A topological duality for Hilbert algebras

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Outline of the talk

1. Overview.
2. Topological duality for distributive meet semi-lattices and implicative meet semi-lattices.
3. Topological duality for Hilbert algebras.

Overview

Hilbert algebras

A Hilbert algebra is a $(\to, 1)$-subreduct of a Heyting algebra. Also a $(\to, 1)$-subreduct of an implicative meet semi-lattice.

The class of Hilbert algebras is definable by the following equations and quasiequation:

H1. $x \to (y \to x) = 1$

H2. $x \to (y \to z) \to (x \to y) \to (x \to z)) = 1$

H3. $x \to y = y \to x = 1$ implies $x = y$

It is a variety [Diego].

The relation $\leq$ defined on a Hilbert algebra $\mathbf{A}$ by

$$a \leq b \quad \Leftrightarrow \quad a \to b = 1$$

is a partial order with greatest element 1.

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Distributive meet semi-lattice:

$$a \wedge b \leq c \quad \Rightarrow \quad (\exists a', b')(a \leq a' \ \& \ b \leq b' \ \& \ c = a' \wedge b')$$

Equivalently: the lattice of filters is distributive (a key property)

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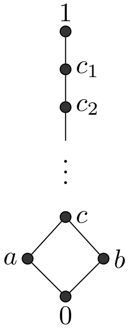
Implicative meet semi-lattice (a.k.a. Browverian semi-lattice):

$$a \wedge b \leq c \quad \Leftrightarrow \quad a \leq b \rightarrow c$$

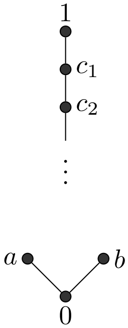
- Implicative meet semi-lattices are distributive as meet semi-lattices.
- They are the $(\wedge, \rightarrow, 1)$-subreducts of Heyting algebras.

L	$\longrightarrow$	$D(L)$
(implicative) distr. meet semi-lattice (bounded)		distr. lattice envelop
$\downarrow$		$\downarrow$
L_*	$\longleftarrow$	$Pr(D(L))$
generalized (Esakia) Priestley space		Priestley space

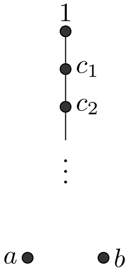
A	$\longrightarrow$	$L(A)$
Hilbert algebra		implicative meet semi-lattice envelop
$\downarrow$		$\downarrow$
A_*	$\longleftarrow$	$L(A)_*$
???		generalized Priestley space



Distributive lattice



(implicative) meet semi-lattice



Hilbert algebra

Duality for distributive meet semi-lattices and implicative meet semi-lattices

The distributive envelop of a distributive meet semi-lattice

Let L, K distributive meet semi-lattices. A map $h : L \to K$ is a sup-homomorphism if

- h is a homomorphism, i.e. $h(1) = 1,\ h(a \wedge b) = h(a) \wedge h(b)$,
- h satisfies

$$\bigcap_{i \leq n} \uparrow c_i \subseteq \uparrow c \quad \Rightarrow \quad \bigcap_{i \leq n} \uparrow h(c_i) \subseteq \uparrow h(c) \qquad \text{(sup)}$$

Condition (sup) is equivalent to:

$$c \in \{c_0, \ldots, c_n\}^{ul} \quad \Rightarrow \quad h(c) \in \{h(c_0), \ldots, h(c_n)\}^{ul}$$

The concept of sup-homomorphism is related to the notion of a Frink ideal.

Let $P = \langle P, \leq \rangle$ be a poset. A Frink ideal is a nonempty set $I \subseteq P$ s.t.

- it is a down-set,
- if $X \subseteq I$ is finite, then $X^{ul} \subseteq I$

The second condition is equivalent to:

$$\bigcap_{i \leq n} \uparrow c_i \subseteq \uparrow c \;\; \& \;\; \{c_0, \ldots, c_n\} \subseteq I \quad \Rightarrow \quad c \in I.$$

Theorem

Let L, K be distributive meet semi-lattices. Let $h : L \to K$ be a homomorphism. The following are equivalent:

- *h is a sup-homomoprhism,*
- *$h^{-1}[I]$ is a Frink-ideal of L, for every Frink ideal I of K.*

Let L be a distributive meet semi-lattice.

