# The prime spectrum of MV-algebras

based on a joint work with A. Di Nola and P. Belluce

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# Contents

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i) $1 \to x = x$
ii) $(x \to y) \to ((y \to z) \to (x \to z))$
iii) $(x \to y) \to y = (y \to x) \to x$
iv) $(\neg y \to \neg x) \to (x \to y)$

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iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

iv) $(\neg y \rightarrow \neg x) \rightarrow (x \rightarrow y)$

$$x \rightarrow y = \neg x \oplus y \quad \text{and} \quad 1 = \neg 0$$

# An Example: the standard MV-algebra

The structure $[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0\rangle$ where the operation are defined as

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*Theorem (Chang 1958)*

*The algebra $[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0 \rangle$ generates the variety of MV-algebras.*

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$$\langle [0, u], \oplus, \neg, 0 \rangle \text{ with } x \oplus y = \min\{x + y, u\} \text{ and } \neg x = u - x$$

is an MV-algebra. Furthermore, **every** MV-algebra can be obtained in this way.

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# Categorical equivalence

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or to **perfect**[1] MV-algebras

*Theorem (Di Nola and Lettieri 1994)*

*There exists a categorical equivalence between perfect MV-algebras and abelian $\ell$-groups.*

[1] An MV-algebra is called **perfect** if it is generated by the intersection of all its maximal ideals.

# Lattice structure

Any MV-algebra has an underlying lattice structure, defined by:

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iii) Define $x \odot y = \neg(\neg x \oplus \neg y)$. The algebra $\langle A, \odot, \rightarrow, 0, 1\rangle$ is a bounded, commutative, **residuated lattice** (or even a bounded commutative **BCK-algebra**).

*Definition*

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*Theorem (McNaughton 1951)*

*The* **free MV-algebra** *over m generators is isomorphic to the algebra McNaughton functions, where the MV operations are defined point-wise.*

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# Ideals

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An ideal $I$ is called **proper** if $I \neq A$. So MV-ideals are also ideals of the lattice reduct (**lattice ideals**.)

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*Lemma*

*An ideal $P$ of an MV-algebra $A$ is prime iff it satisfies the following equivalent conditions:*

1. *for all $a, b \in A$, $a \rightarrow b \in P$ or $b \rightarrow a \in P$;*
2. *for all $a, b \in A$, if $a \wedge b \in P$ then $a \in P$ or $b \in P$;*
3. *for all $I, J$ ideals of $A$, if $I \cap J \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.*

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**Theorem**

*Let $A$ be an MV-algebra and $\operatorname{Spec} A$ the set of its prime ideals. Then $A$ is a subdirect product of the family $\{A/P\}_P$ with $P$ ranging among prime ideals of $A$.*

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*Definition*

A topological space is an **MV-space** if it is, up to homeomorphisms, the spectral space of an MV-algebra.

It is easy to see that there are examples of different MV-algebras with the same Spec.

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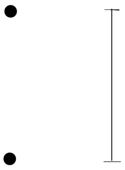
$\{0,1\}$ $[0,1]$



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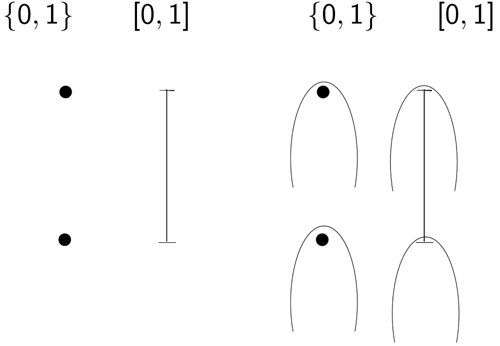

*Definition*

A topological space $X$ is called **spectral** if

1. $X$ is a compact, $T_0$ space,
2. every non-empty irreducible closed subset of $X$ is the closure of a unique point ($X$ is **sober**),
3. and the set $\Omega$ of compact open subsets of $X$ is a basis for the topology of $X$ and is closed under finite unions and intersections.

