessitation (N)

$$(\mathrm{MP})\ \frac{\varphi,\ \varphi \rightarrow \psi}{\psi} \qquad (\mathrm{SUB})\ \frac{\varphi(p)}{\varphi(\psi)} \qquad (\mathrm{N})\ \frac{\varphi}{\Box\varphi}$$

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Some Well-known Modal Logics

1. **K** least modal logic: Logic of all frames
2. $\mathbf{K4} = \mathbf{K} + \Diamond\Diamond p \to \Diamond p$
   logic of transitive ($\forall w \forall v \forall u \ wRv \ \& \ vRu \Rightarrow wRu$) frames
3. $\mathbf{S4} = \mathbf{K4} + p \to \Diamond p$
   logic of reflexive ($\forall w \ wRw$) and transitive frames (a.k.a. quasi-order or preorder)
4. $\mathbf{K4D} = \mathbf{K4} + \Diamond\top$
   logic of serial ($\forall w \exists v \ wRv$) transitive frames

Equivalent Formulas

- $\Box(p \to q) \to (\Box p \to \Box q)$
  $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$
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Equivalent Formulas

- $\Box(p \to q) \to (\Box p \to \Box q)$
  $\Box(p \land q) \leftrightarrow (\Box p \land \Box q)$
  $\Diamond(p \lor q) \leftrightarrow (\Diamond p \lor \Diamond q)$
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Topological Spaces

Definition

Call (X, τ) a topological space if

$\tau \subseteq \mathcal{P}(X)$

1. $X, \varnothing \in \tau$
2. $U, V \in \tau \Rightarrow U \cap V \in \tau$
3. $U_i \in \tau \Rightarrow \bigcup_{i \in I} U_i \in \tau$ for any indexing set I

U open: $U \in \tau$

C closed: $X - C \in \tau$

Recall

For $A \subseteq X$ there are

Interior: The greatest open set contained in A, $int(A)$

Closure: The least closed set containing A, $\overline{A}$

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Properties of Interior and Closure

Set operations

$$\begin{aligned} int(A) &= \bigcup\{U \in \tau : U \subseteq A\} \\ \overline{A} &= \bigcap\{C : A \subseteq C \text{ and } X - C \in \tau\} \end{aligned}$$

Characterization

$$\begin{aligned} x \in int(A) \quad &\text{iff} \quad \exists U \in \tau,\ x \in U \text{ and } \forall y \in U,\ y \in A \\ x \in \overline{A} \quad &\text{iff} \quad \forall U \in \tau,\ x \in U \Rightarrow \exists y \in U,\ y \in A \end{aligned}$$

$$\begin{aligned} int(A) &= X - \overline{X - A} \\ \overline{A} &= X - int(X - A) \end{aligned}$$

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More Properties of Interior and Closure

Set Inclusions

$$
\begin{array}{rcl}
int(A) & \subseteq & A \\
X & \subseteq & int(X) \\
int(A) & \subseteq & int(int(A)) \\
int(A \cap B) & = & int(A) \cap int(B)
\end{array}
\qquad
\begin{array}{rcl}
A & \subseteq & \overline{A} \\
\overline{\varnothing} & \subseteq & \varnothing \\
\overline{\overline{A}} & \subseteq & \overline{A} \\
\overline{A \cup B} & = & \overline{A} \cup \overline{B}
\end{array}
$$

Diamond as Closure
I

Valuations

A function $\nu : \mathfrak{Var} \to \mathcal{P}(X)$ is a valuation

Intuitively, ν indicates where each WFF, φ is true

Formally, for $x \in X$

$$
\begin{array}{llll}
 & x \models p & \text{iff} & x \in \nu(p) \\
 & x \models \neg\varphi & \text{iff} & x \not\models \varphi \\
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 & x \models \Box\varphi & \text{iff} & \exists U \in \tau,\ x \in U \text{ and } \forall y \in U,\ y \models \varphi \\
\text{Hence,} & x \models \Diamond\varphi & \text{iff} & \forall U \in \tau,\ x \in U \Rightarrow \exists y \in U,\ y \models \varphi
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Diamond as Closure II

Observation: φ defines a subset of X

Given ν,
Let $||\varphi|| = \{x \in X : x \models \varphi\}$. Then

$$||\Diamond\varphi|| = \overline{||\varphi||}$$
$$||\Box\varphi|| = int(||\varphi||)$$

Valid Formulas

1. φ is valid in X provided $\forall\nu$, $\forall x \in X$, $x \models \varphi$; write $X \models \varphi$
   Equivalently $||\varphi|| = X$ for each ν
2. For a class of spaces $\mathcal{C}$, $\mathbf{L}_c(\mathcal{C}) = \{\varphi : \forall X \in \mathcal{C},\ X \models \varphi\}$ is a modal logic (Exercise)
3. $\mathbf{L}_c(X) = \{\varphi : X \models \varphi\}$ in case $\mathcal{C}$ is only one space.