A distributive lattice expansion of L is a pair $\langle e, E\rangle$ where E is a distributive lattice and e a sup-embedding from L to E.

The distributive envelop of L is the unique (up to isomorphism) distributive lattice expansion $\langle e, D(L)\rangle$ with the following universal property: for every distributive lattice expansion $\langle h, E\rangle$ of L there is a unique lattice embedding $k : D(L) \to E$ such that $k \circ e = h$.

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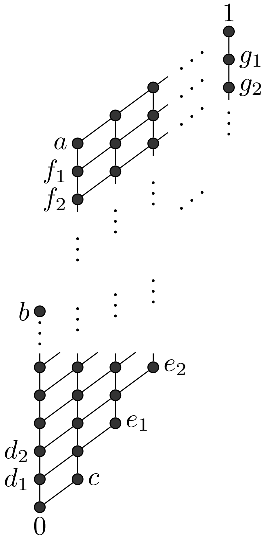
In fact, let

- $\mathsf{DMSLat}^{\mathsf{sup}}$ be the category of distributive meet semi-lattices with sup-homomorphisms
- DLat the category of distributive lattices with $(1, \vee, \wedge)$-homomorphisms.
- $U : \mathsf{DLat} \to \mathsf{DMSLat}^{\mathsf{sup}}$ the forgetful functor that forgets the operation $\vee$.

Then U has a left adjoint and $D(.)$ gives the object part.

- If L is an implicative meet-semilattice, $D(L)$ need not be a Heyting algebra.



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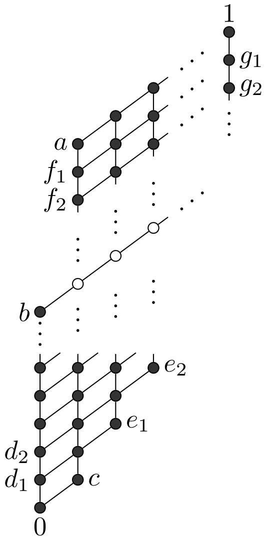

$D(L)$

The topological dual of a distributive and of an implicative meet semi-lattice

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- Points: Optimal filters, $\mathsf{Op}(L)$,

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$$\mathsf{Op}(L) \cong \mathsf{Prime}(D(L))$$

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- Topology: generated by the subbase

$$\{\varphi(a) : a \in L\} \cup \{L - \varphi(a) : a \in L\}$$

where

$$\varphi(a) = \{F \in \mathsf{Op}(L) : a \in F\}$$

- Special dense set: the prime elements of the lattice of filters of L, called prime filters of L.

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The structure $\langle \mathsf{Op}(L), \tau, \subseteq \rangle$ is a Priestley space.

The dual of L is

$$L_* := \langle \mathsf{Op}(L), \tau, \subseteq, \mathsf{Pr}(L) \rangle$$

The clopen up-sets $\varphi(a)$, with $a \in L$, have the following characterization.

Let $U \in ClUp(L_*)$. Then

$$(\exists a \in L)\ U = \varphi(a) \ \Leftrightarrow\ L_* - U = \downarrow(\mathsf{Pr}(L) - U) \ \Leftrightarrow\ \max(L_* - U) \subseteq \mathsf{Pr}(L)$$

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Let:

- $X = \langle X, \tau_X, \leq_X \rangle$ a Priestley space.
- X_0 a dense subset of X.

A clopen up-set U is X_0- admissible (admissible) if $\max(X - U) \subseteq X_0$.

We set

$$X^* := \{U \in ClUp(X) : U \text{ is } X_0\text{-admissible}\}.$$

Definition

A quadruple $X = \langle X, \tau_X, \leq_X, X_0 \rangle$ is a generalized Priestley space if:

1. $\langle X, \tau_X, \leq_X \rangle$ is a Priestley space.
2. X_0 is a dense subset of X.
3. $\forall x \in X \; \exists y \in X_0 \; x \leq y$.
4. $x \in X_0$ iff $\{U \in X^* : x \notin U\}$ is updirected.
5. $\forall x, y \in X$, $x \leq y$ iff $(\forall U \in X^*)(x \in U \Rightarrow y \in U)$.

