On Finitely Valued Bimodal Symmetric Gödel Logics

Phridoni Alshibaia

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Heyting Algebras

A **Heyting** algebra H is a bounded lattice such that for all a and b in H there is a greatest element x of H such that:

$a \wedge x \leq b$.

This element, which is uniquely determined by a and b, is the relative pseudo-complement of a with respect to b, and is denoted $a \rightarrow b$. We write 1 and 0 for the largest and smallest element of H, respectively.

Heyting Algebras

There is another definition of **Heyting** algebras:

An algebra $\langle H, \vee, \wedge, \to, 0, 1\rangle$ with three binary and two nullary operations is a Heyting algebra if it satisfies:

H1: $\langle H, \vee, \wedge\rangle$ is a bounded distributive lattice
H2: $x \to x = 1$
H3: $(x \to y) \wedge y = y$; $x \wedge (x \to y) = x \wedge y$
H4: $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;
$(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

One can easily check that these two definitions are equivalent.

Gödel Algebras

A **Gödel** algebra (G- algebra) is a Heyting algebra with the linearity condition:

$$(x \to y) \vee (y \to x) = 1.$$

Gödel algebras are also called linear Heyting algebras since subdirectly irreducible Gödel algebras are linearly ordered Heyting algebras.
G-algebras are algebraic models of the **Gödel** Logic.

Double-Brouwerian Algebras

An algebra $\langle T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1\rangle$ is a **Double-Brouwerian** algebra (or **Skolem** algebra) if $\langle T, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, $\rightharpoonup$ is an implication (relative pseudo-complement), $\rightharpoondown$ is coimplication (relative pseudo difference) on T.

Heyting algebras are associated with theories in intuitionistic logic (*Int*) in the same way Boolean algebras are associated with theories in classical logic. Heyting-Brouwer logic (alias symmetric Intuitionistic logic Int^2) was introduced by C.Rauszer. Notice that the variety of Skolem algebras are agebraic models for symmetric Intuitionistic logic Int^2.

G_n^2 Algebras

An algebra $\langle T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1\rangle$ is said to be a G^2-algebra if:

(i) $\langle T, \vee, \wedge, \rightharpoonup, 0, 1\rangle$ is a G-algebra, corresponding to a Gödel Logic;
(ii) $\langle T, \vee, \wedge, \rightharpoondown, 0, 1\rangle$ is a dual G-algebra (alias Brouwerian algebra with the linearity condition $(p \rightharpoondown q) \wedge (q \rightharpoondown p) = 0$).

G_n^2-algebras are algebraic models of n-valued symmetric Gödel logics.

KM-Algebras

An algebra $\langle H, \vee, \wedge, \rightarrow, \Box, 0, 1\rangle$ where $\langle H, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra and $\Box$ is subjected to the identities 1-3 below, is called **KM**-algebra:

1.$x \leq \Box x$
2.$\Box x \rightarrow x = x$
3.$\Box x \leq y \vee (y \rightarrow x)$

MG_n^2 Algebras

We investigate the symmetric Gödel logic G_n^2, the language of which is enriched with two modalities $\Box, \Diamond$. (We need here the second modality to preserve the duality principle, as the G_n^2 is symmetric).
We will call it *n*-valued bimodal symmetric Gödel logic and denote by MG_n^2. The MG_n^2-algebra is a finite algebra $\langle T, \vee, \wedge, \rightharpoonup, \rightharpoondown, \Box, \Diamond, 0, 1\rangle$ where $\langle T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1\rangle$ is G_n^2-algebra and the operators $\Box$, $\Diamond$ satisfy the following conditions:

1. $x \leq \Box x$, $\Box x \leq y \vee (y \rightharpoonup x)$, $\Box x \rightharpoonup x = x$ (**KM**-axioms)
2. $\Diamond x \leq x$, $x \rightharpoondown \Diamond x = \Diamond x$, $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$
3. $\Box\Diamond x \leq x$, $\Diamond\Box x \geq x$.

Kripke frames

A **Kripke frame** or modal frame is a pair $\langle X, R \rangle$, where X is a non-empty set, and R is a binary relation on X. Elements of X are called nodes or worlds, and R is known as the accessibility relation.

Let $\langle X, R \rangle$ be a Kripke frame. We shall say a subset $Y \subset X$ is an **upper cone (or cone)** if $x \in Y$ and xRy imply $y \in Y$. The concept of **lower cone** is defined dually. A subset $Y \subset X$ is called a **bicone** if it is an upper cone and a lower cone at the same time.