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Since a spectral space is $T_0$, it is partially ordered by the so-called **specialisation order**: $x \leq y$ iff $x \in \mathrm{cl}(y)$ where $\mathrm{cl}(y)$ is the closure of $y$.

Since MV-space are spectral, one may be tempted to try to characterise them through inverse limit of finite spaces. However in 2004 Di Nola and Grigolia characterised the pro-finite MV-spaces and proved that they do not coincide with the full category of MV-space.

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**Theorem**

*An MV-space is pro-finite if and only if it is a completely normal dual Heyting space.*

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**Theorem**

*There are MV-spaces, as well as completely normal spectral spaces, which are not pro-finite.*

*Definition*

Recall that the triple $\langle X, \leq, \tau \rangle$, where

1. $\langle X, \leq \rangle$ is a poset and
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a) $\tau$ is a Stone space and

b) for any $x, y \in X$ such that $x \not\leq y$ there is a clopen decreasing set $U$ such that $y \in U$ and $x \notin U$.

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Note that each closed subset of a Priestley space is in turn a Priestley space with respect to the inherited topology.

# Priestley duality

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Consider also the functor $\Xi$, assigning to each bounded distributive lattice $L$ its set of prime ideals, ordered by set inclusion and topologised by the basis given by the sets $\tau(a) = \{P \in \mathrm{Spec}(L) \mid a \notin P\}$ and their complements for $a \in L$.
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*Theorem*

*The pair $\Delta, \Xi$ is a categorical duality between BDLat and Pries.*

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More precisely Martinez works on a **particular case of Priestley duality**, developed by Cornish and Fowler, characterising Kleene algebras (which in turn are particular De Morgan algebras.)

*Definition*

A tuple $\langle X, \tau, \leq, g, \{\phi_p\}_{p \in X} \rangle$ is called a **Wajsberg space** if:

1. $\langle X, \tau, \leq, g \rangle$ is a De Morgan space,
2. $\{\phi_p\}_{p \in X}$ is a family of functions $\phi_p : D_p \longrightarrow X$ where $D_p = \{q \in X \mid p \leq g(q)\}$ such that $\forall p, q \in X$:
   a. $\phi_p$ is order-preserving and continuous in the upper topology,
   b. $p \leq g(q)$ implies $p, q \leq \phi_p(q)$,
   c. $p \leq g(q)$ implies $\phi_p(q) = \phi_q(p)$,
   d. $p \leq g(q)$ implies $\phi_p(g(\phi_p(q))) \leq g(q)$,
   e. $p, p' \leq g(q)$ implies $\phi_p(\phi_{p'}(q)) = \phi_{p'}(\phi_p(q))$,
   f. If $U \in Up(X)$ and $q \notin U$, there exists $q_U$, the greatest $p \in X$ such that $p \leq g(q)$ and $\phi_p(q) \notin U$; given $U, V \in Up(X)$ if $q \notin U \cup V$ then $(q_V)_V \notin U$.
3. For every $U, V \in Up(X)$, $\bigcap_{p \in U} \left( D_p^c \cup \phi^{-1}(V) \right) \in Up(X)$.

Where $Up(X)$ is the lattice of clopen increasing subsets of $X$.

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# Gehrke-Priestley duality

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This allows to realise that the **failure of canonicity** for MV-algebras lays on an **"alternation" of operations** in the terms defining the variety.

The problem is overcome by considering class of algebras with a **signature doubled** respect to the initial one and to consider equations as inequalities.

