Topological groupoid quantales: a non étale setting

Alessandra Palmigiano, Riccardo Re

10 June 2010

A Stone-type setting

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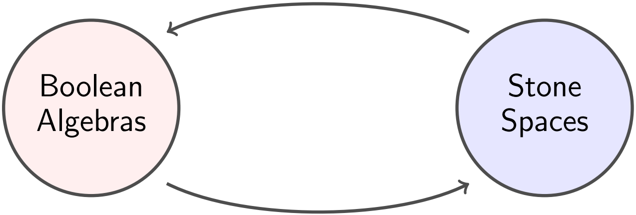

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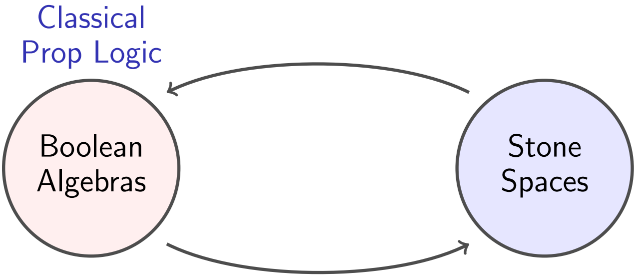
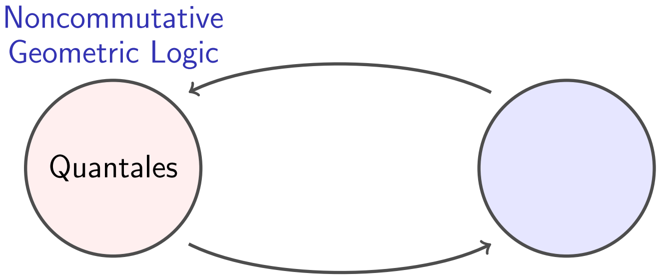
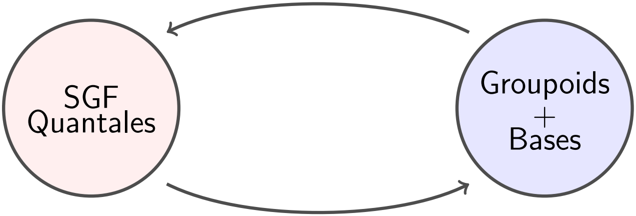



Noncommutative
Geometric Logic
Quantales

A Stone-type setting

A duality on objects:



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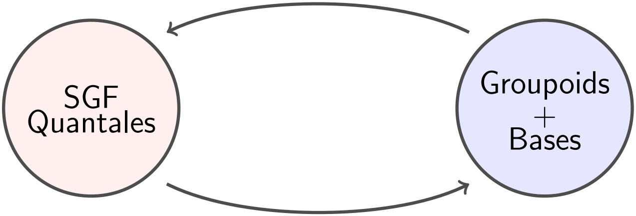

SGF-axioms: 1–3

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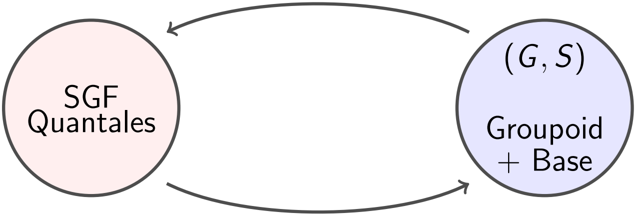

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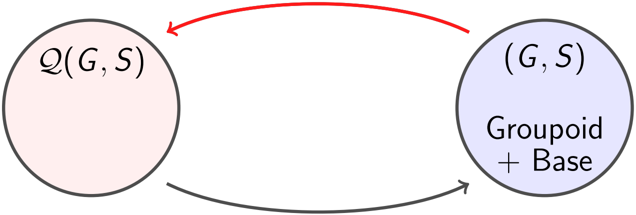

Topological groupoid quantale

$\mathcal{Q}(G,S)$ is SGF

(soundness)

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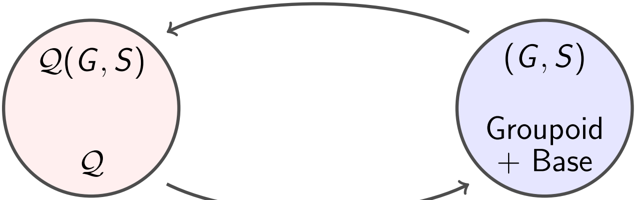

