Completeness and definability of a modal logic interpreted over iterated strict partial orders

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Introduction

Given a topology τ on a nonempty set X

- the τ-derived set $d_\tau(A)$ of a set $A \subseteq X$ of points
  $=$
  the set of all limit points of A with respect to τ

The derivative operator d_τ possesses interesting properties

- a set $A \subseteq X$ of points is τ-closed
  iff
  $d_\tau(A) \subseteq A$

What happens if we iterate the derivative operator d_τ

- considering the sequence $d_\tau, d_\tau \circ d_\tau, \ldots$ of operators

Introduction

If τ is T_D, then

- each element d_τ^α of this sequence is a derivative operator

A question arises

- what is the link between the topologies τ_α corresponding to the elements d_τ^α of the sequence

The answer is simple

- the topologies τ_α are getting finer when α increases

Introduction

The lattice of all T_D topologies on X is complete

- this iteration process should stop

The Cantor-Bendixson rank of (X, τ) is defined as

- the least ordinal α such that $d_\tau(d_\tau^\alpha(X)) = d_\tau^\alpha(X)$

A consequence of Tarski's fixpoint theorem is that

- there exists an ordinal $\alpha^\star$ such that $\alpha \leq \alpha^\star$ and $d_\tau \circ d_\tau^{\alpha^\star} = d_\tau^{\alpha^\star}$

Introduction

Any strict partial order R on X defines a function θ_R which

- associates to each strict partial order $S \subseteq R$ on X the strict partial order $\theta_R(S) = R \circ S$ on X

What happens if we iterate the function θ_R

- considering the sequence R, $\theta_R(R)$, $\ldots$ of partial orders

Simply

- the partial orders $\theta_R^\alpha(R)$ are getting smaller when α increases

Introduction

The lattice of all strict partial orders on X is complete

- this iteration process should stop

There exists an ordinal $\alpha^\star$ such that

- $\theta_R(\theta_R^{\alpha^\star}(R)) = \theta_R^{\alpha^\star}(R)$

If R is the strict partial order on X corresponding to a given Alexandroff T_D derivative operator d, then

- $\theta_R^{\alpha^\star}(R)$ is a strict partial order on X corresponding to the derivative operator $d_\tau^{\alpha^\star}$ considered above

Contents

A **topology** on X is a set τ of subsets of X such that

- $\emptyset \in \tau$
- $X \in \tau$
- if $A, B \in \tau$, then $A \cap B \in \tau$
- if $(A_i)_i$ is a collection of subsets of X such that $A_i \in \tau$ for every i, then $\bigcup_i A_i \in \tau$

We shall say that

- $A \subseteq X$ is τ**-closed** iff $X \setminus A \in \tau$

τ is said to be $\mathbf{T_D}$ iff

- for all $x \in X$, there exists $A, B \in \tau$ such that $A \setminus B = \{x\}$

We shall say that τ is **Alexandroff** iff

- each intersection of members of τ is in τ

Let $\leq$ be the binary relation between topologies on X such that

- $\tau \leq \tau'$ iff $\tau \subseteq \tau'$

Remark that for all topologies τ, τ' on X

- if $\tau \leq \tau'$, then if τ is T_D, then τ' is T_D

Example: the **Sierpiński space**

- $X = \{x, y\}$
- $\tau = \{\emptyset, \{x\}, X\}$

Given a topology τ on X

- let $\mathbf{L}_\tau$ be the set of all topologies τ' on X such that $\tau \leq \tau'$

Remark that

- the least element of L_τ is τ
- the greatest element of L_τ is the topology $\mathcal{P}(X)$
- the least upper bound of a family $\{\tau'_i \colon i \in I\}$ in L_τ is the intersection of all $\tau' \in L_\tau$ such that $\bigcup\{\tau'_i \colon i \in I\} \subseteq \tau'$
- the greatest lower bound of a family $\{\tau'_i \colon i \in I\}$ in L_τ is $\bigcap\{\tau'_i \colon i \in I\}$
- $(L_\tau, \leq)$ is a complete lattice

A **derivative operator** on X is a function $d\colon \mathcal{P}(X) \to \mathcal{P}(X)$ such that

- $d(\emptyset) = \emptyset$
- for all $A, B \subseteq X$, $d(A \cup B) = d(A) \cup d(B)$
- for all $A \subseteq X$, $d(d(A)) \subseteq d(A) \cup A$
- for all $x \in X$, $x \notin d(\{x\})$

