Some Generalizations of the Stone Duality Theorem

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Introduction

In 1937, M. Stone [Stone] proved that there exists a bijective correspondence between the class of all (up to homeomorphism) zero-dimensional locally compact Hausdorff spaces (briefly, *Boolean spaces*) and the class of all (up to isomorphism) generalized Boolean pseudolattices (briefly, GB-PLs) (or, equivalently, Boolean rings with or without unit). In the class of compact Boolean spaces (briefly, *Stone spaces*) this bijection can be extended to a duality $S^t : \mathbf{ZHC} \longrightarrow \mathbf{Bool}$ between the category $\mathbf{ZHC}$ of Stone spaces and continuous maps and the category $\mathbf{Bool}$ of Boolean algebras and Boolean homomorphisms. As far as I know, in the case of Boolean spaces such an extension to a duality does not exist. In fact, there are some obstacles for doing this. Indeed, to every Boolean space X, M. Stone juxtaposes the generalized Boolean pseudolattice $CK(X)$ of all compact open subsets of X and reconstructs from it the space X (up to homeomorphism). If $f : X \longrightarrow Y$ is a continuous map between two Stone spaces then its dual map $\varphi = S^t(f) : CO(Y) \longrightarrow CO(X)$ (where, for every topological space Z, $CO(Z)$ is the set of all clopen subsets of Z) is defined by the formula $\varphi(G) = f^{-1}(G)$, for every $G \in CO(Y)$. If, however, $f : X \longrightarrow Y$ is a continuous map between two Boolean spaces and at least the space X is not compact then the preimages $f^{-1}(G)$ of the elements G of $CK(Y)$ are not obliged to be elements of the set $CK(X)$. These preimages will belong to $CK(X)$ iff the map f is perfect; then it is natural to expect that the category of GBPLs and pseudolattice homomorphisms preserving zero elements (or, equivalently, the category $\mathbf{BoolRng}$ of Boolean rings and ring homomorphisms) will be the dual category of the category $\mathbf{PZHLC}$ of Boolean spaces and perfect maps. However it is not the case. For example, if X and Y are two non-empty Boolean non-compact spaces and the 0-pseudolattice homomorphism $\varphi_0 : CK(Y) \longrightarrow CK(X)$ is defined by $\varphi_0(G) = 0 (= \emptyset)$ for every $G \in CK(Y)$, then there is no one function $f : X \longrightarrow Y$ such that $\varphi_0(G) = f^{-1}(G)$, for every $G \in CK(Y)$. Hence, even in the case of perfect maps, the mentioned homomorphisms are too much. In fact, as it is proved by D. Hofmann [Hof], the category $\mathbf{BoolRng}$ is dually equivalent to the category $\mathbf{pStone}$ of pointed Stone spaces and continuous maps preserving the fixed points. Thus, if one looks for a dual category to the category $\mathbf{PZHLC}$, having GBPLs as objects, then this category has to have as morphisms some subclass of the class of pseudolattice homomorphisms preserving zero elements. Such a category is described here and is named $\mathbf{GBPL}$ (see Theorem 18 below where two duality functors $\Theta_g^t : \mathbf{PZHLC} \longrightarrow \mathbf{GBPL}$ and $\Theta_g^a : \mathbf{GBPL} \longrightarrow \mathbf{PZHLC}$ are defined). Further, we want also to find a dual category to the category $\mathbf{ZHLC}$. It is clear that in this case the preimages of the compact open sets are clopen sets but they are not obliged to be compact sets. In [Stone], M. Stone proves that clopen subsets of a Boolean space X correspond to simple ideals of the GBPL $CK(X)$ (i.e. those ideals of $CK(X)$ which have a complement in the frame $Idl(CK(X))$ of all ideals of $CK(X)$). Therefore one has to use the simple ideals of GBPLs. As it is proved by M. Stone, the set of all simple ideals of a GBPL forms a Boolean algebra. Here we describe the objects of the desired dual category to the category $\mathbf{ZHLC}$ as pairs (B, I), where B is a Boolean algebra and I is a dense (proper or non proper) ideal of it, satisfying a condition of completeness type; this condition is the following: for every simple ideal J of I, the join $\bigvee_B J$ exists; it is fulfilled for every pair (B, B), where B is a Boolean algebra because, as it is shown by M. Stone, an ideal of a Boolean algebra is simple iff it is principal. In this way we build a category named $\mathbf{ZLBA}$ and we prove that it is dually equivalent to the category $\mathbf{ZHLC}$ (see Theorem 11 where two duality functors $\Theta_d^t : \mathbf{ZHLC} \longrightarrow \mathbf{ZLBA}$ and $\Theta_d^a : \mathbf{ZLBA} \longrightarrow \mathbf{ZHLC}$ are defined). The idea of the construction of the category $\mathbf{ZLBA}$ comes from the ideas and results obtained in [D-AMH1-10]. However, the proof that the categories $\mathbf{ZHLC}$ and $\mathbf{ZLBA}$ are dually equivalent can be carried out independently from the results of [D-AMH1-10]; this is the more economical way. Namely, we first construct a category $\mathbf{LBA}$ containing as a subcategory the category $\mathbf{ZLBA}$ and find a contravariant adjunction between the categories $\mathbf{LBA}$ and $\mathbf{ZHLC}$ which leads to the mentioned above duality between the categories $\mathbf{ZHLC}$ and $\mathbf{ZLBA}$. We define also two more categories $\mathbf{PZLBA}$ and $\mathbf{PLBA}$ which are dual to the category $\mathbf{PZHLC}$.