One Generated Free MG_n^2-Algebras

Now at first, we describe the one generated free MG_n^2 algebras
Let (C_n^m, R_n^m) $(0 \leq m \leq n > 0)$ be a Kripke frame, where C_n^m is n-element set $\{c_1^m, ..., c_n^m\}$, R_n^m is an irreflexive and transitive relation such that $c_1^m R_n^m c_2^m ... c_{n-1}^m R_n^m c_n^m$.
Let $X_n = \coprod_{m=0}^n C_n^m$ be a disjoint union of C_n^m, $R_n = \cup_{m=0}^n R_n^m$.
$X = \cup_{i=1}^n X_n$, $R = \cup_{i=1}^n R_n$.

Let g_n^m be m-element upper set of C_n^m and $g_n = \{g_n^0, ..., g_n^n\}$.
$G = \cup_{i=1}^n g_n$.

One Generated Free MG_n^2-Algebras

Let $(Con(X), \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \varnothing, X)$ be the algebra generated by G by means of the following operations:
The union $\cup$, the intersection $\cap$, $A \rightharpoonup B = -R_\rho^{-1} - (-A \cup B)$, $A \rightharpoondown B = R_\rho(A \cap -B), \Box(A) = -R^{-1} - (A), \Diamond(A) = R(A)$ for any upper cones of A and B of X_n, where R_ρ is a reflexive closure of the relation R. Observe, that if A is an upper cone of MG_n^2-frame then $\Box A \supseteq A$ and $\Diamond A \subseteq A$ (because of irreflexivity of R).

One Generated Free MG_n^2-Algebras

Lemma. The MG_n^2-algebra $T_n^m = Con(C_n^m), \cup, \cap, \rightharpoonup, \neg, \Box, \Diamond, \varnothing, C_n^m)$ is generated by any element of T_n^m, where $Con(C_n^m)$ is the set of all upper cones of (C_n^m, R_n^m) .

Theorem. The algebra $(Con(X), \bigcup, \bigcap, \rightharpoonup, \neg, \Box, \Diamond, \emptyset, X)$ is a one generated free MG_n^2-algebra.

m-generated Free MG_n^2-Algebras

Now we shall give a general scheme for constructing a Kripke frame, a set of upper sets of which describes *m*-generated free MG_n^2-algebra.

Denote by $\mathbf{n}$ the set $\{1, ..., n\}$. Given $g_1, ...g_n$ and given $p \subseteq n$, we define G_p to be the set of all $x \in X$ such that for $i = 1, ..., n$, $x \in g_i$ iff $i \in p$, and given $x \in X$ we set $Col(x) = \{i \in \mathbf{n} : x \in g_i\}$. A point $x \in G_p$ is said to have the color p, written as $Col(x) = p$.

Let $X(m)$ be a disjoint union of finite linearly ordered irreflexive and transitive Kripke frames such that for any positive integer n the number of *n*-element chains is defined in the following way: we can color *n*-element linearly ordered Kripke frame with different ways stipulating that if xRy, then $Col(x) \subseteq Col(y)$

m-generated Free MG_n^2-Algebras

Theorem. The algebra $(Con(X(m)), \bigcup, \bigcap, \rightharpoonup, \neg, \Box, \Diamond, \emptyset, X)$ is a m-generated free MG_n^2-algebra.

Projective Algebras

Let **K** be any variety of algebras. Then $F_K(m)$ denotes the m generated free algebra in the variety **K**. An algebra A is said to be a *retract* of the algebra B, if there are homomorphisms $\varepsilon : A \to B$ and $h : B \to A$ such that $h\varepsilon = Id_A$, where Id_A denotes the identity map over A.

An algebra $A \in \mathbf{K}$ is called *projective*,if for any $B, C \in \mathbf{K}$, any epimorphism (that is an onto homomorphism) $\gamma : B \to C$ and any homomorphism $\beta : A \to C$, there exists a homomorphism $\alpha : B \to C$ such that $\gamma\alpha = \beta$.

Projective Algebras

Notice that in varieties, projective algebras are characterized as retract of free algebras. A subalgebra A of $F_K(m)$ is said to be projective subalgebra if there exists an endomorphism $h : F_K(m) \to F_K(m)$ such that $h(F_K(m)) = A$ and $h(x) = x$ for every $x \in A$.

Projective Algebras

Theorem 3. Any subalgebra of m generated free MG_n^2 algebra is projective.

Thank You!