$A \subseteq X$ is said to be

- **d-closed** iff $d(A) \subseteq A$

We shall say that d is $\mathbf{T_D}$ iff

- for all $A \subseteq X$, $d(d(A)) \subseteq d(A)$

d is said to be **Alexandroff** iff

- for all $x \in X$, there exists a greatest $A \subseteq X$ such that A is d-closed and $x \notin A$

Topologies and derivative operators

Derivative operators

Let $\leq$ be the binary relation between derivative operators on X such that

- $d \leq d'$ iff for all $A \subseteq X$, $d(A) \subseteq d'(A)$

Remark that for all derivative operators d, d' on X

- if $d \leq d'$, then if d' is T_D, then d is T_D

Example

- $X = \{x, y\}$
- $d(\emptyset) = \emptyset$
- $d(\{x\}) = \{y\}$
- $d(\{y\}) = \emptyset$
- $d(X) = \{y\}$

Given a derivative operator d on X

- let $\mathbf{L_d}$ be the set of all derivative operators d' on X such that $d' \leq d$

Remark that

- the least element of L_d is the derivative operator $d_\emptyset$: $\mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $d_\emptyset(A) = \emptyset$
- the greatest element of L_d is d
- we do not know any representation of the least upper bound and the greatest lower bound of a family $\{d'_i : i \in I\}$ in L_d
- $(L_d, \leq)$ is a complete lattice

Topologies and derivative operators

Given a topology τ on X

- let $\mathbf{d}_\tau$ be the function $d_\tau\colon \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $d_\tau(A) = \{x\colon x \text{ is a } \tau\text{-limit point of } A\}$

Remark that

- d_τ is a derivative operator on X
- for all $A \subseteq X$, A is d_τ-closed iff A is τ-closed
- d_τ is T_D iff τ id T_D
- d_τ is Alexandroff iff τ is Alexandroff
- $d_{\tau'} \leq d_\tau$ iff $\tau \leq \tau'$

Topologies and derivative operators

Given a derivative operator d on X

- let $\tau_{\mathbf{d}}$ be the set of d-open subsets of X

Remark that

- τ_d is a topology on X
- for all $A \subseteq X$, A is τ_d-closed iff A is d-closed
- τ_d is T_D iff d id T_D
- τ_d is Alexandroff iff d is Alexandroff
- $\tau_{d'} \leq \tau_d$ iff $d \leq d'$

Let us further remark that

- $\tau_{d_\tau} = \tau$
- $d_{\tau_d} = d$

Given a topology τ on X

- the function $f\colon L_\tau \to L_{d_\tau}$ such that $f(\tau') = d_{\tau'}$ is an anti-isomorphism between $(L_{d_\tau}, \leq)$ and $(L_\tau, \leq)$

Given a derivative operator d on X

- the function $f\colon L_d \to L_{\tau_d}$ such that $f(d') = \tau_{d'}$ is an anti-isomorphism between $(L_{\tau_d}, \leq)$ and $(L_d, \leq)$

Alexandroff T_D derivative operators and strict partial orders

Given an Alexandroff T_D derivative operator d on X

- let $\mathbf{L_d^A}$ be the set of all Alexandroff T_D derivative operators d' on X such that $d' \leq d$

Remark that

- the least element of L_d^A is the derivative operator $d_\emptyset$
- the greatest element of L_d^A is d
- we do not know any representation of the least upper bound and the greatest lower bound of a family $\{d_i' : i \in I\}$ in L_d^A
- $(L_d^A, \leq)$ is a complete lattice

A **strict partial order** on X is a binary relation R on X such that

- for all $x \in X$, $x \notin R(x)$
- for all $x \in X$, $R(R(x)) \subseteq R(x)$

We shall say that $A \subseteq X$ is

- **R-closed** iff $R^{-1}(A) \subseteq A$

Alexandroff T_D derivative operators and strict partial orders

Strict partial orders

Let $\leq$ be the binary relation between strict partial orders on X such that

- $R \leq R'$ iff $R \subseteq R'$

Given a strict partial order R on X

- let $\mathbf{L_R}$ be the set of all strict partial orders R' on X such that $R' \leq R$