We now fix the notation.

If $\boldsymbol{C}$ denotes a category, we write $X \in |\boldsymbol{C}|$ if X is an object of $\boldsymbol{C}$, and $f \in \boldsymbol{C}(X,Y)$ if f is a morphism of $\boldsymbol{C}$ with domain X and codomain Y. We will say that a subcategory $\boldsymbol{B}$ of a category $\boldsymbol{A}$ is a *cofull subcategory* if $|\boldsymbol{B}| = |\boldsymbol{A}|$.

The set of all clopen (= closed and open) subsets of a topological space X will be denoted by $CO(X)$ and the set of all compact open subsets of X by $CK(X)$.

The closed maps, as well as open maps, between topological spaces are assumed to be continuous but are not assumed to be onto.

All lattices are with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0. We do not require the elements 0 and 1 to be distinct. Since we follow Johnstone's terminology from [J], we will use the term *pseudolattice* for a poset having all finite non-empty meets and joins; the pseudolattices with a bottom will be called *0-pseudolattices*.

The operation "complement" in Boolean algebras will be denoted by "*".

If A is a Boolean algebra then the set of all ultrafilters of A will be denoted by $Ult(A)$.

We denote by $S^t : \mathbf{ZHC} \longrightarrow \mathbf{Bool}$ and $S^a : \mathbf{Bool} \longrightarrow \mathbf{ZHC}$ the Stone duality functors between the category $\mathbf{ZHC}$ of compact zero-dimensional Hausdorff spaces (= *Stone spaces*) and continuous maps and the category $\mathbf{Bool}$ of Boolean algebras and Boolean homomorphisms. For fixing the notation, recall that the Stone space $S^a(A)$ of a Boolean algebra A is the set $X = Ult(A)$ endowed with a topology $\mathcal{T}$ having as an open base the family $\{\lambda_A^S(a) \mid a \in A\}$, where $\lambda_A^S(a) = \{u \in X \mid a \in u\}$ for every $a \in A$; then $S^a(A) = (X, \mathcal{T})$ is a compact Hausdorff zero-dimensional space, and the map $\lambda_A^S : A \longrightarrow CO(X), \ \ a \mapsto \lambda_A^S(a)$, is a Boolean isomorphism.

Preliminaries

1 Recall that a *frame* is a complete lattice L satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$, for every $a \in L$ and every $S \subseteq L$.

Let A be a distributive 0-pseudolattice and $Idl(A)$ be the frame of all ideals of A. If $J \in Idl(A)$ then we will write $\neg_A J$ (or simply $\neg J$) for the pseudocomplement of J in $Idl(A)$ (i.e. $\neg J = \bigvee\{I \in Idl(A) \mid I \wedge J = \{0\}\}$). Note that $\neg J = \{a \in A \mid (\forall b \in J)(a \wedge b = 0)\}$ (see Stone [ST1]). Recall that an ideal J of A is called *simple* (Stone [ST1]) if $J \vee \neg J = A$. As it is proved in [ST1], the set $Si(A)$ of all simple ideals of A is a Boolean algebra with respect to the lattice operations in $Idl(A)$. Recall also that the regular elements of the frame $Idl(A)$ (i.e. those $J \in Idl(A)$ for which $\neg\neg J = J$) are called *normal ideals* (Stone [ST1]).

2 Let us recall the notion of *lower adjoint* for posets. Let $\varphi : A \longrightarrow B$ be an order-preserving map between posets. If $\psi : B \longrightarrow A$ is an order-preserving map satisfying the following condition

($\wedge$) for all $a \in A$ and all $b \in B$, $b \leq \varphi(a)$ iff $\psi(b) \leq a$

(i.e. the pair (ψ, φ) forms a Galois connection between posets B and A) then we will say that ψ is a *lower adjoint* of φ. It is easy to see that condition ($\wedge$) is equivalent to the following condition:

($\wedge'$) $\forall a \in A$ and $\forall b \in B$, $\psi(\varphi(a)) \leq a$ and $\varphi(\psi(b)) \geq b$.

Note that if $\varphi : A \longrightarrow B$ is an (order-preserving) map between posets, A has all meets and φ preserves them then, by the Adjoint Functor Theorem (see, e.g., [J]), φ has a lower (or *left*) adjoint which will be denoted by $\varphi_\wedge$.