Let X be a g-Priestley space. A clopen subset U is Esakia clopen if U is a *finite union* of sets $U_i - V_i$ and U_i, V_i are X_0-admissible clopen up-sets.

A g-Priestley space is a generalized Esakia space if for every Esakia clopen set U, ${\downarrow}U$ is clopen.

If $X = \langle X, \tau_X, \leq_X, X_0 \rangle$ is a generalized Esakia space, it does not necessarily follow that $\langle X, \tau_X, \leq_X \rangle$ is an Esakia space.

Morphisms

Definition

Let X and Y be g-Priestley spaces. A g-Priestely morphism from X to Y is a relation $R \subseteq X \times Y$ such that

1. If $x \not{R} y$, then there is an Y_0-admissible clopen up-set U of Y such that $R[x] \subseteq U$ and $y \notin U$.
2. If U is an Y_0-admissible clopen up-set U of Y, then

$$\Box_R U = \{x \in X : R[x] \subseteq U\}$$

is an X_0-admissible clopen up-set of X.

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- Composition is defined as follows: Let $R : X \to Y$ and $S : Y \to Z$

$$xR \star Sz \quad \iff \quad (\forall U \in Z^*)((S \circ R)[x] \subseteq U \Rightarrow z \in U)$$

- The identity g-Priestley morphism from $X \to X$ is $\leq_X$.

A g-Priestely morphism $R \subseteq X \times Y$ is functional if for every $x \in X$ there exists $y \in Y$ such that $R[x] = {\uparrow} y$.

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Definition

Let X, Y be g-Esakia spaces. A g-Esakia morphism from X to Y is a relation $R \subseteq X \times Y$ such that

1. R is a g-Priestley morphism,
2. for every $x \in X$ and every $y \in Y_0$, if xRy, then there exists $z \in X_0$ such that $x \leq z$ and $R[z] = \uparrow y$.

Composition of g-Esakia morphisms is $\star$.

The identity g-Esakia morphism is $\leq_X$.

Categorical dualities

BDMSLat	bounded distributive meet semi-lattices	homomorphisms
$\mathsf{BDMSLat}^{sup}$	idem	sup-homomorphisms
BImMSLat	bounded implicative meet semi-lattices	homomorphisms
$\mathsf{BImMSLat}^{sup}$	idem	sup homomorphisms

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GPrSp	g-Priestley spaces	g-Priestley morphisms
$\mathsf{GPrSp}^{\mathsf{F}}$	g-Priestley spaces	functional g-Priestley morphisms
GEsSp	g-Esakia spaces	g-Esakia morphisms
$\mathsf{GPrSp}^{\mathsf{F}}$	g-Esakia spaces	functional g-Esakia morphisms

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GPrSp	g-Priestley spaces	g-Priestley morphisms
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GEsSp	g-Esakia spaces	g-Esakia morphisms
$\mathsf{GPrSp}^{\mathsf{F}}$	g-Esakia spaces	functional g-Esakia morphisms

$$\begin{array}{ccc} \mathsf{BDMSLat} & \cong & (\mathsf{GPrSp})^{\mathsf{op}} \\ \mathsf{BDMSLat}^{\mathsf{sup}} & \cong & (\mathsf{GPrSp}^{\mathsf{F}})^{\mathsf{op}} \quad \text{(Hansoul)} \\ \mathsf{BImMSLat} & \cong & (\mathsf{GEsrSp})^{\mathsf{op}} \\ \mathsf{BImMSLat}^{\mathsf{sup}} & \cong & (\mathsf{GEsrSp}^{\mathsf{F}})^{\mathsf{op}} \end{array}$$

Duality for Hilbert algebras

Deductive filters of a Hilbert algebra

Let $\mathbf{A}$ be a Hilbert algebra.

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A subset F of A is a deductive filter if

1. $1 \in F$,
2. if $a, a \to b \in F$, then $b \in F$.

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A subset F of A is a deductive filter if

1. $1 \in F$,
2. if $a, a \to b \in F$, then $b \in F$.