Remark that

- the least element of L_R is the strict partial order $\emptyset$
- the greatest element of L_R is R
- the least upper bound of a family $\{R'_i\colon i \in I\}$ in L_R is the transitive closure of $\bigcup\{R'_i\colon i \in I\}$
- the greatest lower bound of a family $\{R'_i\colon i \in I\}$ in L_R is $\bigcap\{R'_i\colon i \in I\}$
- $(L_R, \leq)$ is a complete lattice

Alexandroff T_D derivative operators and strict partial orders

Given an Alexandroff T_D derivative operator d on X

- let $\mathbf{R_d}$ be the binary relation on X such that for all $x, y \in X$, $x \; R_d \; y$ iff $x \in d(\{y\})$

Remark that

- R_d is a strict partial order on X
- for all $A \subseteq X$, A is R_d-closed iff A is d-closed
- $R_d \leq R_{d'}$ iff $d \leq d'$

Alexandroff T_D derivative operators and strict partial orders

Given a strict partial order R on X

- let $\mathbf{d_R}$ be the function d_R: $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that for all $A \subseteq X$, $d_R(A) = R^{-1}(A)$

Remark that

- d_R is an Alexandroff T_D derivative operator on X
- for all $A \subseteq X$, A is d_R-closed iff A is R-closed
- $d_R \leq d_{R'}$ iff $R \leq R'$

Alexandroff T_D derivative operators and strict partial orders

Let us further remark that

- $d_{R_d} = d$
- $R_{d_R} = R$

Given an Alexandroff T_D derivative operator d on X

- the function $f\colon L^A_d \to L_{R_d}$ such that $f(d') = R_{d'}$ is an isomorphism between $(L_{R_d}, \leq)$ and $(L^A_d, \leq)$

Given a strict partial order R on X

- the function $f\colon L_R \to L^A_{d_R}$ such that $f(R') = d_{R'}$ is an isomorphism between $(L^A_{d_R}, \leq)$ and $(L_R, \leq)$

Cantor-Bendixson ranks

Cantor-Bendixson ranks of Alexandroff T_D derivative operators

Given an Alexandroff T_D derivative operator d on X

- let $\theta_{\mathbf{d}}$ be the function $\theta_d\colon L_d \to L_d$ such that for all $d' \in L_d$, $\theta_d(d') = d \circ d'$

Clearly

- θ_d is monotonic
- θ_d has a least fixpoint $\text{lfp}(\theta_d)$ and a greatest fixpoint $\text{gfp}(\theta_d)$
- $\text{lfp}(\theta_d) = d_\emptyset$
- $\text{gfp}(\theta_d)$ is the least upper bound of the family $\{d'\colon d' \leq \theta_d(d')\}$ in L_d

Cantor-Bendixson ranks

Cantor-Bendixson ranks of Alexandroff T_D derivative operators

For all ordinals α, we inductively define $\theta_{\mathbf{d}}{\downarrow}\alpha$ as follows

- $\theta_d{\downarrow}0$ is d
- for all successor ordinals α, $\theta_d{\downarrow}\alpha$ is $\theta_d(\theta_d{\downarrow}(\alpha - 1))$
- for all limit ordinals α, $\theta_d{\downarrow}\alpha$ is the greatest lower bound of the family $\{\theta_d{\downarrow}\beta\colon \beta \in \alpha\}$ in L_d

There exists an ordinal α such that

- $\theta_d{\downarrow}\alpha = \mathrm{gfp}(\theta_d)$

Cantor-Bendixson ranks

The least ordinal α such that

- $\theta_d{\downarrow}\alpha = \mathrm{gfp}(\theta_d)$

is called the **Cantor-Bendixson rank** of d

Example

- $X = \mathbb{Z}$
- $d_{\mathbb{Z}}(A) = \{x\colon \text{there exists } y \in A \text{ such that } x <_{\mathbb{Z}} y\}$
- obviously
  - $\theta_{d_{\mathbb{Z}}}(\theta_{d_{\mathbb{Z}}}{\downarrow}\omega) = \theta_{d_{\mathbb{Z}}}{\downarrow}\omega$
- the Cantor-Bendixson rank of $d_{\mathbb{Z}}$ is ω