SGF
+ 2 spatiality axioms
(sound as well)

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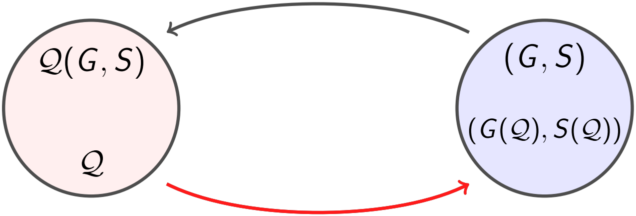

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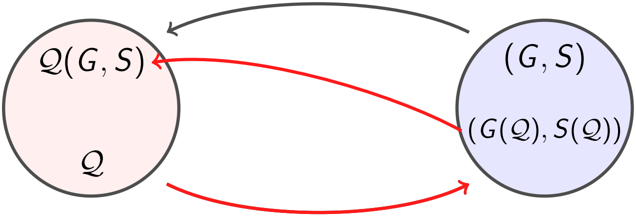

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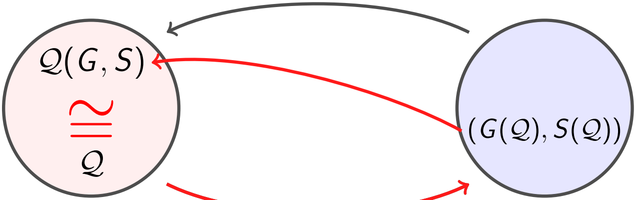

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+ 2 spatiality axioms
(completeness)

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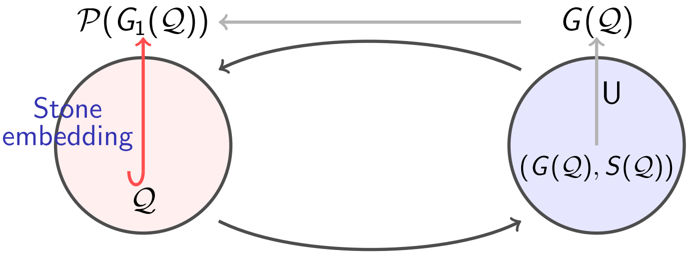
Similarities with Jónsson-Tarski'52:



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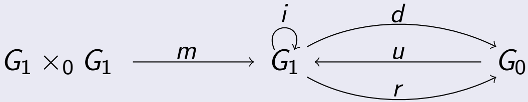
Set Groupoids are tuples

$$G = (G_0, G_1, m, d, r, u, i)$$

s.t. G_0 and G_1 are sets, and:



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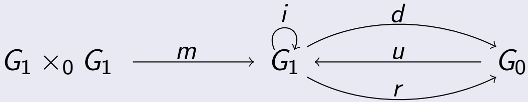

Topological Groupoids: Groupoids in Top.

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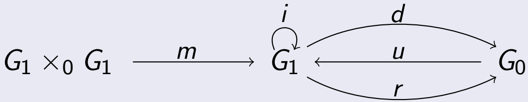

Topological Groupoids: Groupoids in Top.

Our setting is intermediate:

Set groupoids such that G_0 is a sober space.

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Fact: G-sets naturally form a (unital) inverse semigroup.

Groupoids and equivalence relations

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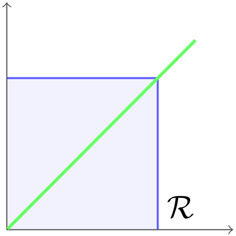

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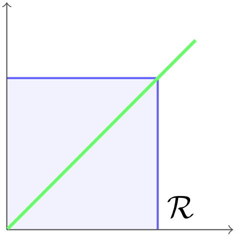

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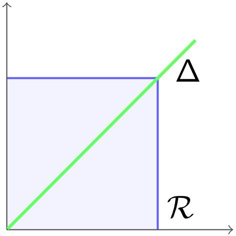

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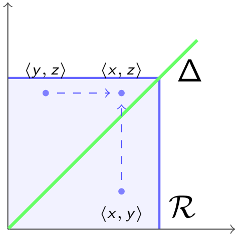

