Coalgebras over Stone spaces and canonical models

Clemens Kupke

Imperial College London

Tbilisi, 10 June 2010

Preliminaries

Structure

- coalgebras over Stone spaces
- final coalgebras and the Hennessy-Milner property
- simulations

Preliminaries

Structure

- coalgebras over Stone spaces
- final coalgebras and the Hennessy-Milner property
- simulations

Acknowledgement

Helle Hvid Hansen (TU Eindhoven), Raul Leal (University of Amsterdam), Alexander Kurz and Yde Venema.

Coalgebra

Definiton

Let C be a category and $T : C \to C$ be a functor. A T-coalgebra is a pair (X, γ) such that

$$\gamma : X \longrightarrow TX \ \in C.$$

Note

In this talk only coalgebras over a concrete base category C will appear, ie., there is a forgetful functor $U : C \to Set$.

Coalgebra and modal logic

Definiton

Let C be a category and $T : C \to C$ be a functor. A T-coalgebra is a pair (X, γ) such that

$$\gamma : X \longrightarrow TX \ \in C.$$

Note

In this talk only coalgebras over a concrete base category C will appear, ie., there is a forgetful functor $U : C \to \mathrm{Set}$.

Connection to ML

Kripke frames are $\mathcal{P}$-coalgebras, (monotone) neighbourhood frames are coalgebras, discrete Markov chains, etc.

Bounded morphisms - coalgebraically

- bounded morphisms $\leftrightarrow$ T-coalgebra morphisms:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \gamma \downarrow & & \downarrow \delta \\ TX & \xrightarrow[Tf]{} & TY \end{array}$$

  Coalg(T): category of T-coalgebras and T-coalgebra morphisms

Bounded morphisms - coalgebraically

- bounded morphisms $\leftrightarrow$ T-coalgebra morphisms:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ TX & \xrightarrow[Tf]{} & TY \end{array}$$

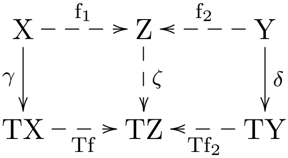
  Coalg(T): category of T-coalgebras and T-coalgebra morphisms

- For T-coalgebras (X, γ) and (Y, δ) we say $x \in X$ and $y \in Y$ are behaviourally equivalent ($x \leftrightarroweq_T y$) if there exists a (third) T-coalgebra (Z, ζ) and T-coalgebra morphisms

$$f_1 : (X, \gamma) \to (Z, \zeta) \quad \text{and} \quad f_2 : (Y, \delta) \to (Z, \zeta)$$

  such that $f_1(x) = f_2(y)$.

Behavioural equivalence: diagram



Remark

Note that there is also a coalgebraic notion of “T-bisimulation”. This notion, however, is not always well-behaved.

Monotone neighbourhood functor

Define

$$
\begin{aligned}
M : \mathrm{Set} &\rightarrow \mathrm{Set} \\
X &\mapsto MX := \{N \subseteq \mathcal{P}(X) \mid V \text{ is upwards-closed.}\} \\
f : X \rightarrow Y &\mapsto Mf : MX \rightarrow MY \\
&\qquad \text{with } Mf(N) := \{V \subseteq Y \mid f^{-1}(V) \in X\}
\end{aligned}
$$

Fact

Coalg(M) is the category of monotone neighbourhood frames

Neighbourhood functor

Define

$$\begin{array}{rcl} 2^2 : \mathrm{Set} & \rightarrow & \mathrm{Set} \\ X & \mapsto & 2^2X := \{N \mid N \subseteq \mathcal{P}(X)\} \\ f : X \to Y & \mapsto & 2^2 f : 2^2 X \to 2^2 Y \\ & & \quad \text{with } 2^2 f(N) := \{V \subseteq Y \mid f^{-1}(V) \in X\} \end{array}$$

Fact

$\mathrm{Coalg}(2^2)$ is the category of neighbourhood frames.

Behavioural equivalence

Theorem(Pauly)

Monotone modal logic is the $\leftrightarrows_{\mathrm{M}}$-invariant fragment of first-order logic.

Theorem(Hansen,K)

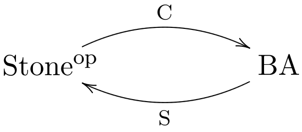
Classical modal logic is the $\leftrightarrows_{2^2}$-invariant fragment of first-order logic.

- generalized by Schröder & Pattinson to coalgebraic modal logic for any functor $\mathrm{T} : \mathrm{Set} \to \mathrm{Set}$,
- closely related to a similar result by ten Cate/Gabelaia/Sustretov on modal logic over topological spaces

The category Stone

In the following we will consider coalgebras over Stone, ie., the category of Stone spaces and continuous functions.

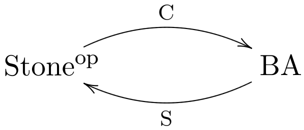
It is well-known that Stone is dually equivalent to BA the category of Boolean algebras and homomorphisms:



The category Stone

In the following we will consider coalgebras over Stone, ie., the category of Stone spaces and continuous functions.