Cantor-Bendixson ranks

Cantor-Bendixson ranks of strict partial orders

Given a strict partial order R on X

- let $\theta_{\mathbf{R}}$ be the function $\theta_R \colon L_R \to L_R$ such that for all $R' \in L_R$, $\theta_R(R') = R \circ R'$

Clearly

- θ_R is monotonic
- θ_R has a least fixpoint $\mathrm{lfp}(\theta_R)$ and a greatest fixpoint $\mathrm{gfp}(\theta_R)$
- $\mathrm{lfp}(\theta_R) = \emptyset$
- $\mathrm{gfp}(\theta_R)$ is the least upper bound of the family $\{R' \colon R' \leq \theta_R(R')\}$ in L_R

Cantor-Bendixson ranks

Cantor-Bendixson ranks of strict partial orders

For all ordinals α, we inductively define $\theta_{\mathbf{R}}{\downarrow}\alpha$ as follows

- $\theta_R{\downarrow}0$ is R
- for all successor ordinals α, $\theta_R{\downarrow}\alpha$ is $\theta_R(\theta_R{\downarrow}(\alpha - 1))$
- for all limit ordinals α, $\theta_R{\downarrow}\alpha$ is the greatest lower bound of the family $\{\theta_R{\downarrow}\beta\colon \beta \in \alpha\}$ in L_R

There exists an ordinal α such that

- $\theta_R{\downarrow}\alpha = \mathrm{gfp}(\theta_R)$

The least ordinal α such that

- $\theta_R{\downarrow}\alpha = \mathrm{gfp}(\theta_R)$

is called the **Cantor-Bendixson rank** of R

Example

- $X = \mathbb{Q}$
- $x \; R_{\mathbb{Q}} \; y$ iff $x <_{\mathbb{Q}} y$
- obviously
  - $\theta_{R_{\mathbb{Q}}}(\theta_{R_{\mathbb{Q}}}{\downarrow}0) = \theta_{R_{\mathbb{Q}}}{\downarrow}0$
- the Cantor-Bendixson rank of $R_{\mathbb{Q}}$ is 0

Cantor-Bendixson ranks

Alexandroff T_D derivative operators v. strict partial orders

Let d be an Alexandroff T_D derivative operator on X and R be a strict partial order on X such that

- for all $x, y \in X$, $x \; R \; y$ iff $x \in d(\{y\})$
- for all $A \subseteq X$, $d(A) = R^{-1}(A)$

One can prove by induction on the ordinal α that

- for all $x, y \in X$, $x \; \theta_R{\downarrow}\alpha \; y$ iff $x \in \theta_d{\downarrow}\alpha(\{y\})$
- for all $A \subseteq X$, $\theta_d{\downarrow}\alpha(A) \supseteq \theta_R{\downarrow}\alpha^{-1}(A)$

Cantor-Bendixson ranks

Alexandroff T_D derivative operators v. strict partial orders

Let

- $\alpha_{\mathbf{d}}$ be be the Cantor-Bendixson rank of d
- $\alpha_{\mathbf{R}}$ be the Cantor-Bendixson rank of R

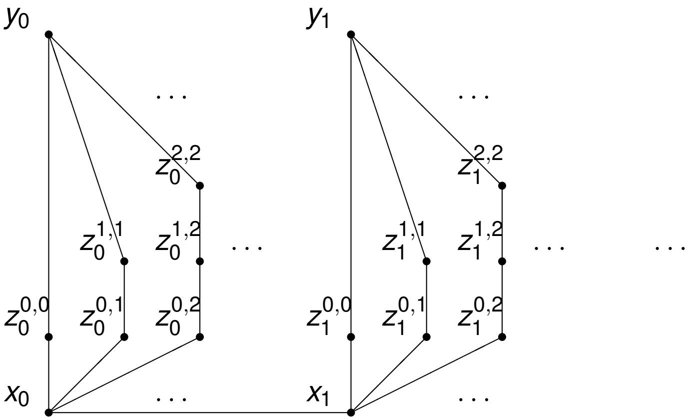
The above considerations prove that

- for all $x, y \in X$, $x \; \theta_R{\downarrow}\alpha \; y$ iff $x \in \theta_d{\downarrow}\alpha(\{y\})$
- for all $A \subseteq X$, $\theta_d{\downarrow}\alpha(A) \supseteq \theta_R{\downarrow}\alpha^{-1}(A)$

