Logics of space with contact and connectedness predicates: complete axiomatizations of the universal fragments

Tinko Tinchev

Faculty of Mathematics and Informatics
Sofia University

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Theories of space and time: algebraic, topological and logical approaches

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and to show their completeness with respect to three different semantics: topological, algebraic and relational (Kripke structures).

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Let $\mathcal{T} = \langle T, \tau \rangle$ be a topological space. The set of regular closed sets in $\mathcal{T}$ form a Boolean algebra under the set-theoretical inclusion, $RC(\mathcal{T})$.
Let us remind: $0 = \varnothing$, $1 = T$, $X_1 \sqcup X_2 = X_1 \cup X_2$,
$X_1 \sqcap X_2 = Cl(Int(X_1 \cap X_2))$, $X^* = Cl(T \setminus X)$

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The regions: the elements of the Boolean algebra $RC(\mathcal{T})$.

When $\mathcal{T}$ is $\mathbb{R}^m$ for some positive integer m, an important kind of regions are the polytops in $\mathbb{R}^m$: the subalgebra, $PRC(\mathbb{R}^m)$, of $RC(\mathbb{R}^m)$ generated by the set of simple polytops. Where a simple polytop is a region which is intersection of finitely many closed half-spaces.

k-contact

Let $a_1, \ldots, a_k$, $k \geq 2$, be regions. They are in *k-contact*, $\mathbf{C}^k(a_1, \ldots, a_k)$ iff $a_1 \cap \cdots \cap a_k \neq \varnothing$.
If $k = 2$, then C^2 is the standard contact relation, C.

$\mathfrak{A}_B = (B, C^2, C^3, \ldots)$, where B is a Boolean algebra of regions or polytops.

The first order language $\mathcal{L}$ is the extension of the language of the Boolean algebras, $0, 1, \sqcup, \sqcap, ^*$ with the binary predicate C^2, $\mathcal{L}'$ is the extension of $\mathcal{L}$ with the set of k-ary predicate symbols C^k for all $k > 2$.

Let $\mathcal{K}$ be a class of Boolean algebras of regions or polytops and

$$Th_\forall(\mathcal{K}, \mathcal{L}') = \{\phi \mid \mathfrak{A}_B \models \phi, B \in \mathcal{K}, \phi \text{ is universal sentence from } \mathcal{L}'\}$$

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Our aim is to axiomatize:

1. $Th_\forall(\mathcal{K}_{all}, \mathcal{L}')$, where $\mathcal{K}_{all}$ is the class of all $RC(\mathcal{T})$
2. $Th_\forall(\mathcal{K}_{connected}, \mathcal{L}')$, where $\mathcal{K}_{connected}$ is the class of all $RC(\mathcal{T})$ for connected topological spaces $\mathcal{T}$
3. $Th_\forall(RC(\mathbb{R}^m), \mathcal{L}')$, $m \geq 1$
4. $Th_\forall(PRC(\mathbb{R}^m), \mathcal{L}')$, $m > 1$
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Let T_{all} be

- the set an universal axiomatization of the Boolean algebras
- the axioms for the equality in $\mathcal{L}' +$
- universal closure of the following formulas

  $C^k(x_1, \ldots, x_k) \rightarrow \bigwedge_{i=1}^{k}(x_i \neq 0)$

  $C^k(x_1, \ldots, x' \sqcup x'', \ldots, x_k) \leftrightarrow$
  $\bigwedge C^k(x_1, \ldots, x', \ldots, x_k) \vee C^k(x_1, \ldots, x'', \ldots, x_k)$, $1 \leq i \leq k$

  $(x \neq 0) \rightarrow C^2(x, \ldots, x)$

  $C^k(x_1, \ldots, x_k) \rightarrow C^k(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$, where σ is a permutation of $1, \ldots, k$

  $C^k(x_1, \ldots, x_k) \rightarrow C^{k+1}(x_1, \ldots, x_k, x_k)$

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Theorem

Let ϕ be an universal sentence from $\mathcal{L}'$. Then

$$T_{all} \vdash \phi \iff \phi \in Th_{\forall}(\mathcal{K}_{all}, \mathcal{L}')$$

Let $T_{connected}$ be $T_{all} + \forall x((x \neq 0) \wedge (x \neq 1) \rightarrow C^2(x, x^*))$

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Theorem (K., P., Z.)

$Th_\forall(\mathcal{K}_{connected}, \mathcal{L}') = Th_\forall(RC(\mathbb{R}^m), \mathcal{L}') = Th_\forall(PRC(\mathbb{R}^m), \mathcal{L}') = Th_\forall(RC(\mathbb{R}), \mathcal{L}')$, $m \geq 2$

Let T_1 be $T_{connected} +$ the universal closure of
$C^3(x_1, x_2, x_3) \to \bigvee_{1 \leq i < j \leq j}(x_i \sqcap x_j \neq 0)$ and
$C^{k+1}(x_1, \ldots, x_{k+1}) \to \bigvee_{1 \leq i < j \leq k+1} C^k(x_i \sqcap x_j, \ldots)$

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Idea I:

To use more or less standard techniques from classical model theory.