It is well-known that Stone is dually equivalent to BA the category of Boolean algebras and homomorphisms:



Consequence

$\mathrm{Alg}(\mathrm{H}) \cong \mathrm{Coalg}(\mathrm{V}^{\mathrm{op}})^{\mathrm{op}}$ with $\mathrm{H} : \mathrm{BA} \to \mathrm{BA}$ some functor and $\mathrm{V}^{\mathrm{op}} : \mathrm{Stone} \to \mathrm{Stone}$ defined by $\mathrm{V} := \mathrm{S} \circ \mathrm{H} \circ \mathrm{C}$.

Modal algebras

Recall

A modal algebra is a pair $\mathcal{A} = (\mathbb{A}, \mathrm{f})$, where $\mathbb{A}$ is a Boolean algebra and $\mathrm{f} : \mathbb{A} \to \mathbb{A}$ is a unary operation such that

- $\mathrm{f}(1) = 1$, and
- $\mathrm{f}(a \wedge b) = \mathrm{f}(a) \wedge \mathrm{f}(b)$.

More categorically

A modal algebra is an algebra for the functor

$$\mathrm{L_K} : \mathrm{BA} \to \mathrm{BA}$$

where $\mathrm{L_K}$ is the functor that maps a Boolean algebra $\mathbb{A}$ to the free Boolean algebra over the meet semilattice underlying $\mathbb{A}$.

Stone coalgebras

Using Stone duality it follows that there is a functor

$$\mathbb{V} : \text{Stone} \to \text{Stone}$$

such that $\text{Coalg}(\mathbb{V})$ is dually equivalent to the category of modal algebras.

Vietoris on Stone

Definiiton

The Vietoris functor $\mathbb{V} : \mathsf{Stone} \to \mathsf{Stone}$ is defined as follows:

$$\begin{aligned} \mathbb{V} : \mathsf{Stone} &\to \mathsf{Stone} \\ \mathbb{X} &\mapsto (\mathrm{K}(\mathbb{X}), \tau_{\mathbb{V}}) \end{aligned}$$

with $\tau_{\mathbb{V}}$ being the topology on $\mathrm{K}(\mathbb{X})$ that is generated by the sets

$$\begin{aligned} [\ni]\mathrm{a} &:= \{\mathrm{F} \in \mathrm{K}(\mathbb{X}) \mid \mathrm{F} \subseteq \mathrm{a}\} \\ \langle\ni\rangle\mathrm{a} &:= \{\mathrm{F} \in \mathrm{K}(\mathbb{X}) \mid \mathrm{F} \cap \mathrm{A} \neq \emptyset\}. \end{aligned}$$

where $\mathrm{a} \in \mathrm{Clp}(\mathbb{X})$.

Summary on Vietoris

Facts

- Coalg($\mathbb{V}$) is dually equivalent to the category of modal algebras

Summary on Vietoris

Facts

- $\mathrm{Coalg}(\mathbb{V})$ is dually equivalent to the category of modal algebras
- $\mathbb{V}$-coalgebras are in fact the well-known descriptive general frames

Summary on Vietoris

Facts

- Coalg($\mathbb{V}$) is dually equivalent to the category of modal algebras
- $\mathbb{V}$-coalgebras are in fact the well-known descriptive general frames
- In other words: Coalg($\mathbb{V}$) is isomorphic to the category of descriptive general frames

Question

What about the algebraic semantics of other modal logics such as classical (or minimal) modal logic?

Descriptive monotone neighbourhood frames

Define

$$
\begin{array}{rcl}
\mathbb{M} : \mathrm{Stone} & \to & \mathrm{Stone} \\
\mathbb{X} & \mapsto & (\{V \mid V \subseteq \mathrm{Clp}(\mathbb{X}) \text{ u.c.}\}, \tau_{\mathbb{M}}) \\
f : \mathbb{X} \to \mathbb{Y} & \mapsto & \mathbb{M}f : \mathbb{M}\mathbb{X} \to \mathbb{M}\mathbb{Y} \\
& \text{where} & \mathbb{M}f(N) := \{a \in \mathrm{Clp}(\mathbb{Y}) \mid f^{-1}(a) \in N\},
\end{array}
$$

Descriptive monotone neighbourhood frames

Define

$$\begin{array}{rcl} \mathbb{M} : \mathrm{Stone} & \to & \mathrm{Stone} \\ \mathbb{X} & \mapsto & (\{V \mid V \subseteq \mathrm{Clp}(\mathbb{X}) \text{ u.c.}\}, \tau_{\mathbb{M}}) \\ f : \mathbb{X} \to \mathbb{Y} & \mapsto & \mathbb{M}f : \mathbb{M}\mathbb{X} \to \mathbb{M}\mathbb{Y} \\ & \text{where} & \mathbb{M}f(N) := \{a \in \mathrm{Clp}(\mathbb{Y}) \mid f^{-1}(a) \in N\}, \end{array}$$

where $\tau_{\mathbb{M}}$ is the topology generate by the sets

$$\begin{array}{rcl} [\nu_m](a) & := & \{V \in \mathbb{M}\mathbb{X} \mid a \in V\} \\ \langle \nu_m \rangle(a) & := & \{V \in \mathbb{M}\mathbb{X} \mid -a \notin V\} \end{array}$$

for $a \in \mathrm{Clp}(\mathbb{X})$.