Cantor-Bendixson ranks

Alexandroff T_D derivative operators v. strict partial orders

Example



A modal logic
Syntax

Formulas are defined as follows

- $\phi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi \mid \Box^\star\phi$

Abbreviations

- Standard definitions for the remaining Boolean operations
- $\Diamond\phi ::= \neg\Box\neg\phi$
- $\Diamond^\star\phi ::= \neg\Box^\star\neg\phi$

A modal logic
Relational semantics

A **relational frame** is a structure of the form $\mathcal{F} = (X, R, S)$ such that

- X is a nonempty set
- R is a strict partial order on X
- S is the greatest fixpoint of the function θ_R in L_R

Lemma: if $\mathcal{F} = (X, R, S)$ is a relational frame, then

1. $R \circ R \leq R$
2. $S \circ S \leq S$
3. $S \leq R$
4. $R \circ S \leq S$
5. $S \circ R \leq S$
6. $S \leq R \circ S$

A modal logic

Relational semantics

A **relational model** is a structure of the form $\mathcal{M} = (X, R, S, V)$ such that

- (X, R, S) is a relational frame
- V is a valuation on X

Satisfiability

- $\mathcal{M}, x \models p$ iff $x \in V(p)$
- $\mathcal{M}, x \not\models \bot$
- $\mathcal{M}, x \models \neg\phi$ iff $\mathcal{M}, x \not\models \phi$
- $\mathcal{M}, x \models \phi \vee \psi$ iff either $\mathcal{M}, x \models \phi$, or $\mathcal{M}, x \models \psi$
- $\mathcal{M}, x \models \Box\phi$ iff for all $y \in X$, if $x\ R\ y$, then $\mathcal{M}, y \models \phi$
- $\mathcal{M}, x \models \Box^{\star}\phi$ iff for all $y \in X$, if $x\ S\ y$, then $\mathcal{M}, y \models \phi$

Lemma: if $\mathcal{F} = (X, R, S)$ is a relational frame, then

1. $\mathcal{F} \models \Box\phi \rightarrow \Box\Box\phi$
2. $\mathcal{F} \models \Box^\star\phi \rightarrow \Box^\star\Box^\star\phi$
3. $\mathcal{F} \models \Box\phi \rightarrow \Box^\star\phi$
4. $\mathcal{F} \models \Box^\star\phi \rightarrow \Box\Box^\star\phi$
5. $\mathcal{F} \models \Box^\star\phi \rightarrow \Box^\star\Box\phi$
6. $\mathcal{F} \models \Box\Box^\star\phi \rightarrow \Box^\star\phi$

A **topological frame** is a structure of the form $\mathcal{F} = (X, d, e)$ such that

- X is a nonempty set
- d is an Alexandroff T_D derivative operator on X
- e is the greatest fixpoint of the function θ_d in L_d

Lemma: if $\mathcal{F} = (X, d, e)$ is a topological frame, then

1. $d \circ d \leq d$
2. $e \circ e \leq e$
3. $e \leq d$
4. $d \circ e \leq e$
5. $e \circ d \leq e$
6. $e \leq d \circ e$

A modal logic

Topological semantics

A **topological model** is a structure of the form $\mathcal{M} = (X, d, e, V)$ such that

- (X, d, e) is a topological frame
- V is a valuation on X

Interpretation

- $\| p \|_{\mathcal{M}} = V(p)$
- $\| \bot \|_{\mathcal{M}} = \emptyset$
- $\| \neg\phi \|_{\mathcal{M}} = X \setminus \| \phi \|_{\mathcal{M}}$
- $\| \phi \vee \psi \|_{\mathcal{M}} = \| \phi \|_{\mathcal{M}} \cup \| \psi \|_{\mathcal{M}}$
- $\| \Box\phi \|_{\mathcal{M}} = X \setminus d(X \setminus \| \phi \|_{\mathcal{M}})$
- $\| \Box^{\star}\phi \|_{\mathcal{M}} = X \setminus e(X \setminus \| \phi \|_{\mathcal{M}})$