- Every deductive filter is an up-set of $\langle A, \leq \rangle$. The converse is not true.

- Every principal up-set $\uparrow a$ of $\langle A, \leq \rangle$ is a deductive filter.

- The deductive filters form a distributive lattice which is complete and the meet is intersection.

Notation: $(a_n, \dots, a_0; b) := a_n \to (a_{n-1} \to (\dots \to (a_0 \to b) \dots)$

We denote by $\langle X \rangle$ the deductive filter generated by $X \subseteq A$. Then

$$a \in \langle X \rangle \quad \text{iff} \quad a = 1 \text{ or } (\exists a_n, \dots, a_0 \in X)\ (a_n, \dots, a_0; 1) = 1$$

A deductive filter is prime if it is a prime element of the lattice of deductive filters.

Prd$(\mathbf{A})$: set of all prime deductive filters of $\mathbf{A}$.

Strong Frink ideals of a Hilbert algebra

Let $\mathbf{A}$ be a Hilbert algebra.
A nonempty set $I \subseteq A$ is called a strong Frink ideal (F-ideal) if

1. I is a down-set,
2. if $X \subseteq I$ and $Y \subseteq A$ are finite and $X^u \subseteq \langle Y \rangle$, then $\langle Y \rangle \cap I \neq \emptyset$.

Equivalently, if for every $a_0, \ldots, a_n \in I$ and every $b_0, \ldots, b_m \in A$,

$$\text{if } \bigcap_{i \leq n} \uparrow a_i \subseteq \langle b_0, \ldots, b_m \rangle, \text{ then } \langle b_0, \ldots, b_m \rangle \cap I \neq \emptyset. \quad (1)$$

An F-ideal I is proper if $I \neq A$.

Definition

A deductive filter F of a $\mathbf{A}$ is optimal if $A - F$ is a strong Frink ideal.

Opd($\mathbf{A}$) denotes the set of all optimal deductive filters of $\mathbf{A}$

Every prime deductive filter is optimal.

A set I is a prime strong Frink ideal iff $A - I$ is an optimal deductive filter.

Let $\mathbf{A}$, $\mathbf{B}$ be Hilbert algebras. A map $h : A \to B$ is a sup-homomorphism if

1. h is a homomorphism, i.e. $h(1) = 1, \quad h(a \to b) = h(a) \to h(b)$
2. h satisfies

$$\bigcap_{i \leq n} \uparrow c_i \subseteq \langle b_0, \ldots, b_m \rangle \quad \Rightarrow \quad \bigcap_{i \leq n} \uparrow h(c_i) \subseteq \langle h(b_0), \ldots, h(b_m) \rangle \qquad \text{(sup)}$$

Condition (sup) is equivalent to:

$$c \in \{c_0, \ldots, c_n\}^{ul} \quad \Rightarrow \quad h(c) \in \{h(c_0), \ldots, h(c_n)\}^{ul}$$

Theorem

Let $h : \mathbf{A} \to \mathbf{B}$ a homomorphism of Hilbert algebras.

1. *h is a sup-homomorphism,*
2. *$h^{-1}[F]$ is an optimal deductive filter of $\mathbf{A}$ for every optimal deductive filter F of $\mathbf{B}$,*
3. *$h^{-1}[I]$ is a strong Frink ideal of $\mathbf{A}$, for every strong Frink ideal I of $\mathbf{B}$.*

The implicative meet semi-lattice envelop of a Hilbert algebra

Let $\mathbf{A}$ be a Hilbert algebra.
An implicative meet semi-lattice envelop of $\mathbf{A}$ is pair $\langle L, e \rangle$, where

1. L is an implicative meet semi-lattice and e is a one-to-one homomorphism from $\mathbf{A}$ to L,
2. for every $a \in L$ there is a finite $X \subseteq A$ such that $a = \bigwedge e[X]$.

- If $\langle L, e \rangle$ is an implicative semi-lattice envelop of $\mathbf{A}$, then e is a sup-homomorphism.