3 Recall that:

(a) a map is *perfect* if it is compact (i.e. point inverses are compact sets) and closed;

(b) a continuous map $f : X \longrightarrow Y$ is called *quasi-open* ([MP]) if for every non-empty open subset U of X, $\mathrm{int}(f(U)) \neq \emptyset$ holds;

(c) a function $f : X \longrightarrow Y$ is called *skeletal* ([MR]) if $\mathrm{int}(f^{-1}(\mathrm{cl}(V))) \subseteq \mathrm{cl}(f^{-1}(V))$ for every open subset V of Y; it is well-known that a function $f : X \longrightarrow Y$ is skeletal iff $\mathrm{int}(\mathrm{cl}(f(V))) \neq \emptyset$ for every non-empty open subset U of X.

The Generalizations of the Stone Duality Theorem

Definition 4 A pair (A, I), where A is a Boolean algebra and I is an ideal of A (possibly non proper) which is dense in A (shortly, dense ideal), is called a *local Boolean algebra* (abbreviated as LBA). An LBA (A, I) is called a *prime local Boolean algebra* (abbreviated as PLBA) if $I = A$ or I is a prime ideal of A. Two LBAs (A, I) and (B, J) are said to be *LBA-isomorphic* (or, simply, *isomorphic*) if there exists a Boolean isomorphism $\varphi : A \longrightarrow B$ such that $\varphi(I) = J$.

Let $\mathbf{LBA}$ be the category whose objects are all LBAs and whose morphisms are all functions φ :

$(A, I) \longrightarrow (B, J)$ between the objects of $\mathbf{LBA}$ such that $\varphi : A \longrightarrow B$ is a Boolean homomorphism satisfying the following condition:

(LBA) For every $b \in J$ there exists $a \in I$ such that $b \leq \varphi(a)$;

let the composition between the morphisms of $\mathbf{LBA}$ be the usual composition between functions, and the $\mathbf{LBA}$-identities be the identity functions.

Remark 5 Note that two LBAs (A, I) and (B, J) are $\mathbf{LBA}$-isomorphic iff they are LBA-isomorphic.

Recall that a distributive 0-pseudolattice A is called a *generalized Boolean pseudolattice* (abbreviated as *GBPL*) if it satisfies the following condition:

(GBPL) for every $a \in A$ and every $b, c \in A$ such that $b \leq a \leq c$ there exists $x \in A$ with $a \wedge x = b$

and $a \vee x = c$ (i.e., x is the *relative complement of a in the interval* $[b, c]$).

We will need a simple lemma.

Lemma 6 *Let A be a Boolean algebra, $M \subseteq A$, $X = S^a(A)$ and $L_M = \{u \in X \mid u \cap M \neq \emptyset\}$ (sometimes we will write L_M^A instead of L_M). Then:*

(a) L_M is an open subset of X and hence the subspace L_M of X is a zero-dimensional locally compact Hausdorff space; $L_M \neq \emptyset$ iff $M \not\subseteq \{0\}$;

(b) If M is an ideal of A then $\lambda_A^S(M) = CK(L_M)$ and hence $\lambda_A^S(M)(= \{\lambda_A^S(a) \mid a \in M\})$ is a base of L_M;

(c) If (A, M) is an LBA then

$$\lambda_{(A,M)} : A \longrightarrow CO(L_M), \quad a \mapsto L_M \cap \lambda_A^S(a),$$

is a dense Boolean embedding;

Recall that a *contravariant adjunction* between two categories $\mathcal{A}$ and $\mathcal{B}$ consists of two contravariant functors $T : \mathcal{A} \longrightarrow \mathcal{B}$ and $S : \mathcal{B} \longrightarrow \mathcal{A}$ and two natural transformations $\eta : Id_{\mathcal{B}} \longrightarrow T \circ S$ and $\varepsilon : Id_{\mathcal{A}} \longrightarrow S \circ T$ such that $T(\varepsilon_A) \circ \eta_{TA} = id_{TA}$ and $S(\eta_B) \circ \varepsilon_{SB} = id_{SB}$, for all $A \in |\mathcal{A}|$ and $B \in |\mathcal{B}|$. The pair (S, T) is a duality iff η and ε are natural isomorphisms.

Theorem 7 *There exists a contravariant adjunction between the category* $\mathbf{LBA}$ *and the category* $\mathbf{ZHLC}$ *of locally compact zero-dimensional Hausdorff spaces and continuous maps.*

Sketch of the proof. We will only describe the contravariant functors $\Theta^a : \mathbf{LBA} \longrightarrow \mathbf{ZHLC}$ and $\Theta^t : \mathbf{ZHLC} \longrightarrow \mathbf{LBA}$ which realize the contravariant adjunction.