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$$\Big(\forall y_1^+,\dots,y_j^+,y_{j+1}^-,\dots,y_k^-,z_1^+,\dots,z_\ell^+,z_{\ell+1}^-,\dots,z_m^- \textit{ all in } X^\Diamond\Big)$$
$$\Bigg[\Bigg(\Big(\bigvee_{\rho_{\mathfrak{s}}(y_i^+)=\alpha_1}\underline{y}_i^+\Big)\vee\Big(\bigvee_{\rho_{\mathfrak{t}}(z_i^-)=\alpha_1}\underline{z}_i^-\Big)\Bigg)\leqslant\Bigg(\Big(\bigwedge_{\rho_{\mathfrak{s}}(y_i^-)=\alpha_1}\overline{y}_i^-\Big)\wedge\Big(\bigwedge_{\rho_{\mathfrak{t}}(z_i^+)=\alpha_1}\overline{z}_i^+\Big)\Bigg)$$
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$$\Longrightarrow \mathfrak{s}'(\underline{y}_1^+,\dots,\underline{y}_j^+,\overline{y}_{j+1}^-,\dots,\overline{y}_k^-)\leqslant\mathfrak{t}'(\overline{z}_1^+,\dots,\overline{z}_\ell^+,\underline{z}_{\ell+1}^-,\dots,\underline{z}_m^-).$$

*Definition*

Let $A$ be an MV-algebra and consider the equivalence relation $\equiv$ defined by

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It is easy to see that $\equiv$ is a congruence on the lattice reduct of $A$ and it also preserves the MV-algebraic sum (indeed it equalises $\vee$ and $\oplus$: $[x \vee y]_{\equiv} = [x \oplus y]_{\equiv}$ for all $x, y \in A$).

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Let us call $[A]$ the quotient set $A/\equiv$ and $[x]$ the equivalence class $[x]_{\equiv}$.

*Lemma*

*The structure* $[A] = \langle [A], \vee, \wedge, [0], [1] \rangle$ *is a bounded distributive lattice, with* $[x] \vee [y] := [x \vee y] = [x \oplus y]$ *and* $[x] \wedge [y] := [x \wedge y]$*, for all* $x, y \in A$*.*

# The Belluce functor

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*Lemma*

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Knowing on which category such a functor is invertible would constitute a key step in the characterisation of MV-spaces.

# The Belluce functor

**Theorem**

*The map* $\gamma : \operatorname{Spec} A \to \operatorname{Spec}[A]$*, defined by* $\gamma(P) = [P]$*, is a (Priestley) homeomorphism between the MV-space* $\operatorname{Spec} A$ *and the spectral space* $\operatorname{Spec}[A]$*.*

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**Theorem**

*Every bounded distributive lattice in the range of $[\cdot]$ is dual completely normal (i.e. the set of prime ideals containing a prime ideal is totally ordered.)*

**Corollary**

*Every MV-space is completely normal.*[a]

[a] $X$ is normal if any two disjoint closed subsets of $X$ are separated by neighbourhoods

# MV-subspaces

*Definition*

$X \subseteq \operatorname{Spec} A$ is an **MV-subspace** if $X$ with the induced topology is homeomorphic to $\operatorname{Spec} A'$ for some MV-algebra $A'$, i.e. if it is an MV-space itself.

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*Proposition*

All closed subspaces of an MV-space are MV-subspaces.

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Let $\tau(a)$ be compact open in $\operatorname{Spec} A$. Then $\tau(a)$ is an MV-space.

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*Lemma*

*Every MV-algebra has the prime extension property (PEP), namely every ideal extending a prime ideal is itself prime.*

*Proposition*

Let $\tau(a)$ be compact open in $\operatorname{Spec} A$. Then $\tau(a)$ is an MV-space.

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An MV-algebra $A$ is called **hyper-archimedean** if for each $x \in A$ $nx \in (A)$ for some integer $n$. Hyper-archimedean MV-algebras are exactly the ones for which $\operatorname{Spec} A = \operatorname{Max} A$.

# The order in MV-spaces

**Proposition**

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MV-spaces are root systems with respect to the specialisation order

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A poset $\langle X, \leq \rangle$ is called a **spectral root system** if it the disjoint union of spectral roots.