- Up to isomorphism there is exactly one implicative meet semi-lattice envelop, denoted $L(\mathbf{A})$, and it is characterized by the universal property:

> *For every implicative meet semi-lattice L' and every homomorphism $g : \mathbf{A} \to \langle L', \to', 1 \rangle$, there is a unique homomorphism $\overline{g} : L(\mathbf{A}) \to L'$ such that $g = \overline{g} \circ e$. Moreover, if g is one-to-one, then $\overline{g}$ is one-to-one; and if g is onto, then $\overline{g}$ is onto.*

ImMSLat: the category of implicative meet semi-lattices with their homomorphisms.

Hil: the category of Hilbert algebras and their homomorphisms

U : ImMSLat $\rightarrow$ Hil, the forgetful functor that forgets the meet operation.

U has a left adjoint and is precisely the functor that maps every Hilbert algebra to its implicative meet semi-lattice envelop.

Let $\mathbf{A}$ be a Hilbert algebra and $L(\mathbf{A})$ its implicative meet semi-lattice envelop.

The relation between the deductive filters of $\mathbf{A}$ and the filters of $L(\mathbf{A})$ is as follows.

A set $F \subseteq A$ is a deductive filter iff $F = G \cap A$ for some filter of $L(\mathbf{A})$.

Let F be a deductive filter of $\mathbf{A}$.

- F is optimal iff $F = G \cap A$ for some optimal filter of $L(\mathbf{A})$.
- F is prime iff $F = G \cap A$ for some prime filter of $L(\mathbf{A})$.

Let I be a strong Frink ideal of $\mathbf{A}$

- I is prime iff $F = G \cap A$ for some prime Frink ideal of $L(\mathbf{A})$.

Augmented Priestley spaces

Let $\mathbf{A}$ be a Hilbert algebra. We add a (new) bottom element 0 and define

$$a \to 0 = 0 \qquad 0 \to a = 1 \qquad 0 \to 0 = 1$$

for every $a \in A$.

The new algebra $\mathbf{A}^0$ is a Hilbert algebra and $L(\mathbf{A}^0) = L(\mathbf{A})^0$.

We build a Priestley space as follows:

- Points: Optimal deductive filters:

$$A_* := \mathsf{Opd}(\mathbf{A}) \cup \{A\} = \mathsf{Opd}(\mathbf{A}^0)$$

- Topology: generated by the subbase

$$\{\varphi(a) : a \in A\} \cup \{A - \varphi(a) : a \in A\}$$

where

$$\varphi(a) = \{F \in \mathsf{Opd}(\mathbf{A}^0) : a \in F\}$$

- Special set of clopen up-sets: $\{\varphi(a) : a \in A\}$

Note that:

- $A \in \varphi(a)$, for all $a \in A$

- $\varphi(0) = \emptyset$

- There is a finite nonempty $X \subseteq A$ such that $\bigcap_{a \in X} \varphi(a) = \{\mathbf{A}\}$ iff $L(\mathbf{A})$ has a bottom element.

Theorem

Let $\mathbf{A}$ *be a Hilbert algebra. Then*

1. $\langle A_*, \subseteq, \tau \rangle$ *is a Priestley space*
2. *for every* $x, y \in A_*$,

$$x \subseteq y \quad \textit{iff} \quad (\forall a \in A)(x \in \varphi(a) \Rightarrow y \in \varphi(a)),$$

3. *every nonempty clopen up-set is a finite union of intersections of a finite number of elements of* $\{\varphi(a) : a \in A\}$.
4. $\mathsf{Prd}(\mathbf{A}) \cup \{A\}$ *is dense in* $\langle A_*, \subseteq, \tau \rangle$,
5. $(\forall x \in A_* - \{A\})(\exists y \in \mathsf{Prd}(\mathbf{A}))\ x \subseteq y$,
6. *for every* $x \in A_*$,

$$x \in \mathsf{Prd}(\mathbf{A}) \;\; \textit{iff} \;\; \{\varphi(a) : x \notin \varphi(a), a \in A\} \textit{ is nonempty and updirected.}$$

Let

$$\mathcal{B}_{\mathbf{A}} = \langle \{\varphi(a) : a \in A\}, \Rightarrow, A_* \rangle.$$

Theorem

$$\varphi{\restriction}A : \mathbf{A} \cong \mathcal{B}_{\mathbf{A}},$$

and so $\mathcal{B}_{\mathbf{A}}$ is a Hilbert algebra.