Let $X \in |\mathbf{ZHLC}|$. Define

$$\Theta^t(X) = (CO(X), CK(X)).$$

Then $\Theta^t(X)$ is an LBA. Let $f \in \mathbf{ZHLC}(X,Y)$. Define $\Theta^t(f) : \Theta^t(Y) \longrightarrow \Theta^t(X)$ by the formula

$$(1) \qquad \Theta^t(f)(G) = f^{-1}(G), \quad \forall G \in CO(Y).$$

For every LBA (A, I), set

$$\Theta^a(A, I) = L_I^A.$$

Then Lemma 6 implies that $L = \Theta^a(A, I)$ is a zero-dimensional locally compact Hausdorff space and $\lambda_{(A,I)}(I)$ is an open base of L. So, $\Theta^a(A, I) \in |\mathbf{ZHLC}|$. Let $\varphi \in \mathbf{LBA}((A, I), (B, J))$. We define the map

$$\Theta^a(\varphi) : \Theta^a(B, J) \longrightarrow \Theta^a(A, I)$$

by the formula

$$(2) \qquad \Theta^a(\varphi)(u') = \varphi^{-1}(u'), \quad \forall u' \in \Theta^a(B, J).$$

We even show that Θ^t is a full and faithful contravariant functor. $\square$

Definition 8 An LBA (A, I) is called a *ZLB-algebra* (briefly, *ZLBA*) if, for every $J \in Si(I)$, the join $\bigvee_A J (= \bigvee_A \{a \mid a \in J\})$ exists.

Let $\mathbf{ZLBA}$ be the full subcategory of the category $\mathbf{LBA}$ having as objects all ZLBAs.

Example 9 Let B be a Boolean algebra. Then the pair (B, B) is a ZLBA.

Remark 10 Note that if A and B are Boolean algebras then any Boolean homomorphism $\varphi : A \longrightarrow B$ is a $\mathbf{ZLBA}$-morphism between the ZLBAs (A, A) and (B, B). Hence, the full subcategory $\boldsymbol{B}$ of the category $\mathbf{ZLBA}$ whose objects are all ZLBAs of the form (A, A) is isomorphic (it can be even said that it coincides) with the category $\mathbf{Bool}$ of Boolean algebras and Boolean homomorphisms.

Theorem 11 *The categories* $\mathbf{ZHLC}$ *and* $\mathbf{ZLBA}$ *are dually equivalent. The corresponding duality functors are* $\Theta_d^a : \mathbf{ZLBA} \longrightarrow \mathbf{ZHLC}$ *and* $\Theta_d^t : \mathbf{ZHLC} \longrightarrow \mathbf{ZLBA}$*, which are restrictions of the contravariant functors* Θ^a *and* Θ^t*, respectively.*

Corollary 12 (Stone Duality Theorem [Stone]) *The categories* $\mathbf{Bool}$ *and* $\mathbf{ZHC}$ *are dually equivalent.*

Definition 13 Let $\mathbf{PZLBA}$ be the cofull subcategory of the category $\mathbf{ZLBA}$ whose morphisms $\varphi : (A, I) \longrightarrow (B, J)$ satisfy the following additional condition:

(PLBA) $\varphi(I) \subseteq J$.

Theorem 14 *The category* $\mathbf{PZHLC}$ *of all locally compact Hausdorff zero-dimensional spaces and all perfect maps between them is dually equivalent to the category* $\mathbf{PZLBA}$*. The corresponding duality functors are* $\Theta^a_p : \mathbf{ZLBA} \longrightarrow \mathbf{ZHLC}$ *and* $\Theta^t_p : \mathbf{ZHLC} \longrightarrow \mathbf{ZLBA}$*, which are restrictions of the contravariant functors* Θ^a_d *and* Θ^t_d*, respectively.*

The above theorem can be stated in a better form. We will do this now.

Definition 15 Let $\mathbf{PLBA}$ be the subcategory of the category $\mathbf{LBA}$ whose objects are all PLBAs and whose morphisms are all $\mathbf{LBA}$-morphisms $\varphi : (A, I) \longrightarrow (B, J)$ between the objects of $\mathbf{PLBA}$ satisfying condition (PLBA).

Theorem 16 *The category* $\mathbf{PLBA}$ *is dually equivalent to the category* $\mathbf{PZHLC}$.

Corollary 17 *There exists a bijective correspondence between the classes of all PLBAs (up to* $\mathbf{PLBA}$*-isomorphism), all ZLBAs (up to* $\mathbf{ZLBA}$*-isomorphism) and all locally compact zero-dimensional Hausdorff spaces (up to homeomorphism).*

We can even express Theorem 16 in a more simple form which is very close to the results obtained by M. Stone in [Stone].

Let $\mathbf{GBPL}$ be the category whose objects are all generalized Boolean pseudolattices and whose morphisms are all 0-pseudolattice homomorphisms $\varphi : I \longrightarrow J$ between its objects satisfying condition (LBA) (i.e., $\forall b \in J \ \exists a \in I$ such that $b \leq \varphi(a)$).