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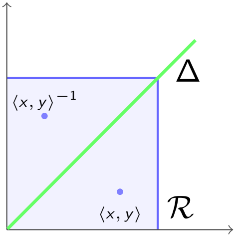

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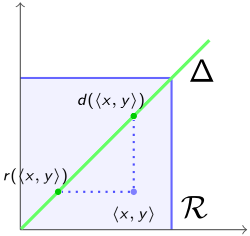

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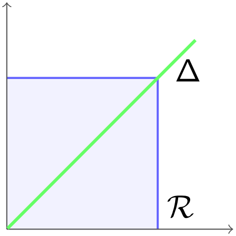

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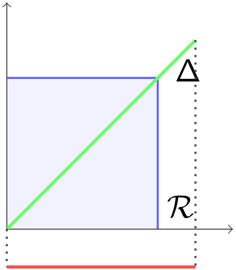

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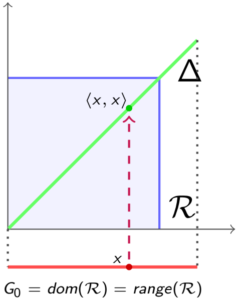

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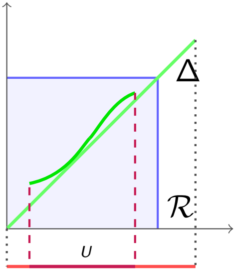

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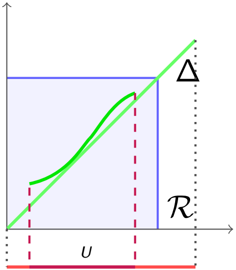

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A restriction of our setting:

G_1 is covered by G-sets.

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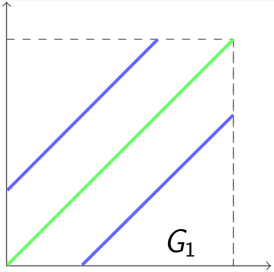

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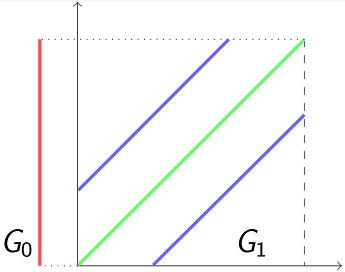

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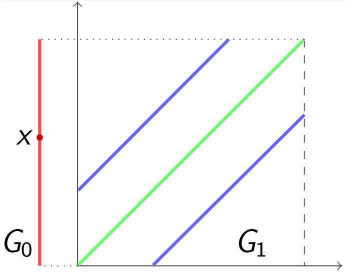

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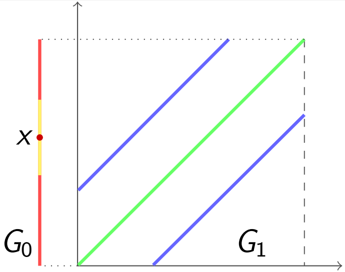

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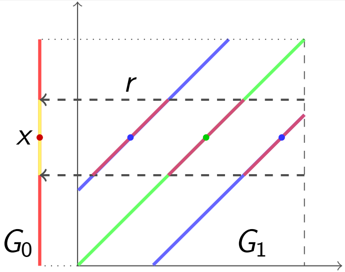

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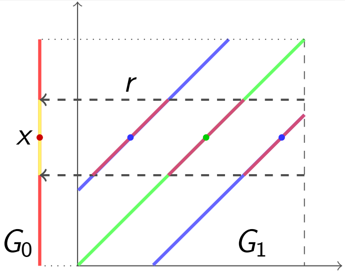

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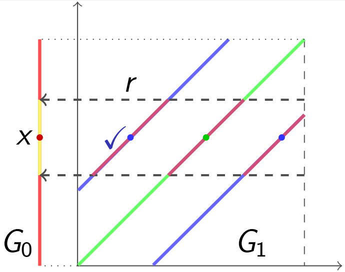

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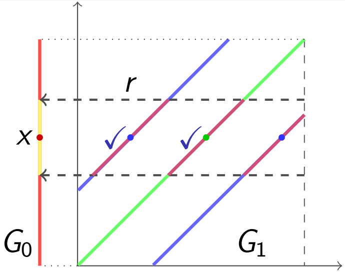