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Expressivity and A Basic Result

Formulas and Properties

$p \to \Diamond p$	$A \subseteq \overline{A}$
$\Diamond\bot \to \bot$	$\overline{\varnothing} \subseteq \varnothing$
$\Diamond\Diamond p \to \Diamond p$	$\overline{\overline{A}} \subseteq \overline{A}$
$\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$	$\overline{A \cup B} = \overline{A} \cup \overline{B}$

Theorem

Let **Top** be class of all topological spaces, $\mathbf{L}_c(\mathbf{Top}) = \mathbf{S4}$.

$$\mathbf{L}_c(\mathbf{Top}) \supseteq \mathbf{S4} \qquad \text{(Sound)}$$

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Specialization Order and **S4**-frames I

Definition

Let $(X, \tau) \in \mathbf{Top}$
Put $xR_\tau y$ iff $x \in \overline{\{y\}}$
Call R_τ the specialization order on X (generated by τ)

Basic Results (exercises)

1. R_τ is a quasi-order
2. If τ is T_1 (points are closed) then $R_\tau = \{(x, x) : x \in X\}$

Examples: τ to R_τ

1. Two point spaces: trivial, Sierpinski, discrete ($(R_\tau)^{-1}$ is closure)
2. Real line $\mathbb{R}$ ($(R_\tau)^{-1}$ is not closure)

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Specialization Order and S4-frames II

Definition

Let (W, R) be a quasi-order (reflexive and transitive)
Call $U \subseteq W$ an R-upset if $w \in U$ & $wRv \Rightarrow v \in U$
Let $\tau_R = \{U \subseteq W : U \text{ is } R\text{-upset}\}$
Call τ_R the Alexandrov topology on W (generated by R)

Examples: R to τ_R

1. Two point frames: cluster, chain, anti-chain (closure is R^{-1})
2. Two Fork (closure is R^{-1})
3. $(\mathbb{R}, \leq)$ (closure is $\leq^{-1}$)

Specialization Order and S4-frames II

Definition

Let (W, R) be a quasi-order (reflexive and transitive)
Call $U \subseteq W$ an R-upset if $w \in U$ & $wRv \Rightarrow v \in U$
Let $\tau_R = \{U \subseteq W : U \text{ is } R\text{-upset}\}$
Call τ_R the Alexandrov topology on W (generated by R)

Examples: R to τ_R

1. Two point frames: cluster, chain, anti-chain (closure is R^{-1})
2. Two Fork (closure is R^{-1})
3. $(\mathbb{R}, \leq)$ (closure is $\leq^{-1}$)

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Specialization Order and S4-frames III

Basic Results (exercises)

1. τ_R is a topology satisfying $U_i \in \tau_R \Rightarrow \bigcap_{i \in I} U_i \in \tau_R$
2. In (W, τ_R), $\overline{A} = R^{-1}(A) = \{w \in W : \exists v \in A \; wRv\}$
3. If $R = \{(w, w) : w \in W\}$ then $\tau_R = \mathcal{P}(W)$
4. R is partial order iff τ_R is T_0
   - Partial order is a quasi-order that is antisymmetric ($\forall w \forall v \; wRv \; \& \; vRw \Rightarrow w = v$)
   - In a T_0 space for each pair of distinct points there is an open set that contains exactly one of the pair ($\forall x \forall y, \; \exists U \in \tau, x \in U \; \& \; y \notin U$ or $x \notin U \; \& \; y \in U$)

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Alexandrov Spaces

Definition

Call $(X, \tau) \in \mathbf{Top}$ an Alexandrov space provided

$$U_i \in \tau \Rightarrow \bigcap_{i \in I} U_i \in \tau \text{ for any indexing set } I$$

E.g. For (W, R) a quasi-order, (W, τ_R) is an Alexandrov space

Let **Alex** be the class of all Alexandrov spaces

Theorems (exercise)