Define a contravariant functor

$$\Theta_g^t : \mathbf{PZHLC} \longrightarrow \mathbf{GBPL}$$

setting $\Theta_g^t(X) = CK(X)$, for every $X \in |\mathbf{PZHLC}|$, and if $f \in \mathbf{PZHLC}(X, Y)$ then

$$\varphi = \Theta_g^t(f) : CK(Y) \longrightarrow CK(X)$$

is defined by the formula $\varphi(G) = f^{-1}(G)$, for every $G \in CK(Y)$.

Let us recall the original Stone's construction of the dual space of a GBPL I (see [Stone]). Let I be a GBPL. Set $\Theta_s^a(I)$ to be the set X of all prime ideals of I endowed with a topology $\mathcal{O}$ having as an open base the set $\{\gamma_I(b) \mid b \in I\}$ where, for every $b \in I$, $\gamma_I(b) = \{i \in X \mid b \notin i\}$ (see M. Stone [Stone]).

Now, for every $I \in |\mathbf{GBPL}|$, set $\Theta_g^a(I) = \Theta_s^a(I)$. Further, if $\varphi \in \mathbf{GBPL}(I, J)$ then set $X = \Theta_g^a(I)$, $Y = \Theta_g^a(J)$ and define a map $f = \Theta_g^a(\varphi) : Y \longrightarrow X$ by the formula $f(j) = \varphi^{-1}(j)$, for every $j \in Y$.

Then $\Theta_g^a : \mathbf{GBPL} \longrightarrow \mathbf{PZHLC}$ is a contravariant functor and we obtain the following theorem:

Theorem 18 *The category* $\mathbf{PZHLC}$ *is dually equivalent to the category* $\mathbf{GBPL}$ *and the corresponding duality functors are* Θ_g^a *and* Θ_g^t.

Corollary 19 (M. Stone [Stone]) *There exists a bijective correspondence between the class of all (up to 0-pseudolattice isomorphism) generalized Boolean pseudolattices and all (up to homeomorphism) locally compact zero-dimensional Hausdorff spaces.*

Note that in [ST1], M. Stone proves that there exists a bijective correspondence between generalized Boolean pseudolattices and Boolean rings (with or without unit).

Some Other Stone-type Duality Theorems

Recall that a homomorphism φ between two Boolean algebras is called *complete* if it preserves all joins (and, consequently, all meets) that happen to exist; this means that if $\{a_i\}$ is a family of elements in the domain of φ with join a, then the family $\{\varphi(a_i)\}$ has a join and that join is equal to $\varphi(a)$.

Definition 20 We will denote by $\mathbf{SZHLC}$ the category of zero-dimensional locally compact Hausdorff spaces and skeletal maps.

Let $\mathbf{SZLBA}$ be the cofull subcategory of the category $\mathbf{ZLBA}$ whose morphisms are, in addition, complete homomorphisms.

Theorem 21 *The categories* $\mathbf{SZHLC}$ *and* $\mathbf{SZLBA}$ *are dually equivalent.*

Remarks 22 Note that in the definition of the category $\mathbf{SZLBA}$ the requirement that the morphisms $\varphi : (A, I) \longrightarrow (B, J)$ are complete can be replaced by the following condition:

(SkeZLBA) For every $b \in J \setminus \{0\}$ there exists $a \in I \setminus \{0\}$ such that $(\forall c \in A)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$.

Moreover, condition (SkeZLBA) can be replaced by the following one:

(CEP) For every $b \in B \setminus \{0\}$ there exists $a \in A \setminus \{0\}$ such that $(\forall c \in A)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$.

The assertion (c) of the next corollary is a zero-dimensional analogue of the Fedorchuk Duality Theorem [F].

Corollary 23 *(a) Let f be a* $\mathbf{PZHLC}$*-morphism. Then f is a quasi-open map iff $\Theta^t(f)$ is complete. In particular, if f is a* $\mathbf{ZHC}$*-morphism then f is a quasi-open map iff $S^t(f)$ is complete.*

(b) The cofull subcategory $\mathbf{QPZLC}$ *of the category* $\mathbf{PZHLC}$ *(see 14) whose morphisms are, in addition, quasi-open maps, is dually equivalent to the cofull subcategory* $\mathbf{QPZLBA}$ *of the category* $\mathbf{PZLBA}$ *whose morphisms are, in addition, complete homomorphisms;*

(c) The category $\mathbf{QZHC}$ *of compact zero-dimensional Hausdorff spaces and quasi-open maps is dually equivalent to the category* $\mathbf{CBool}$ *of Boolean algebras and complete Boolean homomorphisms.*

The last corollary together with Fedorchuk Duality Theorem [F] imply the following assertion in which the equivalence $(a) \iff (b)$ is a special case of a much more general theorem due to Monk [Monk].