Descriptive neighbourhood frames

Define

$$\begin{array}{rcl} \mathbb{N} : \mathrm{Stone} & \to & \mathrm{Stone} \\ \mathbb{X} & \mapsto & (\{V \mid V \subseteq \mathrm{Clp}(\mathbb{X})\}, \tau_{\mathbb{N}}) \\ f : \mathbb{X} \to \mathbb{Y} & \mapsto & \mathbb{N}f : \mathbb{N}\mathbb{X} \to \mathbb{N}\mathbb{Y} \\ & \text{where} & \mathbb{N}f(N) := \{a \in \mathrm{Clp}(\mathbb{Y}) \mid f^{-1}(a) \in N\}, \end{array}$$

Descriptive neighbourhood frames

Define

$$\begin{array}{rcl} \mathbb{N} : \mathrm{Stone} & \to & \mathrm{Stone} \\ \mathbb{X} & \mapsto & (\{V \mid V \subseteq \mathrm{Clp}(\mathbb{X})\}, \tau_{\mathbb{N}}) \\ f : \mathbb{X} \to \mathbb{Y} & \mapsto & \mathbb{N}f : \mathbb{N}\mathbb{X} \to \mathbb{N}\mathbb{Y} \\ & \text{where} & \mathbb{N}f(N) := \{a \in \mathrm{Clp}(\mathbb{Y}) \mid f^{-1}(a) \in N\}, \end{array}$$

where $\tau_{\mathbb{N}}$ is the topology generate by the sets

$$\begin{array}{rcl} [\nu](a) & := & \{V \in \mathbb{N}\mathbb{X} \mid a \in V\} \\ \langle\nu\rangle(a) & := & \{V \in \mathbb{N}\mathbb{X} \mid -a \notin V\} \end{array}$$

for $a \in \mathrm{Clp}(\mathbb{X})$.

Dualities

Proposition

$$\mathrm{Coalg}(\mathbb{M})^{\mathrm{op}} \cong \mathrm{BAM},$$

where BAM is the category of Boolean algebras with a monotone (unary) operator.

Dualities

Proposition

$$\mathrm{Coalg}(\mathbb{M})^{\mathrm{op}} \cong \mathrm{BAM},$$

where BAM is the category of Boolean algebras with a monotone (unary) operator.

Proposition

$$\mathrm{Coalg}(\mathbb{N})^{\mathrm{op}} \cong \mathrm{BAE},$$

where BAE is the category of Boolean algebra extensions (with a unary operator).

(Coalgebraic) Semantics of modal logic

Language

$$\mathcal{L} \ni \varphi ::= \bot \mid \mathrm{p} \in \mathrm{Prop} \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Box\varphi.$$

Semantics (modalities)

Given $\mathrm{T} : \mathrm{Stone} \to \mathrm{Stone}$ (we think of $\mathrm{T} \in \{\mathbb{V}, \mathbb{M}, \mathbb{N}\}$), define

$$[\![\Box]\!] : \mathrm{C} \Rightarrow \mathrm{C} \circ \mathrm{T}.$$

In our examples:

T	$[\![\Box]\!]_{\mathbb{X}}(\mathrm{a})$
$\mathbb{V}$	$[\ni](\mathrm{a})$
$\mathbb{M}$	$[\nu_{\mathrm{m}}](\mathrm{a})$
$\mathbb{N}$	$[\nu](\mathrm{a})$

Semantics (formulas)

A T-model $\mathfrak{M} = (\mathbb{X}, \gamma, \mathrm{h})$ is a T-coalgebra $(\mathbb{X}, \gamma)$ together with a valuation $\mathrm{h} : \mathbb{X} \to \prod_{\mathrm{p} \in \mathrm{Prop}} 2$. We define

$$
\begin{aligned}
[\![\bot]\!]_{\mathfrak{M}} &:= \emptyset \\
[\![\mathrm{p}]\!]_{\mathfrak{M}} &:= \{\mathrm{x} \mid \pi_{\mathrm{p}}(\mathrm{h}(\mathrm{x})) = 1\} \\
[\![\varphi_1 \wedge \varphi_2]\!]_{\mathfrak{M}} &:= [\![\varphi_1]\!]_{\mathfrak{M}} \cap [\![\varphi_2]\!]_{\mathfrak{M}} \\
[\![\neg\varphi]\!]_{\mathfrak{M}} &:= -[\![\varphi]\!]_{\mathfrak{M}} \\
[\![\Box\varphi]\!]_{\mathfrak{M}} &:= [\![\Box]\!]_{\mathbb{X}}([\![\varphi]\!]_{\mathfrak{M}})
\end{aligned}
$$

Obviously $[\![\varphi]\!]_{\mathfrak{M}} \in \mathrm{Clp}\mathbb{X}$ for all $\varphi \in \mathcal{L}$.