1. $(X, \tau) \in \mathbf{Alex}$ iff $\forall x \in X$ there is least $U \in \tau$ with $x \in U$
2. $R = R_{\tau_R}$ and $\tau \subseteq \tau_{R_\tau}$
3. If $(X, \tau) \in \mathbf{Alex}$ then $\tau = \tau_{R_\tau}$

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c-Semantics Realizing Kripke Semantics

Theorem

Let (W, R) be a quasi-order

$$(W, R) \models \varphi \text{ iff } (W, \tau_R) \models \varphi$$

- Frame semantics for quasi-orders is special case of c-semantics So frame completeness moves to topological completeness
- $\mathbf{L}_c(\mathbf{Alex}) = \mathbf{S4}$
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1. $\mathbf{L}_c(\mathbf{Top}) = \mathbf{S4}$
2. Let X be (separable) metrizable dense-in-itself space, $\mathbf{L}_c(X) = \mathbf{S4}$
   dense-in-itself: X has no isolated points, that is $\{x\} \notin \tau$
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Remark

Idea is to move frame completeness to topological completeness via functions that make R^{-1} coincide with closure; i.e.

$$f^{-1}(R^{-1}(A)) = \overline{f^{-1}(A)}$$

Such functions are called interior functions; some examples for $\mathbb{R}$

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Topological Derivative

Limit Point Operator

Recall

Definition: For $A \subseteq X$,

$$x \in d(A) \text{ iff } \forall U \in \tau,\ x \in U \Rightarrow \exists y \in U - \{x\},\ y \in A$$

Properties:

$$\begin{aligned} \overline{A} &= A \cup d(A) \\ d(\varnothing) &\subseteq \varnothing \\ d(A \cup B) &= d(A) \cup d(B) \\ d(d(A)) &\subseteq A \cup d(A) \end{aligned}$$

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Coderivative t is dual to derivative; so ...

$$t(A) = X - d(X - A)$$
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Also

$$\begin{aligned} d(A) &= X - t(X - A) \\ int(A) &= A \cap t(A) \\ t(X) &\supseteq X \\ t(A \cap B) &= t(A) \cap t(B) \\ A \cap t(A) &\subseteq t(t(A)) \end{aligned}$$

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Diamond as Derivative I

Valuations

A valuation is $\nu : \mathfrak{Var} \to \mathcal{P}(X)$

$$
\begin{array}{lll}
x \models p & \text{iff} & x \in \nu(p) \\
x \models \neg\varphi & \text{iff} & x \not\models \varphi \\
x \models \varphi \wedge \psi & \text{iff} & x \models \varphi \text{ and } x \models \psi \\
x \models \Box\varphi & \text{iff} & \exists U \in \tau,\ x \in U \text{ and } \forall y \in U - \{x\},\ y \models \varphi
\end{array}
$$

Hence,

$$
x \models \Diamond\varphi \quad \text{iff} \quad \forall U \in \tau,\ x \in U \Rightarrow \exists y \in U - \{x\},\ y \models \varphi
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Diamond as Derivative II

As Before: φ defines a subset of X

Given ν,
Put $||\varphi|| = \{x \in X : x \models \varphi\}$. Then

$$\begin{aligned} ||\Diamond\varphi|| &= d(||\varphi||) \\ ||\Box\varphi|| &= t(||\varphi||) \end{aligned}$$

Validity

1. φ is valid in X provided $\forall\nu,\ ||\varphi|| = X$
2. $\mathbf{L}_d(\mathcal{C}) = \{\varphi : \forall X \in \mathcal{C},\ X \models \varphi\}$ is a modal logic for any class of spaces $\mathcal{C}$ (Exercise)
3. If $\mathbf{L} = \mathbf{L}_d(\mathcal{C})$ for some class $\mathcal{C}$ of spaces, call $\mathbf{L}$ a d-logic
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Some Topological Properties

Recall (Exercises)

1. (X, τ) is dense-in-itself (dii) if X has no isolated points $(\forall x \in X,\ \{x\} \notin \tau)$
   Equivalently...
   $$d(X) = X$$
2. (X, τ) is T_d provided points are locally closed $(\forall x \in X\ \exists U \in \tau,\ \{x\} = U \cap \overline{\{x\}})$
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Expressivity in d-Semantics

Formulas and Properties

Always Valid:

$\Diamond\bot \to \bot$	$d(\varnothing) \subseteq \varnothing$	
$\Diamond\Diamond p \to p \vee \Diamond p$	$d(d(A)) \subseteq A \cup d(A)$	
$\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$	$d(A \cup B) = d(A) \cup d(B)$	

Sometimes Valid:

$\Diamond\Diamond p \to \Diamond p$	$d(d(A)) \subseteq d(A)$	(T_d)
$\Diamond\top$	$d(X) = X$	(dii)

Never Valid:

$p \to \Diamond p$	$A \subseteq d(A)$	

So d-semantics is strictly more expressive than c-semantics!