A modal logic

Topological semantics

Lemma: if $\mathcal{F} = (X, d, e)$ is a topological frame, then

1. $\mathcal{F} \models \Box\phi \rightarrow \Box\Box\phi$
2. $\mathcal{F} \models \Box^\star\phi \rightarrow \Box^\star\Box^\star\phi$
3. $\mathcal{F} \models \Box\phi \rightarrow \Box^\star\phi$
4. $\mathcal{F} \models \Box^\star\phi \rightarrow \Box\Box^\star\phi$
5. $\mathcal{F} \models \Box^\star\phi \rightarrow \Box^\star\Box\phi$
6. $\mathcal{F} \models \Box\Box^\star\phi \rightarrow \Box^\star\phi$

Axiomatization and completeness

Axiomatization

Let **L** be the least normal logic in our language containing

1. $\Box\phi \rightarrow \Box\Box\phi$
2. $\Box^\star\phi \rightarrow \Box^\star\Box^\star\phi$
3. $\Box\phi \rightarrow \Box^\star\phi$
4. $\Box^\star\phi \rightarrow \Box\Box^\star\phi$
5. $\Box^\star\phi \rightarrow \Box^\star\Box\phi$
6. $\Box\Box^\star\phi \rightarrow \Box^\star\phi$

Proposition (Soundness)

- if $\phi \in L$, then ϕ is valid in all relational frames
- if $\phi \in L$, then ϕ is valid in all topological frames

Axiomatization and completeness

Axiomatization

Proposition (Completeness)

- if ϕ is valid in all relational frames, then $\phi \in L$

Example

- if $\phi = \Box(p \to \Diamond p) \to (\Diamond p \to \Diamond^{\star} p)$, then
  - ϕ is valid in all topological frames
  - ϕ is not valid in all relational frames

A set Γ of formulas is said to be an **L-theory** iff

- Γ contains *L*
- Γ is closed under the rule of modus ponens

We shall say that an *L*-theory Γ

- is **consistent** iff $\bot \notin \Gamma$
- is **maximal** iff for all formulas ϕ, either $\phi \in \Gamma$, or $\neg\phi \in \Gamma$

Given an *L*-theory Γ and a formula ϕ

- $\Gamma + \phi = \{\psi\colon \phi \to \psi \in \Gamma\}$
- $\Box\Gamma = \{\phi\colon \Box\phi \in \Gamma\}$
- $\Box^{\star}\Gamma = \{\phi\colon \Box^{\star}\phi \in \Gamma\}$

Lemma

1. $\Gamma + \phi$ is the least L-theory containing Γ and ϕ
2. $\Gamma + \phi$ is consistent iff $\neg\phi \notin \Gamma$
3. $\Box\Gamma$ is an L-theory
4. $\Box^\star\Gamma$ is an L-theory

Lemma (Lindenbaum's Lemma)

- if Γ is a consistent L-theory, then there exists a maximal consistent L-theory Δ such that $\Gamma \subseteq \Delta$

Lemma (Existence Lemma)

1. if Γ is a maximal consistent L-theory such that $\Box\phi \notin \Gamma$, then there exists a maximal consistent L-theory Δ such that $\Box\Gamma \subseteq \Delta$ and $\phi \notin \Delta$
2. if Γ is a maximal consistent L-theory such that $\Box^\star\phi \notin \Gamma$, then there exists a maximal consistent L-theory Δ such that $\Box^\star\Gamma \subseteq \Delta$ and $\phi \notin \Delta$

Lemma

- if $\Box^\star\Gamma \subseteq \Delta$ then there exists a maximal consistent L-theory Λ such that $\Box\Gamma \subseteq \Lambda$ and $\Box^\star\Lambda \subseteq \Delta$

Axiomatization and completeness

Axiomatization

A **subordination structure** is a structure of the form $\mathcal{S} = (X, R, S, \mu)$ such that

- X is a finite nonempty set
- R and S are strict partial orders on X
- $S \subseteq R$
- $R \circ S \subseteq S$
- $S \circ R \subseteq S$
- μ is an interpretation on X, i.e. μ associates a maximal consistent L-theory $\mu(x)$ to any $x \in X$

Proposition

- if ϕ is true in the class of all subordination structures of cardinality 1 then $\phi \in L$

Axiomatization and completeness

Axiomatization

Given a subordination structure $\mathcal{S} = (X, R, S, \mu)$, it may contain imperfections

$\Box$-imperfections: triples of the form $(x, \Box, \phi)$ where $x \in X$ is such that

- $\Box\phi \notin \mu(x)$
- for all $y \in X$, if $x\ R\ y$, then $\phi \in \mu(y)$