Definition

An augmented Priestley space is a tuple $\langle X, \leq, \tau, S \rangle$ such that

1. $\langle X, \leq, \tau \rangle$ is a Priestley space,
2. $\langle X, \leq \rangle$ has a greatest element, t.
3. S is a family of nonempty clopen up-sets.
4. $x \leq y$ iff $(\forall U \in S)(x \in U \Rightarrow y \in U)$, for every $x, y \in X$.
5. the set

$$X_S = \{x \in X : \{U \in S : x \notin U\} \text{ is nonempty and updirected}\} \cup \{t\}$$

   is dense in X,
6. for every nonempty clopen up-set $U \subseteq X$, $\max(X - U) \subseteq X_S$ iff U is the intersection of a nonempty finite subset of S.
7. for every $U, V \in S$, $[\downarrow(U - V)]^c \in S$.

Fact: The structure $\langle X, \leq, \tau, X_S \cup \{t\} \rangle$ is a generalized Priestley space, in fact a generalized Esakia space.

Theorem

Let $\mathbf{A}$ *be a Hilbert algebra. Then*

$$(\mathbf{A})_* = \langle A_*, \subseteq, \tau, \varphi[A] \rangle,$$

where τ *is the topology generated by the family of the sets* $\varphi(a)$, *with* $a \in A$, *and their complements, taken as a generating subbase, is an augmented Priestley space.*

Let $X = \langle X, \leq, \tau, S\rangle$ be an augmented Priestley space.

For $U, V \subseteq X$ let

$$U \Rightarrow V = [{\downarrow}(U - V)]^c = \{x \in X : {\uparrow}x \cap U \subseteq V\}$$

Then the algebra

$$(X)^* = \langle S, \Rightarrow, X\rangle$$

is a Hilbert algebra.

The closure of S under finite intersections is the set

$$X^* = \{U : U \text{ is a clopen up-set and } \max(X - U) \subseteq X_S\}$$

This set is closed under $\Rightarrow$ and it is the implicative meet semi-lattice envelop of $\mathbf{A}$.
Notice that $\max{\downarrow}(U - V) = \max(U - V)$.
For every $x \in X$, $x \in X_S \cup \{t\}$ iff
$\{U \subseteq X : x \notin U, U$ is a clopen up-set and $\max(X - U) \subseteq X_S\}$ is updirected.

Let $X = \langle X, \leq, \tau, S \rangle$ be an augmented Priestley space. Let

$$\varepsilon : X \to ((X)^*)_*$$

be the map defined by

$$\varepsilon(x) = \{U \in S : x \in U\}.$$

Note that $\varepsilon(t) = S$.

Theorem

If $x \in X - \{t\}$, then $\varepsilon(x)$ is an optimal deductive filter of S.
If $x \in X_S$, then $\varepsilon(x)$ is a prime deductive filter of S.

Theorem

- $\varepsilon : X \to ((X)^*)_*$ *is an order isomorphism and a homeomorphism*
- $\varepsilon[X_S] = \mathsf{Prd}(S)$
- $S_{((X)^*)_*} = \{\varepsilon[U] : U \in S\}$.

Morphism of augmented Priestley spaces

Let $R \subseteq X \times Y$, for $U \subseteq Y$ we set

$$\Box_R U := \{x \in X : R[x] \subseteq U\}.$$

Notice that for every $U, V \subseteq Y$

$$\Box_R(U \cap V) = \Box_R U \cap \Box_R V$$

and

$$\Box_R(U \Rightarrow V) \subseteq \Box_R U \Rightarrow \Box_R V.$$

Let $\mathbf{A}, \mathbf{B}$ be Hilbert algebras and $h : \mathbf{A} \to \mathbf{B}$ a homomorphism.
We define $R_h \subseteq B_* \times A_*$ by

$$xR_h y \quad \text{iff} \quad h^{-1}[x] \subseteq y$$

Note that BR_hA and $R_h[B] = \{A\}$
Notation:

- $\subseteq_{B_*}$ denotes the inclusion relation restricted to B_*
- $\subseteq_{A_*}$ denotes the inclusion relation restricted to A_*.