Expressivity in d-Semantics

Formulas and Properties

Always Valid:		
$\Diamond\bot \to \bot$	$d(\varnothing) \subseteq \varnothing$	
$\Diamond\Diamond p \to p \vee \Diamond p$	$d(d(A)) \subseteq A \cup d(A)$	
$\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$	$d(A \cup B) = d(A) \cup d(B)$	
Sometimes Valid:		
$\Diamond\Diamond p \to \Diamond p$	$d(d(A)) \subseteq d(A)$	(T_d)
$\Diamond\top$	$d(X) = X$	(dii)
Never Valid:		
$p \to \Diamond p$	$A \subseteq d(A)$	

So d-semantics is strictly more expressive than c-semantics!

Some Results in d-Semantics

Theorem

1. $\mathbf{L}_d(\mathbf{Top}) = \mathbf{wK4} = \mathbf{K} + \Diamond\Diamond p \to p \vee \Diamond p$
2. $\mathbf{L}_d(\{X \in \mathbf{Top} : X \text{ is } \mathrm{T}_d\}) = \mathbf{K4}$
3. $\mathbf{L}_d(\{X \in \mathbf{Top} : X \text{ is dii and } \mathrm{T}_d\}) = \mathbf{K4D}$

As Before:

Utilize results in frame semantics

But the new situation is more delicate

Recall closure 'was' R^{-1}... Want similar for d

Examples:

1. 2 point spaces: trivial and Sierpinski
2. Distinguish between line and plane

$$d(A) \cap d(X - A) \subseteq d(\overline{A} \cap \overline{X - A})$$

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Analogue to Specialization Order I

Definition

Call (W, R) weakly transitive if
$\forall w \forall v \forall u \; wRv \;\&\; vRu \;\&\; w \neq u \Rightarrow wRu$

Lemma (Exercise)

(W, R) is weakly transitive iff $(W, R) \models \Diamond\Diamond p \rightarrow (p \vee \Diamond p)$

Definition

For $(X, \tau) \in \mathbf{Top}$, put $xS_\tau y$ iff $x \in d(\{y\})$

Basic Results (Exercises)

1. (X, S_τ) is weakly transitive and irreflexive (no point is related to itself)
2. $S_\tau = R_\tau - \{(x, x) : x \in X\}$ (recall R_τ is specialization order)

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More Basic Results (Exercises)

1. $S_{\tau_R} \subseteq R$
2. $\tau_{S_\tau} = \tau_{R_\tau}$
   Hence...
   - $\tau \subseteq \tau_{S_\tau}$
   - τ_{S_τ} is Alexandrov topology
3. If R is irreflexive and weakly transitive then
   - $S_{\tau_R} = R$
   - $d(A) = R^{-1}(A)$ in (W, τ_R)
4. If X is Alexandrov then $\tau = \tau_{S_\tau}$

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d-Semantics Realizing Kripke Semantics

Theorem

Let (W, R) be an irreflexive weakly transitive frame
In d-semantics

$$(W, R) \models \varphi \text{ iff } (W, \tau_R) \models \varphi$$

- Frame semantics for irreflexive weakly transitive frames is special case of d-semantics
- $\mathbf{L}_d(\mathbf{Alex}) = \mathbf{wK4}$
  (Note: **wK4** is logic of irreflexive weakly transitive frames)
- $\mathbf{L}_d(\mathbf{Alex}_{fin}) = \mathbf{wK4}$

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Completeness in d-Semantics

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1. For a separable metrizable dense-in-itself 0-dimensional space X, $\mathbf{L}_d(X) = \mathbf{K4D}$

   0-dimensional: clopens form basis for τ
2. $\mathbf{L}_d(\mathbb{Q}) = \mathbf{L}_d(\mathbf{C}) = \mathbf{K4D}$
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Remark

As before, move frame completeness to d-semantics via functions

$$f^{-1}(R^{-1}(A)) = d(f^{-1}(A))$$

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