$\Box^\star$-imperfections: triples of the form $(x, \Box^\star, \phi)$ where $x \in X$ is such that

- $\Box^\star\phi \notin \mu(x)$
- for all $y \in X$, if $x\ S\ y$, then $\phi \in \mu(y)$

imperfections of density pairs of the form (x, y) where $x, y \in X$ are such that

- $x\ S\ y$
- for all $z \in X$, either not $x\ R\ z$, or not $z\ S\ y$

Axiomatization and completeness

Repairing imperfections

Lemma: Given a $\Box$-imperfection $(x, \Box, \phi)$ in a subordination structure $\mathcal{S}$

- there exists a subordination structure $\mathcal{S}'$ such that $\mathcal{S}'$ contains $\mathcal{S}$ and $(x, \Box, \phi)$ is not a $\Box$-imperfection in $\mathcal{S}'$

Proof

Axiomatization and completeness

Repairing imperfections

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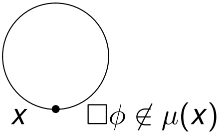
Proof

$\mathcal{S} = (X, R, S, \mu)$



Axiomatization and completeness

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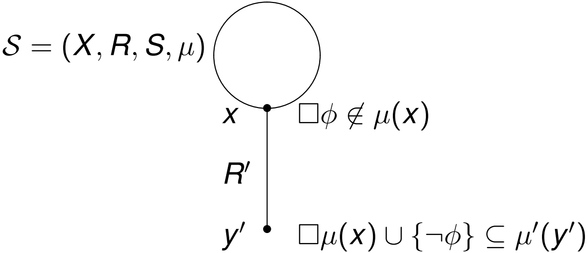

Axiomatization and completeness

Repairing imperfections

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Axiomatization and completeness

Repairing imperfections

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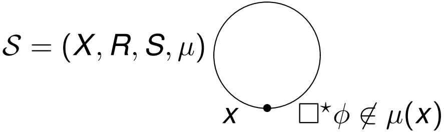
- there exists a subordination structure $\mathcal{S}'$ such that $\mathcal{S}'$ contains $\mathcal{S}$ and $(x, \Box^\star, \phi)$ is not a $\Box^\star$-imperfection in $\mathcal{S}'$

Proof



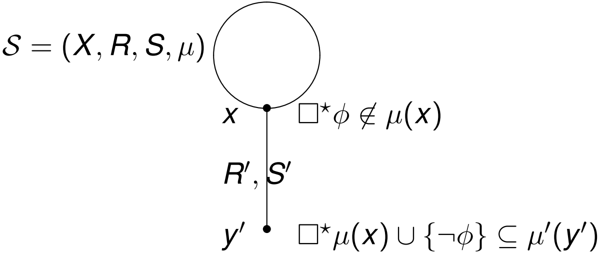
Axiomatization and completeness

Repairing imperfections

Lemma: Given a $\Box^\star$-imperfection $(x, \Box^\star, \phi)$ in a subordination structure $\mathcal{S}$

- there exists a subordination structure $\mathcal{S}'$ such that $\mathcal{S}'$ contains $\mathcal{S}$ and $(x, \Box^\star, \phi)$ is not a $\Box^\star$-imperfection in $\mathcal{S}'$

Proof



Axiomatization and completeness

Repairing imperfections

Lemma: Given an imperfection of density (x, y) in a subordination structure $\mathcal{S}$

- there exists a subordination structure $\mathcal{S}'$ such that $\mathcal{S}'$ contains $\mathcal{S}$ and (x, y) is not an imperfection of density in $\mathcal{S}'$

Proof

Axiomatization and completeness

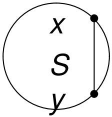
Repairing imperfections

Lemma: Given an imperfection of density (x, y) in a subordination structure $\mathcal{S}$

- there exists a subordination structure $\mathcal{S}'$ such that $\mathcal{S}'$ contains $\mathcal{S}$ and (x, y) is not an imperfection of density in $\mathcal{S}'$

Proof

$$\mathcal{S} = (X, R, S, \mu)$$



Axiomatization and completeness

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Lemma: Given an imperfection of density (x, y) in a subordination structure $\mathcal{S}$

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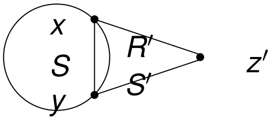
Proof