Theorem

1. $\subseteq_{B_*} \circ R_h \subseteq R_h$
2. $R_h \circ \subseteq_{A_*} \subseteq R_h$
3. *if* $x \in B_*$, $y \in A_*$ *and* $x \not R_h y$, *then there is* $a \in A$ *such that* $y \notin \varphi(a)$ *and* $R_h[x] \subseteq \varphi(a)$
4. $\varphi(h(a)) = \Box_{R_h} \varphi(a)$
5. $\varphi(h(a \to b)) = \varphi(h(a)) \Rightarrow \varphi(h(b))$.
6. *If* $x \in B_*$, $y \in \mathrm{Prd}(\mathbf{A})$ *and* $x R_h y$, *then there is* $z \in \mathrm{Prd}(\mathbf{B})$ *such that* $x \subseteq z$ *and* $R_h[z] = \uparrow y$

Theorem

h *is a sup-homomorphism iff* $R_h[x]$ *has a least element for every* $x \in B_* - \{B\}$, *namely* $h^{-1}[x]$.

Definition

Let X and Y be augmented Priestley spaces. A relation $R \subseteq X \times Y$ is called an *augmented Priestley morphism* if

1. if $x \not R y$, then there is $U \in S_Y$ such that $y \notin U$ and $R[x] \subseteq U$,
2. if $x \in X$, $y \in Y_{S_Y}$ and xRy then there is $z \in X_{S_X}$ such that $x \leq z$ and $R[z] = \uparrow y$.
3. if $U \in S_Y$, then $\Box_R U \in S_X$.

An augmented Priestley morphism R is *functional* if for every $x \in X$, $R[x]$ has a least element.

Let $R \subseteq X \times Y$ be an augmented Priestley morphism. The map $h_R : S_Y \to S_X$ defined by

$$h_R(U) = \Box_R U.$$

is a homomorphism from $\langle S_X, \Rightarrow, X \rangle$ to $\langle S_Y, \Rightarrow, Y \rangle$.

Let $\mathbf{A}, \mathbf{B}$ be Hilbert algebras and h a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. For every $a \in A$

$$\varphi(h(a)) = h_{R_h}(\varphi(a)).$$

Let R be an augmented Priestley morphism from X to Y. Then for every $x \in X$ and every $y \in Y$

$$xRy \quad \text{iff} \quad \varepsilon(x) R_{h_R} \varepsilon(y).$$

Composition of augmented Priestley morphisms

Let

- X, Y, Z be augmented Priestley spaces,
- R an augmented Priestley morphism from X to Y,
- S an augmented Priestley morphism from Y to Z.

The composition $S \circ R$ may not be an augmented Priestley morphism.

We define the relation $S{*}R \subseteq X \times Z$ as follows

$$xS{*}Rz \quad \text{iff} \quad \forall U \in S_Z((S \circ R)[x] \subseteq U \Rightarrow z \in U).$$

Then $S{*}R$ is an augmented Priestely morphism from X to Z.

If X is an augmented Priestley space, the order $\leq_X$ of X is an augmented Priestely morphism and it is the identity morphism on X.

Definition

Let APS be the category of augmented Priestley spaces as objects and augmented Priestley morphisms as arrows, with composition the operation $*$.

We define the functors

$$(.)_* : \mathsf{Hil} \rightleftarrows \mathsf{APS} : (.)^*$$

as follows:

- $(\mathbf{A})_* = \langle A_*, \subseteq, \tau, \varphi[A] \rangle$
- $(h : \mathbf{A} \to \mathbf{B})_* = R_h : (\mathbf{B})_* \to (\mathbf{A})_*$

- $(X)^* = S_X$
- $(R : X \to Y)^* = h_R$

These functors establish a dual equivalence between $\mathbf{Hil}$ and $\mathbf{APS}$.

