Vietoris via ∇
(the pointfree case)

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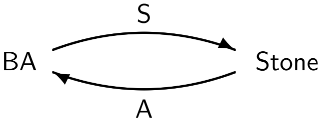
This talk is based on joint work with many colleagues, including:

Marta Bílková, Christian Kissig, Clemens Kupke, Alexander Kurz, Larry Moss, Alessandra Palmigiano, Luigi Santocanale, Steve Vickers, Jacob Vosmaer

Stone duality



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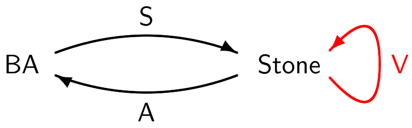

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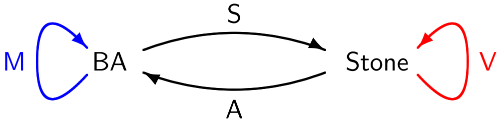

Modal Logic dualizes/axiomatizes the Vietoris functor
(Abramksy)

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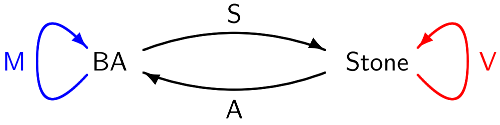

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This provides an Algebra|Coalgebra duality.

Variants of Stone duality

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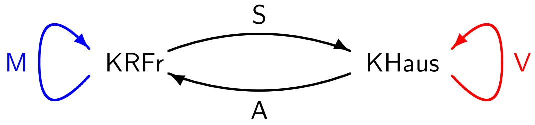

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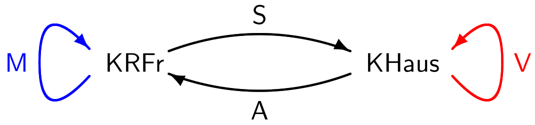

Geometric modal logic dualizes/axiomatizes the Vietoris functor
(Johnstone)

Aim of talk

Generalize this picture from power set functor to arbitrary set functor
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Overview

- Introduction
- Background
- Vietoris via ∇
- (Preservation) Results
- Final remarks

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  - Frames
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Objects & arrows

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- Frame: complete lattice where finite meets distribute over all joins.
- Fix signature: $\mathbb{L} = \langle \bigvee, \bigwedge, 0, 1 \rangle$, with $\bigvee : \mathsf{P}L \to L$ and $\bigwedge : \mathsf{P}_\omega L \to L$.
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- Truth: $\mathbb{L} \models_V s \approx t$ if $[\![s]\!]^{\mathbb{L}}_V \approx [\![t]\!]^{\mathbb{L}}_V$

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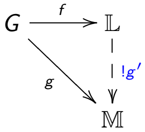

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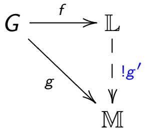

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Fact: Every frame presentation presents a (modulo isos, unique) frame!

The Vietoris construction

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- Let $\mathbb{X} = \langle X, \tau \rangle$ be a topological space.
- $K(\mathbb{X})$ denotes the collection of compact sets, and for $a \in \tau$, define

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\langle \ni \rangle a &:= \{F \in K(\mathbb{X}) \mid F \cap a \neq \varnothing\} \\
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Fact The Vietoris construction preserves various properties, including:
- compactness
- compact Hausdorfness
- Stone-ness

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Fact
V is a functor on the categories KHaus and Stone.

Vietoris pointfree (Johnstone)

Given a frame $\mathbb{L}$, define $L_\Box := \{\Box a \mid a \in L\}$ and $L_\Diamond := \{\Diamond a \mid a \in L\}$.

$$
\mathrm{V}\mathbb{L} := \mathrm{Fr}\langle L_\Box \uplus L_\Diamond \mid \quad
\begin{array}{ll}
\Box(\bigwedge A) = \bigwedge_{a\in A} \Box a & (A \in \mathrm{P}_\omega L) \\
\Diamond(\bigvee A) = \bigvee_{a\in A} \Diamond a & (A \in \mathrm{P}_\omega L) \\
\\
\Box a \wedge \Diamond b \leq \Diamond(a \wedge b) & \\
\Box(a \vee b) \leq \Box a \vee \Diamond b & \\
\\
\Box(\bigsqcup A) = \bigsqcup_{a\in A} \Box a & (A \in \mathrm{P}L \text{ directed}) \\
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Semantics Fix a Kripke model $\mathbb{S} = \langle S, R, V \rangle$.

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Informally: α and $R[s]$ cover one another.

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History

- model theory: Hintikka, Scott, . . .
- modal logic: Fine's normal forms
- ∇ as primitive: Barwise & Moss/Janin & Walukiewicz

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∇ via relation lifting

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Apply relation lifting to the binary relation $\Vdash_{\mathbb{S}} \subseteq S \times \mathrm{ML}$:

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This paves the way for coalgebraic generalizations of modal logic!

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- Lifted membership relation: $\overline{\mathsf{T}}{\in_L} \subseteq \mathsf{T}L \times \mathsf{T}\mathsf{P}L$.
- Given $\Phi \in \mathsf{T}\mathsf{P}L$, define $\lambda(\Phi) := \{\alpha \in \mathsf{T}L \mid \alpha \overline{\mathsf{T}}{\in_L} \Phi\}$.

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$\mathsf{V_T}$ generalizes Johnstone's J: $\mathsf{J} = \mathsf{V_P}$.

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In particular, if $\Phi \in \mathit{SRD}(\Gamma)$ then $(\mathsf{T}\bigwedge)\Phi \ \overline{\mathsf{T}}{\leq} \ \gamma$ for all $\gamma \in \Gamma$.