Final Coalgebras

Definition

A T-coalgebra $(\mathbb{Y}, \upsilon)$ is called final if for all T-coalgebras $(\mathbb{X}, \gamma)$ there is a unique coalgebra morphism

$$\begin{array}{ccc} \mathbb{X} & \stackrel{!}{\dashrightarrow} & \mathbb{Y} \\ \gamma \downarrow & & \downarrow \upsilon \\ \mathrm{T}\mathbb{X} & \underset{\mathrm{T}!}{\dashrightarrow} & \mathrm{T}\mathbb{Y} \end{array}$$

Final Coalgebras $\leftrightarrow$ canonical models

Fact

Let $\mathrm{T} = \mathbb{V}/\mathbb{M}/\mathbb{N}$ and let Prop be a set (of propositional variables). Then the final coalgebra for the functor

$$\mathrm{T} \times \prod_{\mathrm{p} \in \mathrm{Prop}} 2$$

is the categorical dual of the free algebra in BAO/BAM/BAE over the set Prop.

In other words, we can represent canonical models as final coalgebras.

Next

Final Coalgebras via Logic

Theorem (Goldblatt, KL)

For any functor $T : \mathrm{Set} \to \mathrm{Set}$, there exists a final T-coalgebra iff there exists an adequate language for T coalgebras with the Hennessy-Milner property.

Reference

R. Goldblatt, Final coalgebras and the Hennessy-Milner property., Annals of Pure and Applied Logic 138 (2006), no. 1-3, 77–93.

In this talk I will use our simplification of Goldblatt's proof to show the analogue for functors $T : \mathrm{Stone} \to \mathrm{Stone}$.

Final Coalgebras and the Hennessy-Milner property

Definition

Let $\mathrm{T} : \mathsf{Stone} \to \mathsf{Stone}$ be a functor. An abstract language for T is a pair

$$\mathrm{L} = (\mathcal{L}, \{\mathrm{Th}_{(\mathbb{X},\gamma)}\}_{(\mathbb{X},\gamma)\in\mathrm{Coalg(T)}})$$

with $\mathcal{L} \in \mathrm{BA}$ and for all $(\mathbb{X},\gamma) \in \mathrm{Coalg(T)}$ we have

$$\mathrm{Th}_{(\mathbb{X},\gamma)} : \mathbb{X} \to \mathrm{S}\mathcal{L} \in \mathsf{Stone}.$$

Abstract Languages

Let $\mathrm{T} : \mathsf{Stone} \to \mathsf{Stone}$ be a functor. Think of a language for T-coalgebras as a set $\mathcal{L}$ (of formulas) together with a semantic map

$$[\![\cdot]\!]_{(\mathbb{X},\gamma)} : \mathcal{L} \to \mathrm{C}\mathbb{X} \quad \text{for each } (\mathbb{X},\gamma) \in \mathsf{Coalg}(\mathrm{T}).$$

Abstract Languages

Let $\mathrm{T} : \mathrm{Stone} \to \mathrm{Stone}$ be a functor. Think of a language for T-coalgebras as a set $\mathcal{L}$ (of formulas) together with a semantic map

$$[\![\cdot]\!]_{(\mathbb{X},\gamma)} : \mathcal{L} \to \mathrm{C}\mathbb{X} \quad \text{for each } (\mathbb{X},\gamma) \in \mathrm{Coalg}(\mathrm{T}).$$

We extend $[\![\cdot]\!]$ to the free Boolean algebra over $\mathcal{L}$:

$$\widehat{[\![\cdot]\!]} : \mathrm{F}_{\mathrm{BA}}(\mathcal{L}) \to \mathrm{C}\mathbb{X} \in \mathrm{BA}$$

and let $\mathrm{Th}_{(\mathbb{X},\gamma)} : \mathbb{X} \to \mathrm{SF}_{\mathrm{BA}}(\mathcal{L})$ be the dual map.

$(\mathrm{F}_{\mathrm{BA}}(\mathcal{L}), \{\mathrm{Th}_{(\mathbb{X},\gamma)}\}_{(\mathbb{X},\gamma)})$ is an abstract language for T.