The natural transformations

The natural transformation from

$$\mathrm{Id}_{\mathbf{Hil}} : \mathbf{Hil} \to \mathbf{Hil} \quad \text{to} \quad ((.)_*)^* : \mathbf{Hil} \to \mathbf{Hil}$$

is given by the φ maps. For every $\mathbf{A} \in \mathbf{Hil}$,

$$\varphi_{\mathbf{A}} = \varphi{\restriction}A : \mathbf{A} \cong ((\mathbf{A})_*)^*.$$

To define the natural transformation from

$$\mathrm{Id}_{\mathbf{APS}} : \mathbf{APS} \to \mathbf{APS} \quad \text{to} \quad ((.)^*)_* : \mathbf{APS} \to \mathbf{APS}.$$

we use for every $X \in \mathbf{APS}$ the map $\varepsilon_X : X \to ((X)^*)_*$. We need to trun this map into a isomorphism of $\mathbf{APS}$ between X and $((X)^*)_*$.
Let X be an augmented Priestely space. Let $\overline{\varepsilon}_X \subseteq X \times ((X)^*)_*$ and $\underline{\varepsilon}_X \subseteq ((X)^*)_* \times X$ be the relations defined by

$$x \, \overline{\varepsilon}_X \, \varepsilon(y) \quad \text{iff} \quad \varepsilon(x) \subseteq \varepsilon(y) \qquad \varepsilon(x) \, \underline{\varepsilon}_X \, y \quad \text{iff} \quad x \leq_X y$$

Lemma

The relations $\overline{\varepsilon}_X$ and $\underline{\varepsilon}_X$ are augmented Priestley morphisms and

$$\overline{\varepsilon}_X * \underline{\varepsilon}_X = \leq_{((X)^*)_*} \text{ and } \underline{\varepsilon}_X * \overline{\varepsilon}_X = \leq_X .$$

Let X be an augmented Priestely space. The X-component of the natural transformation from

$$\mathrm{Id}_{\mathbf{APS}} : \mathbf{APS} \to \mathbf{APS} \quad \text{to} \quad ((.)^*)_* : \mathbf{APS} \to \mathbf{APS}.$$

is the relation $\overline{\varepsilon}_X \subseteq X \times ((.)^*)_*$.

Lemma

For every Hilbert algebra $\mathbf{A}$ and every augmented Priestley space X,

1. $((\mathbf{A})_*)^* \cong \mathbf{A}$
2. $((X)^*)_* \cong X$

Theorem

$\mathbf{Hil}$ *is dually equivalent to* $\mathbf{APS}$.

Let $\mathbf{Hil^{sup}}$ the category of Hilbert algebras with sup-homomorphisms.
Let $\mathbf{APS^{F}}$ the category of augmented Priestley spaces with morphims the functional augmented Priestley morphisms.

Theorem

$\mathbf{Hil^{sup}}$ *is dually equivalent to* $\mathbf{APS^{F}}$.

Let $\mathbf{A}$, $\mathbf{B}$ be Hilbert algebras. A semi-homomorphism from $\mathbf{A}$ to $\mathbf{B}$ is a map $h : A \to B$ that

- $h(1) = 1$
- $h(a \to b) \leq h(a) \to h(b)$.

The relation $R_h : (\mathbf{A})_* \to (\mathbf{B})_*$ satisfies

- if $x \in B_*$, $y \in A_*$ and $x \not R_h y$, then there is $a \in A$ such that $y \notin \varphi(a)$ and $R_h[x] \subseteq \varphi(a)$
- $\varphi(h(a)) = \Box_{R_h} \varphi(a)$
- $\varphi(h(a \to b)) \subseteq \varphi(h(a)) \Rightarrow \varphi(h(b))$.

Let X, Y augmented Priestley spaces. A relation $R \subseteq X \times Y$ is a semi-augmented Priestley morphism if

- if $x \not R y$, then there is $U \in S_Y$ such that $y \notin U$ and $R[x] \subseteq U$,
- if $U \in S_Y$, then $\Box_R U \in S_X$.

Let $\mathbf{Hil^{sem}}$ the category of Hilbert algebras with semi-homomorphisms.
Let $\mathbf{APS^{sem}}$ the category of augmented Priestley spaces with morphims the semi-augmented Priestley morphisms.

Theorem

$\mathbf{Hil^{sem}}$ *is dually equivalent to* $\mathbf{APS^{sem}}$.

Thank you !