# A characterisation of the order

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Recalling that an MV-algebra is called **local** if it has a unique maximal ideal, an MV-space is called **local** if it has a greatest element or, equivalently, if its corresponding MV-algebra is local.

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Recalling that an MV-algebra is called **local** if it has a unique maximal ideal, an MV-space is called **local** if it has a greatest element or, equivalently, if its corresponding MV-algebra is local.

*Corollary*

*Every MV-space is a disjoint union of local MV-spaces.*

Let $A$ be a linearly ordered MV-algebra, and let $\operatorname{Pr} A$ be the set of principal ideals of $A$.

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For each $P \in \Pr A$ let us choose a generator $u_P$ of $P$, and set $A_0 = \langle u_P \mid P \in \Pr A \rangle$, the subalgebra generated by the $u_P$'s.

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A linearly ordered algebra $A$ is **reduced** if $A = \langle u_P \mid (u_P] = P \in \operatorname{Pr} A \rangle$. In this case, the set $\{u_P \mid P \in \operatorname{Pr} A\}$ will be called a set of **principal generators** of $A$. Note that a set of principal generators is not unique in general.

*Lemma*

*Let $A$ be a reduced MV-algebra with principal generators $\{u_P \mid P \in \operatorname{Pr} A\}$. Then for every $a \neq 1$ in $A$ there is some principal generator $u_P$, $\tau(a) = \tau(u_P)$.*

# Reduced algebra are sufficient

### *Lemma*

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### *Proposition*

Any reduced MV-algebra is perfect.

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**Proposition**

Any reduced MV-algebra is perfect.

**Lemma**

$\operatorname{Spec} A_0 \cong \operatorname{Spec} A$

**Corollary**

*Let $X$ be a spectral space such that, for each $x \in X$, $\mathrm{cl}(x)$ is an upward chain under the specialisation order (i.e., if $y, z \in \mathrm{cl}(x)$, then $x \leq y \leq z$ or $x \leq z \leq y$) then there is a reduced MV-algebra $A_x$ such that $\operatorname{Spec} A_x \cong \mathrm{cl}(x)$.*

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**Lemma**

*In a reduced MV-algebra $A$ there is a bijection between the set of proper compact open sets of $\operatorname{Spec} A$ and $\operatorname{Pr} A$.*

# Three simple results and one remark

**Corollary**

*Let $X$ be a spectral space such that, for each $x \in X$, $\mathrm{cl}(x)$ is an upward chain under the specialisation order (i.e., if $y, z \in \mathrm{cl}(x)$, then $x \leq y \leq z$ or $x \leq z \leq y$) then there is a reduced MV-algebra $A_x$ such that $\mathrm{Spec}\, A_x \cong \mathrm{cl}(x)$.*

**Lemma**

*In a reduced MV-algebra $A$ there is a bijection between the set of proper compact open sets of $\mathrm{Spec}\, A$ and $\mathrm{Pr}\, A$.*

**Remark**

Note that, in a reduced algebra, $u_P \leq u_Q$ iff $\tau(u_P) \subseteq \tau(u_Q)$ iff $(u_P] \subseteq (u_Q]$.

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Starting from $X$ we seek for a construction that yields an MV-algebra $A$ such that $X$ and $\operatorname{Spec} A$ are homeomorphic. The theory of springs below gives a partial solution to this problem.

Given $X$ as above, let $\Omega$ be the set of its compact open subsets.
For any $x \in X$ we set $\Omega_x = \{\omega \in \Omega \mid x \in \omega\}$.

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*In a spectral space $X$, the intersection of a compact open set $U$ and a closed set $V$ is a compact open subset of $V$.*

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**Lemma**

*In a spectral space $X$, the intersection of a compact open set $U$ and a closed set $V$ is a compact open subset of $V$.*

Taking, in particular $V = \mathrm{cl}(x)$, this suggests an equivalence relation on $\Omega$ of being *indistinguishable over* $x$, namely

$$\omega \equiv_x \omega' \text{ if, and only if, } \omega \cap \mathrm{cl}(x) = \omega' \cap \mathrm{cl}(x).$$

This is an equivalence relation and, so let $[\omega]_x$ denote the class of $\omega$.