HM property

Definition

We say L is adequate if for all T-coalgebras $(\mathbb{X}, \gamma)$ and $(\mathbb{Y}, \delta)$ and all states $x \in X$, $y \in Y$ we have

$$\mathrm{Th}_{(\mathbb{X},\gamma)}(x) = \mathrm{Th}_{(\mathbb{Y},\delta)}(y) \quad \text{if} \quad x \leftrightarroweq_T y.$$

HM property

Definition

We say L is adequate if for all T-coalgebras $(\mathbb{X}, \gamma)$ and $(\mathbb{Y}, \delta)$ and all states $x \in X$, $y \in Y$ we have

$$\mathrm{Th}_{(\mathbb{X},\gamma)}(x) = \mathrm{Th}_{(\mathbb{Y},\delta)}(y) \quad \text{if} \quad x \leftrightarroweq_{T} y.$$

Definition

We say L has the Hennessy-Milner property (HM property) if for all T-coalgebras $(\mathbb{X}, \gamma)$ and $(\mathbb{Y}, \delta)$ and all states $x \in X$, $y \in Y$ we have

$$\mathrm{Th}_{(\mathbb{X},\gamma)}(x) = \mathrm{Th}_{(\mathbb{Y},\delta)}(y) \quad \text{implies} \quad x \leftrightarroweq_{T} y.$$

Theorem

Theorem

Let $T : \mathrm{Stone} \to \mathrm{Stone}$ be a functor and let L be an adequate language for T. The following are equivalent:

1. L has the HM property,
2. the set (of all satisfiable theories)

$$Y := \{u \in S\mathcal{L} \mid \exists\, (\mathbb{X}_u, \gamma_u) \in \mathrm{Coalg}(T)\ \exists x_u \in X \\ \mathrm{Th}_{(\mathbb{X}_u, \gamma_u)}(x_u) = u\}$$

   is the carrier of the final T-coalgebra.

Proof

$1 \Rightarrow 2$: Let L be a language for T with the HM property. We put

$$Y := \{u \in S\mathcal{L} \mid \exists(\mathbb{X}_u, \gamma_u)\ \exists x \in X\ (Th_{(\mathbb{X}_u,\gamma_u)}(x) = u)\}.$$

Proof

$1 \Rightarrow 2$: Let L be a language for T with the HM property. We put

$$Y := \{u \in S\mathcal{L} \mid \exists(\mathbb{X}_u, \gamma_u)\ \exists x \in X\ (Th_{(\mathbb{X}_u,\gamma_u)}(x) = u)\}.$$

Y is the image of the theory map of $\coprod(\mathbb{X}_u, \gamma_u)$ and thus a closed subset of $S\mathcal{L}$.

Proof

1 ⇒ 2: Let L be a language for T with the HM property. We put

$$Y := \{u \in S\mathcal{L} \mid \exists(\mathbb{X}_u, \gamma_u)\ \exists x \in X\ (Th_{(\mathbb{X}_u,\gamma_u)}(x) = u)\}.$$

Y is the image of the theory map of $\coprod(\mathbb{X}_u, \gamma_u)$ and thus a closed subset of $S\mathcal{L}$.

Therefore $\mathbb{Y} := (Y, \tau_Y)$ is a Stone space, where τ_Y is the subspace topology and the maps $Th_{(\mathbb{X},\gamma)}$ restrict to continuous functions $!_{(\mathbb{X},\gamma)} : \mathbb{X} \to \mathbb{Y}$.

Proof (continued)

Define a function $\upsilon : \mathbb{Y} \to \mathrm{T}\mathbb{Y}$ by putting $\upsilon(\mathrm{y}) = \mathrm{t}$ if there exists some T-coalgebra $(\mathbb{X}, \gamma)$ and some $\mathrm{x} \in \mathrm{X}$ with $!_{(\mathbb{X},\tau)}(\mathrm{x}) = \mathrm{u}$ and $\mathrm{T}!_{(\mathbb{X},\gamma)}(\gamma(\mathrm{x})) = \mathrm{t}$:

$$
\begin{array}{ccc}
\mathbb{X} & \xrightarrow{\;!_{(\mathbb{X},\gamma)}\;} & \mathbb{Y} \\
\downarrow & & \downarrow \\
\mathrm{T}\mathbb{X} & \xrightarrow[\mathrm{T}!_{(\mathbb{X},\gamma)}]{} & \mathrm{T}\mathbb{Y}
\end{array}
$$

Proof(continued).

- v is well-defined and continuous: Well-definedness follows from adequacy and HM property. Continuity of v can be checked by chasing the diagram for $!_{\coprod(\mathbb{X}_u, \gamma_u)}$.
- It follows that $(\mathbb{Y}, v)$ is the final T-coalgebra.

□

Proof(continued).

- υ is well-defined and continuous: Well-definedness follows from adequacy and HM property. Continuity of υ can be checked by chasing the diagram for $!_{\coprod(\mathbb{X}_u,\gamma_u)}$.
- It follows that $(\mathbb{Y}, \upsilon)$ is the final T-coalgebra.

□

Corollary

A functor T has a final coalgebra iff there exists an abstract language L for T that is adequate and that has the HM property w.r.t. T-coalgebras.