Corollary 24 *Let* $\varphi \in \mathbf{Bool}(A, B)$ *and* A', B' *be minimal completions of* A *and* B *respectively. We can suppose that* $A \subseteq A'$ *and* $B \subseteq B'$*. Then the following conditions are equivalent:*

(a) φ *can be extended to a complete homomorphism* $\psi : A' \longrightarrow B'$*;*

(b) φ *is a complete homomorphism;*

(c) φ *satisfies condition (CEP) (see 22 above).*

Now, using Theorem 18, we will present in a simpler form the result established in Corollary 23(b).

Theorem 25 *The category* $\mathbf{QPZLC}$ *is dually equivalent to the cofull subcategory* $\mathbf{QGBPL}$ *of the category* $\mathbf{GBPL}$ *whose morphisms, in addition, preserve all meets that happen to exist.*

Remark 26 The proof of Theorem 25 shows that in the definition of the category $\mathbf{QPZLBA}$ the requirement that its morphisms $\varphi : I \longrightarrow J$ preserve all meets that happen to exist can be replaced by the following condition:

(QGBPL) For every $b \in J \setminus \{0\}$ there exists $a \in I \setminus \{0\}$ such that $(\forall c \in I)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$.

Theorem 27 *(a) Let* $f \in \mathbf{ZHLC}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ *and* $(B, J) = \Theta^t(Y)$. *Then the map* f *is open iff there exists a map* $\psi : I \longrightarrow J$ *which satisfies the following conditions:*

(OZL1) *For every* $b \in J$ *and every* $a \in I$, $(a \wedge \varphi(b) = 0) \rightarrow (\psi(a) \wedge b = 0)$, *and*

(OZL2) *For every* $a \in I$, $\varphi(\psi(a)) \geq a$

(such a map ψ *will be called a* lower pre-adjoint *of* φ*).*

(b) The cofull subcategory $\mathbf{OZHLC}$ *of the category* $\mathbf{ZHLC}$ *whose morphisms are open maps is dually equivalent to the cofull subcategory* $\mathbf{OZLBA}$ *of the category* $\mathbf{ZLBA}$ *whose morphisms have, in addition, lower pre-adjoints.*

Theorem 28 *(a) Let* $f \in \mathbf{PZHLC}(X, Y)$, $(A, I) = \Theta^t(X)$, $(B, J) = \Theta^t(Y)$ *and* $\varphi = \Theta^t(f)$*. Then the map* f *is open iff* $\varphi : B \longrightarrow A$ *has a lower adjoint* $\psi : A \longrightarrow B$*.*

(b) The cofull subcategory $\mathbf{OPZHLC}$ *of the category* $\mathbf{PZHLC}$ *whose morphisms are, in addition, open maps is dually equivalent to the cofull subcategory* $\mathbf{OPZLBA}$ *of the category* $\mathbf{PZLBA}$ *whose morphisms have, in addition, lower adjoints.*

Definition 29 Let $\varphi \in \mathbf{GBPL}(J, I)$. If $\psi : I \longrightarrow J$ is a map which satisfies conditions (OZL1) and

(OZL2) (see 27) then ψ is called a *lower preadjoint* of φ.

Let $\mathbf{OGBPL}$ be the cofull subcategory of the category $\mathbf{GBPL}$ whose morphisms have, in addition, lower preadjoints.

Corollary 30 *The category* $\mathbf{OGBPL}$ *is dually equivalent to the category* $\mathbf{OPZHLC}$.

Characterizations of the embeddings and of surjective and injective maps by means of their dual maps

In this section we will investigate the following problem: characterize the injective and surjective morphisms of the category $\mathbf{ZHLC}$ and its subcategories $\mathbf{PZHLC}$, $\mathbf{OZHLC}$ by means of some properties of their dual morphisms. Such a problem was considered by M. Stone in [Stone] for surjective continuous maps and for closed embeddings (i.e. for injective morphisms of the category $\mathbf{PZHLC}$). An analogous problem will be investigated for the homeomorphic embeddings and dense embeddings.

We start with a simple observation.

Proposition 31 *Let* $f \in \mathbf{ZHLC}(X,Y)$*,* $(A,I) = \Theta^t(X)$*,* $(B,J) = \Theta^t(Y)$ *and* $\varphi = \Theta^t(f)$*. Then* φ *is an injection* $\iff$ $\varphi_{|J}$ *is an injection* $\iff$ $\mathsf{cl}_Y(f(X)) = Y$*.*

Proposition 32 *Let* $f \in \mathbf{ZHLC}(X,Y)$*,* $\varphi = \Theta^t(f)$*,* $(A,I) = \Theta^t(X)$*,* $(B,J) = \Theta^t(Y)$ *and* $\varphi(B) \supseteq I$ *(or* $\varphi(J) \supseteq I$*). Then* f *is an injection.*

Theorem 33 *Let* $f \in \mathbf{ZHLC}(X,Y)$*,* $\varphi = \Theta^t(f)$*,* $(A,I) = \Theta^t(X)$ *and* $(B,J) = \Theta^t(Y)$*. Then* f *is an injection iff* $\varphi : (B,J) \longrightarrow (A,I)$ *satisfies the following condition:*