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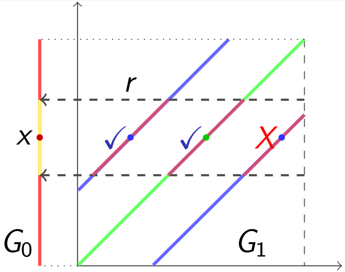

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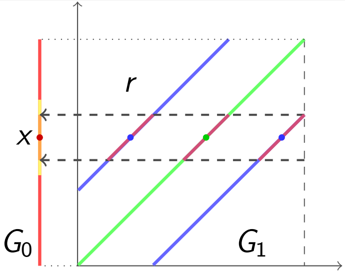

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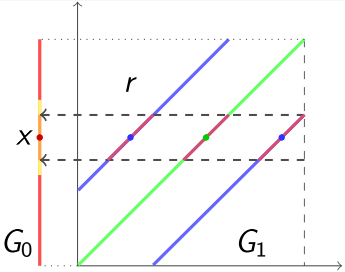

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- if G is étale, the G-sets form a base for the topology of G_1.
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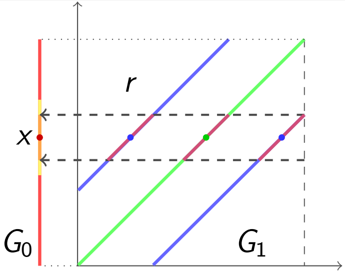

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If $\mathcal{R}$ étale, partial homeom's of Δ can only intersect over opens.

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Selection bases: in general not topological bases

Example: non étale groupoid

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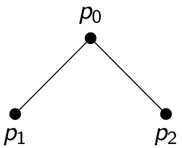

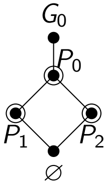

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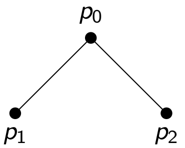

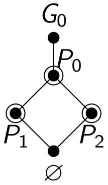

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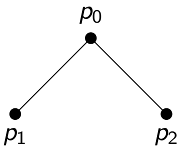

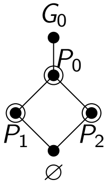

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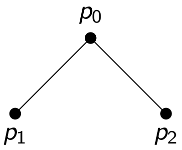

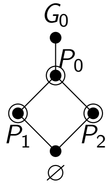

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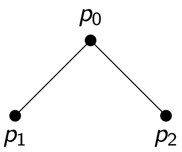

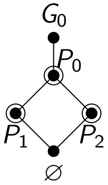

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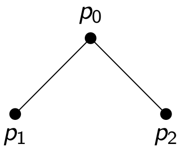

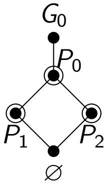

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($\{p_0\}$ closed not open)

Unital involutive quantales

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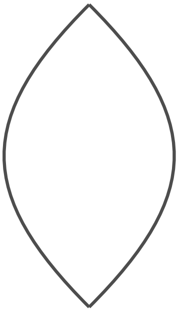

$\mathcal{Q} = (Q, \bigvee, \cdot)$
quantale

Quantales: complete $\bigvee$-semilattices

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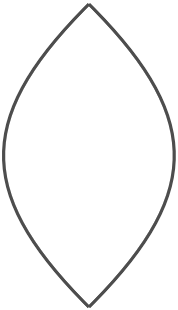

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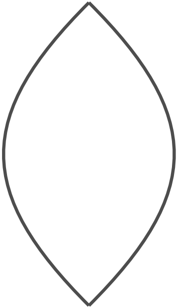

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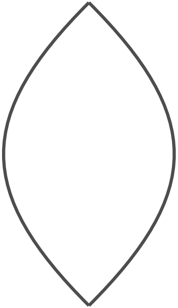

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Every quantale is a complete (non distributive) lattice

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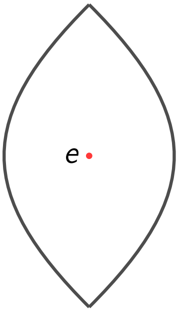

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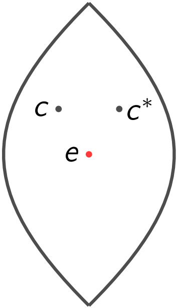

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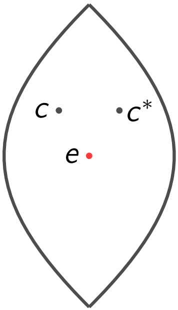

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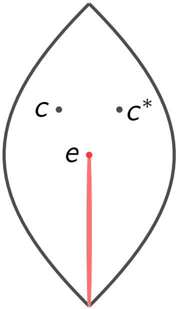
For every groupoid G, $(\mathcal{P}(G_1), \bigcup, \cdot, ^{-1}, E)$ is a unital invol. quantale.