Therefore

Facts

- modal logic is adequate and has the HM property w.r.t. $(\mathbb{V} \times \prod_{p \in \mathrm{Prop}} 2)$-coalgebras,
- monotone modal logic is adequate and has the HM property w.r.t. $(\mathbb{M} \times \prod_{p \in \mathrm{Prop}} 2)$-coalgebras, and
- classical modal logic is adequate and has the HM property w.r.t. $(\mathbb{N} \times \prod_{p \in \mathrm{Prop}} 2)$-coalgebras,

Consequence

The functors $\mathbb{V}, \mathbb{M}$ and $\mathbb{N}$ have final coalgebras.

Regularly algebraic over Set

Definition

A concrete category C is regulary algebraic if the forgetful functor $U : C \to Set$

- has a left adjoint and
- creates regular factorizations

Regularly algebraic over Set

Definition

A concrete category C is regulary algebraic if the forgetful functor $U : C \to Set$

- has a left adjoint and
- creates regular factorizations

Example

- any category monadic over Set
- category of Stone spaces

Generalization

Theorem

Let C be a category that is regularly algebraic over Set with forgetful functor $U : C \to Set$ and let $T : C \to C$ be a functor. The functor T has a final coalgebra iff there exists an "adequate object" $\mathcal{L}$ for T-coalgebras that has the Hennessy-Milner property.

Problematic: notion of an abstract language

Definition

Let T be a functor $T : C \to C$. An object $\mathcal{L}$, in C is an adequate object for T-coalgebras if there exists a natural transformation

$$\mathrm{Th} : U \to \Delta_{\mathcal{L}},$$

where $U : \mathrm{Coalg}(T) \to C$ is the forgetful functor and $\Delta_{\mathcal{L}} : \mathrm{Coalg}(T) \to C$ is the constant functor with value $\mathcal{L}$.

Stone coalgebras & Simulations

- only a very small observation
- Idea: simulate a (non-normal) modal logic by transforming their (general) frames into "polynomial Vietoris-coalgbras"
- possible pay-off: simpler simulations

Descriptive monotone neighbourhood frames (again)

Definition

A set $N \subseteq \mathbb{V}\mathbb{X}$ is called $[\ni]$-closed if for all $F \in K\mathbb{X}$ we have

$$F \in N \text{ if for all } a \in \text{Clp}\mathbb{X}\,(F \subseteq a \rightarrow a \in N).$$

Descriptive monotone neighbourhood frames (again)

Definition

A set $N \subseteq \mathbb{V}\mathbb{X}$ is called $[\ni]$-closed if for all $F \in K\mathbb{X}$ we have

$$F \in N \text{ if for all } \ a \in \mathrm{Clp}\mathbb{X}\, (F \subseteq a \rightarrow a \in N).$$

For a Stone space $\mathbb{X}$ we put

$$\mathrm{Up}\mathbb{V}\,\mathbb{X} := \{N \subseteq \mathbb{V}\mathbb{X} \mid N \text{ u.c. } \& \ N \text{ is } [\ni]\text{-closed}\},$$

and for $f : \mathbb{X} \rightarrow \mathbb{Y} \in \mathsf{Stone}$ we define $\mathrm{Up}\mathbb{V}\,f : \mathrm{Up}\mathbb{V}\,\mathbb{X} \rightarrow \mathrm{Up}\mathbb{V}\,\mathbb{Y}$ by putting

$$\mathrm{Up}\mathbb{V}\,f(N) := \{F \in \mathbb{V}\mathbb{Y} \mid f^{-1}(F) \in N\}.$$

Descriptive monotone neighbourhood frames (again)

Definition

A set $N \subseteq \mathbb{V}\mathbb{X}$ is called $[\ni]$-closed if for all $F \in K\mathbb{X}$ we have

$$F \in N \text{ if for all } a \in \mathrm{Clp}\mathbb{X}\ (F \subseteq a \to a \in N).$$

For a Stone space $\mathbb{X}$ we put

$$\mathrm{Up}\mathbb{V}\,\mathbb{X} := \{N \subseteq \mathbb{V}\mathbb{X} \mid N \text{ u.c. } \& \ N \text{ is } [\ni]\text{-closed}\},$$

and for $f : \mathbb{X} \to \mathbb{Y} \in \mathrm{Stone}$ we define $\mathrm{Up}\mathbb{V}\,f : \mathrm{Up}\mathbb{V}\,\mathbb{X} \to \mathrm{Up}\mathbb{V}\,\mathbb{Y}$ by putting

$$\mathrm{Up}\mathbb{V}\,f(N) := \{F \in \mathbb{V}\mathbb{Y} \mid f^{-1}(F) \in N\}.$$

Then $\mathrm{Up}\mathbb{V} : \mathrm{Stone} \to \mathrm{Stone}$ is a functor and

$$\mathrm{Coalg}(\mathrm{Up}\mathbb{V}) \cong \mathrm{Coalg}(\mathbb{M}).$$

Why is $\mathrm{Up}\mathbb{V}$ interesting?