Idea II: modal approach (Vakarelov)

We consider the open formulas of the language $\mathcal{L}$ as modal formulas:

there are two type of syntactical objects — Boolean terms constructed from the Boolean constants 0, 1 and countably many Boolean variables by means of the Boolean operations. the (modal) formulas are build from atomic formulas — $a \leq b$

and $C(a, b)$, where a and b are Boolean terms — by means of the propositional connections

Relational semantics

Kripke frame and model are standard notions.

Any Boolean term a has a value $V(a) \subseteq W$ in the model $\mathcal{M} = (W, R, V)$ defined by induction.

Truth value of the atomic formula:

$\mathcal{M} \models (a \leq b)$ iff $V(a) \subseteq V(b)$

$\mathcal{M} \models C(a, b)$ iff $\exists x \exists y (x \in V(a) \& y \in V(b) \& xRy)$

Now, the truth value of an arbitrary formula is defined as usual by induction.

Algebraic semantics is given in Boolean algebras with binary relation in a standard way.

Topological semantics is given by the Boolean algebras of the type $RC(\mathcal{T})$ in a standard way.

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Formal system L for the modal formulas:
Axioms — corresponding to the matrices of the given axioms
Rule of inference: (MP)

Theorem (B., T., V.)

For any modal formula ϕ the following conditions are equivalent
(i) ϕ is a theorem of L
(ii) ϕ is true in the class of all (finite) Kripke frames with reflexive and symmetric relation
(iii) ϕ is true in the class of all (finite) contact Boolean algebras
(iv) ϕ is true in the class of all $RC(\mathcal{T})$.

The same is true when $L_1 = L + (a \neq 0, 1 \rightarrow C(a, a^*))$ and we add in (ii) connectedness of the frames, in (iii) — the corresponding condition, and in (iv) — connectedness of the topological spaces.

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The above theorem(s) can be easily modified for the language $\mathcal{L}'$. But the cases concerning $RC(\mathbb{R}^m)$ and $PRC(\mathbb{R}^m)$ require additional efforts.

Roughly speaking, for a given finite model over the frame for the logic we have to find appropriated p-morphic preimage which can be “realized” in $PRC(\mathbb{R}^m)$.

The predicate connectedness

In the $RC(\mathcal{T})$ one of the predicates with a good meaning is the unary predicate c: $c(a)$, where $a \in RC(\mathcal{T})$, iff a is connected subset.

Now in the modal approach we add one more type of atomic formulas: $c(a)$

Kripke semantics: $\mathcal{M} \models c(a)$ iff $V(a)$ is a connected subset of W (in the graph theory sense).

Let L^c be the extension of L with the axiom

$c(a) \wedge a = p \sqcup q \wedge p, q \neq 0 \rightarrow C(p, q)$

and the rule

$$\frac{\alpha \wedge p \sqcup q = a \wedge p, q \neq 0 \rightarrow C(p, q)}{\alpha \rightarrow c(a)},$$

where p, q are different Boolean variables not occurring in the term a and in the formula α.

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Theorem

The logic L^c is complete with respect to
(i) the class of all (finite) Kripke frames
(ii) the class of all algebras of the type $RC(\mathbb{R}^3)^A$, where A is a polytop in $\mathbb{R}^3$
(iii) the class of all topological spaces

Let $L_1^c = L^c + (a \neq 0, 1 \rightarrow C(a, a^*))$.

Theorem

The logic L_1^c is complete with respect to
(i) the class of all (finite) Kripke frames connected in graph theory sense
(ii) the class of all algebras of the type $RC(\mathbb{R}^3)^A$, where A is a connected polytop in $\mathbb{R}^3$
(iii) the class of all connected topological spaces.

One application of L_1^c

Let us define the predicates SC and Sc in $PRC(\mathbb{R}^2)$ in the following way

$Sc(A)$ iff $Int(A)$ is connected in topological sense

$SC(A, B)$ iff $(\exists A' \leq A)(\exists B' \leq B)(A, B \neq 0 \wedge Sc(A' \cup B'))$

Let $L_2 = L_1^c +$ the following two axioms

$$\neg(\bigwedge_{1 \leq i \leq 5} (x_i \neq 0 \wedge c(x_i)) \wedge \bigwedge_{1 \leq i < j \leq 5} (C(x_i, x_j) \wedge (x_i \sqcap x_j = 0)))$$

$$\neg(\bigwedge_{1 \leq i \leq 6} (x_i \neq 0 \wedge c(x_i)) \wedge \bigwedge_{1 \leq i < j \leq 6} (x_i \sqcap x_j = 0) \wedge \bigwedge_{\substack{1 \leq i \leq 3 \\ 4 \leq j \leq 6}} (C(x_i, x_j)))$$

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Theorem

The logic L_2 *is complete with respect to the structure* $\langle PRC(\mathbb{R}^2), SC, Sc\rangle$