Functorial properties

Proposition Let $f : \mathbb{L} \to \mathbb{M}$ be a frame homomorphism.
The map $\nabla \circ \mathsf{T}f : \mathsf{T}L \to \mathsf{V}_{\mathsf{T}}M$ given by $\alpha \mapsto \nabla(\mathsf{T}f)\alpha$ is compatible with $(\nabla 1)$, $(\nabla 2)$ and $(\nabla 3)$.

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There is a natural transformation $\epsilon_{\mathsf{T}} : \mathsf{V_T} \to \mathsf{Id}$.

Overview

- Introduction
- Background
- Vietoris via ∇
- (Preservation) Results
- Final remarks

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Theorem
Let $\mathbb{L}$ be a frame. If $\mathbb{L}$ is regular, then so is $\mathsf{V}_\mathsf{T}\mathbb{L}$.

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Let $\mathbb{L}$ be a frame, and $\alpha, \beta \in \mathsf{T}L$. If $\alpha \; \overline{\mathsf{T}}(\leqslant) \; \beta$

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Proof of Key Lemma (if $\alpha \,\overline{\mathsf{T}}(\leqslant)\, \beta$ then $\nabla\alpha \leqslant \nabla\beta$)

WLOG $\alpha \in \mathsf{T}A$ for $A \subseteq_\omega L$, and $\beta = (\mathsf{T}f)\alpha$, some $f : A \to L$ so that $a \leqslant fa$ for each $a \in A$. Define $h : \mathsf{P}A \to L$ by

$$h(B) := \bigwedge \left(\{\neg a \mid a \in B\} \cup \{fa \mid a \notin B\}\right).$$

Then

$$\begin{aligned} 1_{\mathbb{L}} &= \bigwedge\{\neg a \vee fa \mid a \in A\} && \text{(definition of } \leqslant\text{)} \\ &= \bigvee\{h(B) \mid B \in \mathsf{P}A\} && \text{(frame distributivity)} \end{aligned}$$

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In other words: $\nabla\alpha \leqslant \nabla(\mathsf{T}f)\alpha$.

Preservation of compactness

A frame $\mathbb{L}$ is compact if every $S \subseteq L$ with $1_{\mathbb{L}} = \bigvee S$ has a finite subset $S_0 \subseteq_\omega S$ with $1_{\mathbb{L}} = \bigvee S_0$.

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Theorem
Assume that T restricts to finite sets, and let $\mathbb{L}$ be a regular frame.
If $\mathbb{L}$ is compact, then so is $\mathsf{V}_{\mathsf{T}}\mathbb{L}$.

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A frame $\mathbb{L}$ is zero-dimensional if $a = \bigvee\{b \leq a \mid b \lll b\}$.

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Theorem
Let $\mathbb{L}$ be a frame. If $\mathbb{L}$ is zero-dimensional, then so is $\mathsf{V}_{\mathsf{T}}\mathbb{L}$.

Overview

- Introduction
- Background
- Vietoris via ∇
- (Preservation) Results
- Final remarks

Conclusions

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Conclusions

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  - to a functor $V_T : Fr \to Fr$, indexed by $T : Set \to Set$, (where T preserves weak pullbacks).
  - in the sense that $J = V_P$
- The construction V_T preserves the following properties:
  - regularity
  - regularity + compactness (provided T retricts to finite sets)
  - zero-dimensionality
  - ...

Ongoing/Future work

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- Is there a natural transformation $\mathsf{V_P} \to \mathsf{V_T}$?

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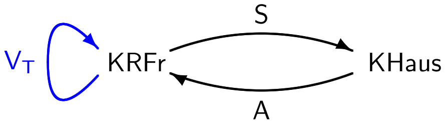
- 

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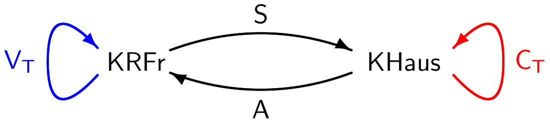
- 

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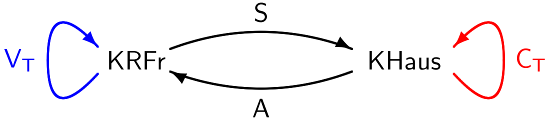
- 

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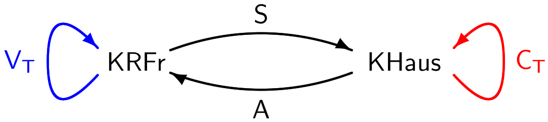
- 

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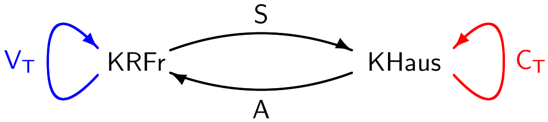
- 

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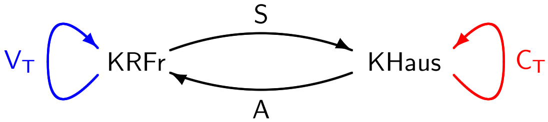
- 

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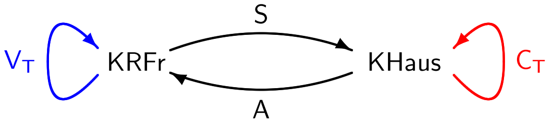
- 

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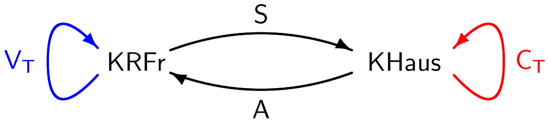
- 

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- Describe final coalgebras over KHaus using geometric ∇-logic.

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