$\mathcal{S} = (X, R, S, \mu)$



Theorem: The following conditions are equivalent

1. $\phi \in L$
2. ϕ is valid in the class of all relational frames
3. ϕ is true in the class of all subordination structures of cardinality 1

Proposition

- $\Box^{\star}$ is not definable in the ordinary language of modal logic with respect to L

Proof:

1. assume there exists a formula ϕ in the ordinary language of modal logic defining $\Box^*$ with respect to L
2. let $\mathcal{M} = (\mathbb{Z}, <_{\mathbb{Z}}, \emptyset, V)$ and $\mathcal{M}' = (\mathbb{Q}, <_{\mathbb{Q}}, <_{\mathbb{Q}}, V')$ with $V(q) = \emptyset$ and $V'(q) = \emptyset$ for all Boolean variables q
3. remark that for all formulas ψ in the ordinary language of modal logic, for all $x \in \mathbb{Z}$ and for all $x' \in \mathbb{Q}$, $\mathcal{M}, x \models \psi$ iff $\mathcal{M}', x' \models \psi$
4. hence, $\mathcal{M}, 0 \models \phi$ iff $\mathcal{M}', 0 \models \phi$
5. remark that $\mathcal{M}, 0 \models \Box^* p$ and $\mathcal{M}', 0 \not\models \Box^* p$
6. since ϕ defines $\Box^*$ with respect to L, $\mathcal{M}, 0 \models \phi$ and $\mathcal{M}', 0 \not\models \phi$: a contradiction

Proposition

- the class of all relational frames is not first-order definable

Proof:

1. assume there exists a first-order sentence ϕ defining the class of all relational frames
2. for all $n \in \mathbb{N}$, let $\mathcal{F}_n = (X_n, R_n, S_n)$ be the relational frame defined by $X_n = \{0, \ldots, n\}$, $R_n = <_{X_n}$ and $S_n = \emptyset$
3. obviously, for all $n \in \mathbb{N}$
   1. $\mathcal{F}_n \models \phi$
   2. $\mathcal{F}_n \models \exists y \, \forall x \, (R(x, y) \vee x \equiv y)$
   3. $\mathcal{F}_n \models \forall x \, \forall y \, \neg S(x, y)$

4. let U be an ultrafilter over $\mathbb{N}$ and $\mathcal{F}_U = (X_U, R_U, S_U)$ be the ultraproduct of the family $\{\mathcal{F}_n\colon n \in \mathbb{N}\}$ modulo U
5. by 3
   1. $\mathcal{F}_U \models \phi$
   2. $\mathcal{F}_U \models \exists y\, \forall x\, (R(x, y) \vee x \equiv y)$
   3. $\mathcal{F}_U \models \forall x\, \forall y\, \neg S(x, y)$
6. for all $i \in \mathbb{N}$, let $[i]$ be the class of $(i, i, \ldots)$ modulo U
7. remark that for all $i, j \in \mathbb{N}$, $[i]\ R_U\ [j]$ iff $i < j$
8. by 5.2, there exists $M_U \in X_U$ such that for all $i \in \mathbb{N}$, either $[i]\ R_U\ M_U$, or $[i] = M_U$
9. by 7, for all $i \in \mathbb{N}$, $[i]\ R_U\ M_U$

10. let R'_U be the binary relation on X_U such that for all $x, y \in X_U$, $x \; R'_U \; y$ iff there exists $i \in \mathbb{N}$ such that $x = [i]$ and $y = M_U$
11. remark that R'_U is a strict partial order on X_U, $R'_U \subseteq R_U$ and $R'_U \neq \emptyset$
12. claim: $R'_U \leq \theta_{R_U}(R'_U)$
13. hence, $R'_U \leq \text{gfp}(\theta_{R_U})$
14. by 5.1 and 5.3, $\text{gfp}(\theta_{R_U}) = \emptyset$
15. by 13, $R'_U = \emptyset$: a contradiction

Notes

Open problems:

1. Philosophical interpretation of $\Box^*$ in terms of beliefs ?
2. What is the logic of $\Box^*$ alone ? $K4$?
3. Finite model property of L ?
4. Decidability/complexity of the membership problem in L ?
5. Modal definability of the class of all relational frames ?
6. Generalization to other monotonic functions θ_R: $L_R \to L_R$

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