SGF quantales and Topological Groupoid Quantales



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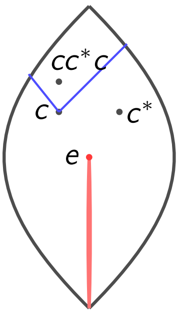

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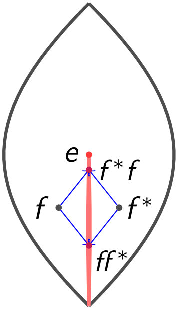

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$\mathcal{I}(\mathcal{Q})$: functional invertible elements

$ff^* \leq e \quad f^*f \leq e$

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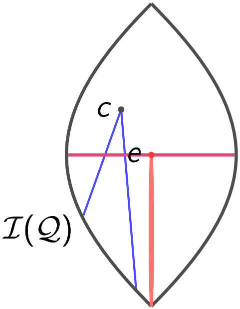

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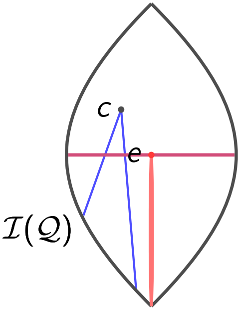

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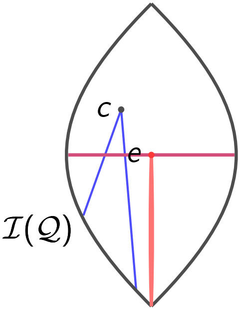

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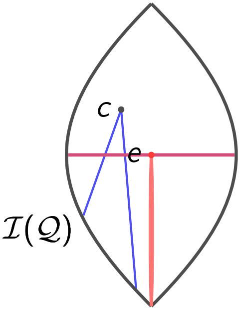

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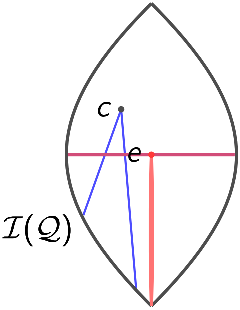

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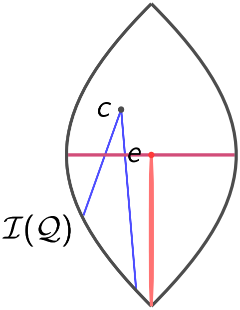

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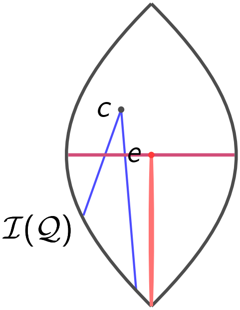

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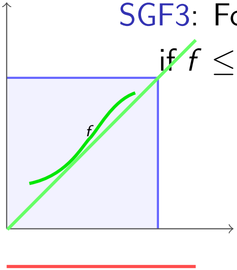

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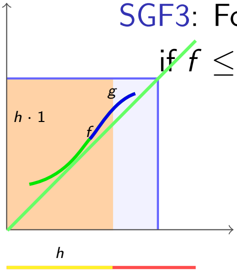

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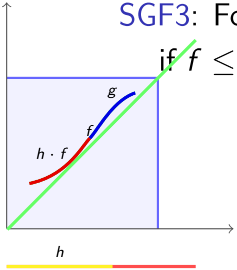

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$$d([p, f]) = p, \quad r([p, f]) = f[p], \quad u(p) = [p, e],$$

$$[p, f][q, g] = [p, fg] \quad \text{only if} \quad q = f[p]$$

$$[p, f]^{-1} = [f[p], f^*].$$

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Prop: If $\mathcal{Q}$ spatial, then $\mathcal{Q}_e$ spatial frame.

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$$\mathcal{Q} \cong \mathcal{Q}(G(\mathcal{Q}), \mathcal{I}(\mathcal{Q}))$$