Some facts

- $\mathrm{Up}\mathbb{V}\,\mathbb{X} \subseteq \mathbb{V}\mathbb{V}\mathbb{X}$ (a closed subspace),
- the $\Box$ of monotone modal logic is interpreted by sets of the form $\langle\ni\rangle_1[\ni]_2(\mathrm{a}) \subseteq \mathrm{Up}\mathbb{V}\,\mathbb{X}$, ie.,

$$[\![\Box]\!](\mathrm{a}) = \langle\ni\rangle_1[\ni]_2(\mathrm{a}).$$

- This looks like the usual simulation of monotone modal logic by two normal modalities.

Polynomial Vietoris functors

Definition

$$T \;::=\; \mathbb{V} \mid T \times T \mid \mathbb{V}T.$$

Corresponding logics

- inductively defined syntax: one $[\ni]$-operator for each occurrence of $\mathbb{V}$ in the functor plus one $[\mathtt{next}]$-operator,
- inductively defined many-sorted semantics, e.g. for a $\mathbb{V}\mathbb{V}$-model $\mathfrak{M} = (\mathbb{X}, \gamma, h)$ we have

$$\begin{array}{rcll}
\mathrm{Clp}\mathbb{X} \ni & [\![p]\!] & := & \{x \in X \mid \pi_p(h(x)) = 1\} \\
\mathrm{Clp}\mathbb{V}\mathbb{X} \ni & [\![[\ni]\varphi]\!] & := & \{F \mid F \subseteq [\![\varphi]\!]\} \\
\mathrm{Clp}\mathbb{V}\mathbb{V}\mathbb{X} \ni & [\![\langle\ni\rangle[\ni]\varphi]\!] & := & \{N \mid \exists F \in N (F \subseteq [\![\varphi]\!])\} \\
\mathrm{Clp}\mathbb{X} \ni & [\![[\mathtt{next}]\langle\ni\rangle[\ni]\varphi]\!] & := & \{x \in X \mid \gamma(x) \in [\![\langle\ni\rangle[\ni]\varphi]\!]\}
\end{array}$$

Towards a generic simulation result(informally)

Facts

- It is possible to simulate the logic of a Vietoris polynomial functor T in K_n where n is the number of occurrences of $\mathbb{V}$ in T.
- This is quite simple but extremely technical. Is it useful?
- Instead I will only treat two examples.

Let M be the smallest monotone modal logic, K_2 the bimodal version of the normal modal logic K.

We define a translation $(\cdot)^t$ of formulas of monotone modal logic into bimodal modal logic.

$$
\begin{aligned}
(p)^t &:= p \qquad\qquad (\bot)^t := \bot \\
(\varphi_1 \wedge \varphi_2)^t &:= (\varphi_1)^t \wedge (\varphi_2)^t \\
(\neg\varphi)^t &:= \neg(\varphi)^t \\
(\Box\varphi)^t &:= \Diamond_1(\Box_2(\varphi)^t)
\end{aligned}
$$

Theorem[Kracht & Wolter 99]

For all formulas $\varphi \in \mathcal{L}$ we have $\varphi \in \mathrm{M} + \Gamma$ iff $\varphi^{\mathrm{t}} \in \mathrm{K}_2 + \Gamma^{\mathrm{t}}$.

Proof

The direction from right to left is easy (using the fact that K_2 is closed under the rule $\varphi_1 \to \varphi_2 / \Diamond_1 \Box_2 \varphi_1 \to \Diamond_1 \Box_2 \varphi_2$).

The opposite direction in Kracht & Wolter is quite complicated.

Simple simulation

- Given a descriptive neighbourhood model $\mathfrak{M}$

$$\mathbb{X} \xrightarrow{\langle \gamma, h \rangle} \mathrm{Up}\mathbb{V}\,\mathbb{X} \times \prod_{p \in \mathrm{Prop}} 2$$

Simple simulation

- Given a descriptive neighbourhood model $\mathfrak{M}$

$$\mathbb{X} \xrightarrow{\langle \gamma, h \rangle} \mathrm{Up}\mathbb{V}\,\mathbb{X} \times \prod_{p \in \mathrm{Prop}} 2$$

- Let $\mathbb{X}' := \mathbb{X} \times \mathbb{V}\mathbb{X}$.

Simple simulation

- Given a descriptive neighbourhood model $\mathfrak{M}$

$$\mathbb{X} \xrightarrow{\langle\gamma,h\rangle} \mathrm{Up}\mathbb{V}\,\mathbb{X} \times \prod_{p\in\mathrm{Prop}} 2$$

- Let $\mathbb{X}' := \mathbb{X} \times \mathbb{V}\mathbb{X}$.
- Define descriptive bimodal Kripke model $\mathfrak{M}^{\bullet}$

$$\mathbb{X}' \xrightarrow{\langle\Gamma_1,\Gamma_2,h'\rangle} \mathbb{V}(\mathbb{X}') \times \mathbb{V}(\mathbb{X}') \times \prod_{p\in\mathrm{Prop}} 2$$

by putting

$$\begin{aligned} \Gamma_1(x,F) &:= \{(x,F') \mid F' \in \gamma(x)\}, \text{ and} \\ \Gamma_2(x,F) &:= \{(x',F) \mid x' \in F\}, \\ h'(x,F) &:= h(x). \end{aligned}$$

Simulation Theorem (continued)

Proposition

For every formula φ, any $\mathrm{Up}\mathbb{V}$-model $\mathfrak{M} = (\mathbb{X}, \gamma, h)$ any $x \in X$ and $F \in K\mathbb{X}$ we have

$$x \in [\![\varphi]\!]_{\mathfrak{M}} \quad \text{iff} \quad (x, F) \in [\![\varphi^t]\!]_{\mathfrak{M}^\bullet}.$$

This shows that $\varphi^t \in K_2 + \Gamma^t$ implies $\varphi \in M + \Gamma$ and finishes the proof of the simulation theorem.