(InZLC) *For any* $a, b \in I$ *such that* $a \wedge b = 0$ *there exists* $a', b' \in J$ *with* $a' \wedge b' = 0$*,* $\varphi(a') \geq a$ *and* $\varphi(b') \geq b$*.*

Corollary 34 *The cofull subcategory* $\mathbf{InZHLC}$ *of the category* $\mathbf{ZHLC}$ *whose morphisms are, in addition, injective maps, is dually equivalent to the cofull subcategory* $\mathbf{DInZHLC}$ *of the category* $\mathbf{ZLBA}$ *whose morphism satisfy condition (InZLC) as well.*

In the sequel, we will not formulate corollaries like that because they follow directly from the respective characterization of injectivity or surjectivity and the corresponding duality theorems.

In the next theorem we will assume that the ideals and prime ideals could be non-proper.

Theorem 35 *Let $f \in \mathbf{ZHLC}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then the following conditions are equivalent:*

(a) f is a surjection;

(b) $\varphi : B \longrightarrow A$ is an injection and for every bounded ultrafilter v in (B, J) there exists $a \in I$ such that $a \wedge \varphi(v) \neq 0$ (i.e. $a \wedge \varphi(b) \neq 0$ for any $b \in v$);

(c) $\varphi : B \longrightarrow A$ is an injection and for every prime ideal J_1 of J, we have that $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$ implies $J_1 = J$ (where $I_{\varphi(b)} = \{a \in I \mid a \leq \varphi(b)\}$);

(d) $\varphi : B \longrightarrow A$ is an injection and for every ideal J_1 of J, $[(\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I) \rightarrow (J_1 = J)]$.

Remark 36 In [[Stone],Theorem 7] M. Stone proved a result which is equivalent to our assertion that (a)⇔(d) in the previous theorem.

Proposition 37 *Let (A, I) be a ZLBA, (B, J) be an LBA and $\psi : J \longrightarrow A$ be a 0-pseudolattice homomorphism satisfying condition (LBA) (i.e., $\forall a \in I\ \exists b \in J$ such that $a \leq \psi(b)$). Then ψ can be extended to a homomorphic map $\varphi : B \longrightarrow A$.*

Remark 38 Note that 36 and 37 imply that in Theorem 35 we can obtain new conditions equivalent to the condition (a) by replacing in (b), (c) and (d) the phrase "φ is an injection" by the phrase "$\varphi_{|J}$ is an injection".

Theorem 39 *Let $f \in \mathbf{OZHLC}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then f is an injection $\iff \varphi(J) \supseteq I \iff \varphi(B) \supseteq I$.*

Theorem 40 *Let $f \in \mathbf{PZHLC}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then f is a* *surjection* $\iff$ φ *is an injection* $\iff$ $\varphi_{|J}$ *is an injection.*

Theorem 41 *Let* $f \in \mathbf{PZHLC}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$ *and* $(B,J) = \Theta^t(Y)$. *Then* f *is an injection iff* $\varphi(J) = I$.

Obviously, the last two theorems imply the well-known Stone's results that a $\mathbf{ZHC}$-morphism f is an injection (resp., a surjection) iff $\varphi = S^t(f)$ is a surjection (resp., an injection).

Now we will be occupied with the homeomorphic embeddings. We will call them shortly *embeddings*.

Theorem 42 *Let* $f \in \mathbf{ZHLC}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$ *and* $(B,J) = \Theta^t(Y)$. *Then* f *is a dense embedding iff* φ *is an injection and* $\varphi(J) \supseteq I$.

Corollary 43 ([Stone]) *Let* $f \in \mathbf{ZHLC}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$ *and* $(B,J) = \Theta^t(Y)$. *Then* f *is a closed embedding iff* $\varphi(J) = I$.

Proposition 44 *Let* $f \in \mathbf{ZHLC}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ *and* $(B, J) = \Theta^t(Y)$. *Then* f *is an embedding iff there exists a ZLBA* (A_1, I_1) *and two* $\mathbf{ZLBA}$*-morphisms* $\varphi_1 : (A_1, I_1) \longrightarrow (A, I)$ *and* $\varphi_2 : (B, J) \longrightarrow (A_1, I_1)$ *such that* $\varphi = \varphi_1 \circ \varphi_2$, φ_1 *is an injection,* $\varphi_1(I_1) \supseteq I$ *and* $\varphi_2(J) = I_1$.

The construction of the dual objects of the closed, regular closed and open subsets

The next theorem is the well-known Stone's result [Stone] (written in our terms and notation) that open sets correspond to the ideals.