Then $[\omega]_x$ corresponds to a unique principal ideal $u_{[\omega]_x}$ in $A_x$ via the homeomorphism between $\operatorname{Spec} A_x$ and $\operatorname{cl}(x)$.

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The correspondence is (strictly) order preserving.
Observe that, if $x \notin \omega$, then $\omega \cap \mathrm{cl}(x) = \varnothing$, so we may limit ourselves to $\omega \in \Omega_x$.

*Definition*

A triple $\langle X, \{A_x\}_{x\in X}, A\rangle$ is an **MV-spring** provided $X$ is a spectral space, each $A_x$ is a reduced MV-algebra and $A$ is a subdirect product of the family of $A_x$'s

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*Example*

Let $A_i$ be a family of reduced MV-algebras and let $A$ be a subdirect product of the $A_i$. Then $\langle \operatorname{Spec} A, \{A_i\}_{i \in I}, A\rangle$ is an MV-spring.

Let $\Omega$ as above and $\mathfrak{F}$ be the free MV-algebra generated by $\Omega$.

Let $\chi_\omega \in \mathfrak{F}^X$ be defined by $\chi_\omega(x) = \begin{cases} \omega & \text{if } x \in \omega \\ 0 & \text{otherwise} \end{cases}$

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Let $A_1 = \langle \chi_\omega \mid \omega \in \Omega \rangle$ be the subalgebra of $\mathfrak{F}^X$ generated by the $\chi_\omega$, and, for each $x \in X$, let $\mathbf{F}_x = \langle u_{[\omega]_x} \mid \omega \in \Omega_x \rangle$.

Fix an $x \in X$, the algebra $A_1$ can be projected into $\mathbf{F}_x$ by the following function.

$$\mu_x : A_1 \overset{\mathsf{ev}_x}{\twoheadrightarrow} \mathfrak{F}_x \overset{\eta_x}{\twoheadrightarrow} \mathbf{F}_x,$$

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Now consider $J_1 = \bigcap_x \ker \mu_x$ and define $\hat{A} = A_1/J_1$.

*Proposition*

The triple $\langle X, \{\mathbf{F}_x\}_{x \in X}, \hat{A} \rangle$ is a spring.

*Proposition*

The triple $\langle X, \{\mathbf{F}_x\}_{x\in X}, \hat{A}\rangle$ is a spring.

Given a spring $\langle X, \{A_x\}_{x\in X}, A\rangle$, we have a family of projections $\pi_x \ : \ A \longrightarrow A_x$ and we can define a new map

$$\varphi_A : X \longleftrightarrow \operatorname{Spec} A \qquad \text{by setting} \qquad \varphi_A(x) = \ker \pi_x .$$

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*Definition*

An MV-spring $\langle X, \{A_x\}_{x \in X}, A \rangle$ will be called an **affine MV-spring** provided $\varphi_A$ is continuous.

# Dense subset of Spec $A$

Given the MV-spring above, $\langle X, \{\mathbf{F}_x\}_{x \in X}, \hat{A} \rangle$, we will write $\varphi$ for $\varphi_{\hat{A}}$; so we have the map $\varphi : X \longleftrightarrow \operatorname{Spec} \hat{A}$ given by $\varphi(x) = \ker \pi_x$.

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**Theorem**

*In $\langle X, \{\mathbf{F}_x\}_{x\in X}, \hat{A}\rangle$ the following properties hold.*

(i) *$\varphi$ is injective;*

(ii) *$\varphi^{-1} : \varphi(X) \longrightarrow X$ is continuous;*

(iii) *$\varphi^{-1}$ is order preserving;*

(iv) *$\varphi(X)$ is a dense subspace of $\operatorname{Spec}\hat{A}$.*

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