Simulation Theorem (continued)

Proposition

For every formula φ, any $\mathrm{Up}\mathbb{V}$-model $\mathfrak{M} = (\mathbb{X}, \gamma, h)$ any $x \in X$ and $F \in K\mathbb{X}$ we have

$$x \in [\![\varphi]\!]_{\mathfrak{M}} \qquad \text{iff} \qquad (x, F) \in [\![\varphi^t]\!]_{\mathfrak{M}^\bullet}.$$

This shows that $\varphi^t \in K_2 + \Gamma^t$ implies $\varphi \in M + \Gamma$ and finishes the proof of the simulation theorem.

However: The exact "strength" of this simulation has still to be investigated.

Simulating classical modal logic

Consider an $\mathbb{N}$-model $\mathfrak{M}$:

$$\mathbb{X} \xrightarrow{\langle \gamma, h \rangle} \mathbb{N}\mathbb{X} \times \prod_{p \in \mathrm{Prop}} 2 \; .$$

Simulating classical modal logic

Consider an $\mathbb{N}$-model $\mathfrak{M}$:

$$\mathbb{X} \xrightarrow{\langle\gamma,h\rangle} \mathbb{N}\mathbb{X} \times \prod_{p\in \mathrm{Prop}} 2 \; .$$

Define a corresponding $\mathbb{V}(\mathbb{V}\times\mathbb{V})$-model:

$$\mathbb{X} \xrightarrow{\langle\overline{\gamma},h\rangle} \mathbb{V}(\mathbb{V}\mathbb{X} \times \mathbb{V}\mathbb{X}) \times \prod_{p\in \mathrm{Prop}} 2 \; ,$$

by putting $\gamma(x) := \overline{\{(a, -a) \mid a \in \gamma(x)\}}$.

Simulating classical modal logic

Finally we put $\mathbb{X}' := \mathbb{X} \times \mathbb{V}\mathbb{X} \times \mathbb{V}\mathbb{X}$ define the K_3-model $\mathfrak{M}^\bullet$ that corresponds to $\mathfrak{M}$:

$$\mathbb{X}' \xrightarrow{\langle \Gamma_1, \Gamma_2, \Gamma_3, h' \rangle} \mathbb{V}(\mathbb{X}') \times \mathbb{V}(\mathbb{X}') \times \mathbb{V}(\mathbb{X}') \times \prod_{p \in \mathrm{Prop}} 2$$

where

$$\begin{array}{rcl} \Gamma_1(x, F_1, F_2) & := & \{(x, F_1', F_2') \mid (F_1', F_2') \in \overline{\gamma}(x)\} \\ \Gamma_2(x, F_1, F_2) & := & \{(x', F_1, F_2) \mid x' \in F_1\} \\ \Gamma_3(x, F_1, F_2) & := & \{(x', F_1, F_2) \mid x' \in F_2\} \\ h'(x, F_1, F_2) & := & h(x) \end{array}$$

Simulation

Let E be the smallest classical modal logic. We define a translation

$$\begin{aligned} (p)^t &:= p \qquad\qquad (\bot)^t := \bot \\ (\varphi_1 \wedge \varphi_2)^t &:= (\varphi_1)^t \wedge (\varphi_2)^t \\ (\neg\varphi)^t &:= \neg(\varphi)^t \\ (\Box\varphi)^t &:= \Diamond_1(\Box_2(\varphi)^t \wedge \Box_3(\neg\varphi)^t). \end{aligned}$$

Proposition

For all $\mathbb{N}$-models $\mathfrak{M}$ and all $x \in X$, $F_1, F_2 \in K\mathbb{X}$ we have $x \in [\![\varphi]\!]_{\mathfrak{M}}$ iff $(x, F_1, F_2) \in [\![(\varphi)^t]\!]_{\mathfrak{M}^\bullet}$.

Summary

- Coalgebraic representation of modal algebras: Stone coalgebras
- Existence of final coalgebras via Hennessy-Milner property
- straightforward (naive?) simulations using combinations of the Vietoris functor

Questions

- Does the "construction" of the final coalgebra tell us something about the structure of the canonical model(s)?
- How well-behaved are the proposed simulations?
- What about other simulations (eg. Thomason simulation or the simulation of polyadic modalities)?