Theorem 45 ([Stone]) *Let* I *be a GBPL and* $(X, \mathcal{O}) = \Theta^a_s(I)$. *Then there exists a frame isomorphism*

$$\iota_s : (Idl(I), \leq) \longrightarrow (\mathcal{O}, \subseteq),\ \ J \mapsto \bigcup\{\gamma_I(a) \mid a \in J\}.$$

If $U \in \mathcal{O}$ *then* $J = \iota_s^{-1}(U)) = \{b \in I \mid \gamma_I(b) \subseteq U\}$, J *is isomorphic to the ideal* $J_U = \{F \in CK(X) \mid F \subseteq$

$U\}$ *of* $CK(X)$ $(= \Theta^t_g(X))$ *and* $J_U = CK(U)$*, i.e.* $J_U = \Theta^t_g(U)$*.*

Corollary 46 *Let* (A, I) *be a ZLBA and* $(X, \mathcal{O}) = \Theta^a(A, I)(= \Theta^a_g(I))$*. Then there exists a frame isomorphism*

$$\iota : (Idl(I), \leq) \longrightarrow (\mathcal{O}, \subseteq),\ J \mapsto \bigcup\{\lambda_{(A,I)}(a) \mid a \in J\}.$$

If $U \in \mathcal{O}$ *then* $J = \iota^{-1}(U) = \{b \in I \mid \lambda_{(A,I)}(b) \subseteq U\}$*, J is isomorphic to the ideal* $J_U = \{F \in CK(X) \mid F \subseteq U\}$ *of* $CK(X)$ $(= \Theta^t_g(X))$ *and* $J_U = CK(U)$*, i.e.* $J_U = \Theta^t_g(U)$*.*

Corollary 47 ([[Stone],Theorem 5]) *Let* I *be a GBPL,* $(X, \mathcal{O}) = \Theta^a_s(I)$*,* J *be an ideal of* I *and* $U = \iota_s(J)$*. Then:*

(a) U *is a clopen set* $\iff$ J *is a simple ideal of* I*;*

(b) U *is a regular open set iff* J *is a normal ideal of* I*;*

(c) U is a compact open set iff J is a principal ideal of I.

If (A, I) is an LBA and $a \in A$ then the ideal $I_a = \{b \in I \mid b \leq a\}$ of I will be called an *A-principal ideal of I*.

Corollary 48 *Let (A, I) be a ZLBA, $(X, \mathcal{O}) = \Theta^a(A, I)$ ($= \Theta^a_g(I)$), J be an ideal of I and $U = \iota(J)$. Then:*

(a) U is a clopen set $\iff$ J is a simple ideal of I $\iff$ J is an A-principal ideal;

(b) U is a regular open set iff J is a normal ideal of I;

(c) U is a compact open set iff J is a principal ideal of I.

The above results show that if $X \in |\mathbf{ZHLC}|$ and U is an open subset of X then $\iota^{-1}(U)$ (or, equivalently, $\iota_s^{-1}(U)$) is $\mathbf{GBPL}$-isomorphic to $\Theta^t_g(U)$.

Now, for every $X \in |\mathbf{ZHLC}|$, we will find the connections between the dual objects $\Theta_g^t(F)$ of the closed or regular closed subsets F of X and the dual object $\Theta_g^t(X)$ of X. The obtained result for regular closed subsets of X seems to be new even in the compact case.

Theorem 49 *Let $I, J \in |\mathbf{GBPL}|$, $X = \Theta_g^a(I)$ and $F = \Theta_g^a(J)$. Then:*

(a)([[Stone],Theorem 4(4)]) *F is homeomorphic to a closed subset of X iff there exists a 0-pseudolattice epimorphism $\varphi : I \longrightarrow J$ (i.e. iff J is a quotient of I);*

(b) F is homeomorphic to a regular closed subset of X if and only if there exists a 0-pseudolattice epimorphism $\varphi : I \longrightarrow J$ which preserves all meets that happen to exist in I.

We will finish with mentioning some assertions about isolated points. All these statements have easy proofs.

Proposition 50 *Let* (A, I) *be a ZLBA,* $a \in A$ *and* $X = \Theta^a(A, I)$*. Then* a *is an atom of* A *iff* $\lambda_{(A,I)}(a)$ *is an isolated point of the space* X*. Also, for every isolated point* x *of* X *there exists an* $a \in I$ *such that* a *is an atom of* I *(equivalently, of* A*) and* $\{x\} = \lambda_{(A,I)}(a)$*.*

Proposition 51 *Let* (A, I) *be a ZLBA and* $X = \Theta^a(A, I)(= \Theta^a_g(I))$*. Then* X *is a discrete space* $\Longleftrightarrow$ *the elements of* I *are either atoms of* I *or finite sums of atoms of* I*.*

Proposition 52 (M. Stone [Stone]) *Let* (A, I) *be a ZLBA and* $X = \Theta^a(A, I)(= \Theta^a_g(I))$*. Then* X *is an extremally disconnected space iff* A *is a complete Boolean algebra.*

Proposition 53 *Let* (A, I) *be a ZLBA and* $X = \Theta^a(A, I)(= \Theta^a_g(I))$*. Then the set of all isolated points of* X *is dense in* X *iff* A *is an atomic Boolean algebra iff* I *is an atomic 0-pseudolattice.*

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THANKS!