Sahlqvist theorem for modal fixed point logics

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Joint work with

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Overview

In classical modal logic Sahlqvist's theorem provides an axiomatically defined class of logics sound and complete wrt to first-order definable classes of frames.

Sambin and Vaccaro (1989) gave a proof of Sahlqvist completeness and correspondence theorems using descriptive frames and topology.

Our goal is to extend the method of Sambin and Vaccaro from modal logics to modal fixed point logics and see what consequences this method has for completeness and correspondence of modal fixed point logics.

Outline

1. An overview of the existing dualities.
2. Generalized semantics for modal fixed point logics.
3. Sahlqvist's theorem.

Part I: Duality

Language of the modal μ-calculus

- countably infinite set of propositional variables,
- constants $\bot$ and $\top$,
- connectives $\wedge$, $\vee$, $\neg$,
- modal operators $\Diamond$ and $\Box$,
- $\mu x \varphi(x, x_1, \ldots, x_n)$ for all formulas $\varphi(x, x_1, \ldots, x_n)$, where x occurs under the scope of an even number of negations.

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Theorem. Every modal logic is complete wrt modal algebras.

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Theorem. The category of modal algebras and corresponding homomorphisms is dually equivalent to the category of descriptive frames and continuous p-morphisms.

Corollary. Every modal logic is complete wrt descriptive frames.

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Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. A map h from propositional variables to B is called an algebra assignment. We define a (possibly partial) semantics for modal μ-formulas by the following inductive definition.

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- $[\bot]_h = 0$
- $[\top]_h = 1$
- $[x]_h = h(x)$, where x is a propositional variable,
- $[\varphi \wedge \psi]_h = [\varphi]_h \wedge [\psi]_h$,
- $[\varphi \vee \psi]_h = [\varphi]_h \vee [\psi]_h$,
- $[\neg\varphi]_h = \neg[\varphi]_h$,
- $[\Diamond\varphi]_h = \Diamond[\varphi]_h$,
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We denote by h_x^a a new algebra assignment such that $h_x^a(x) = a$ and $h_x^a(y) = h(y)$ for each propositional variable $y \neq x$ and $a \in B$.

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- If $\varphi(x, x_1, \ldots, x_n)$ is positive in x then

  $$[\mu x \varphi(x, x_1, \ldots, x_n)]_h = \bigwedge \{a \in B : [\varphi(x, x_1, \ldots, x_n)]_{h^a_x} \leq a\},$$

  if this meet exists; otherwise, the semantics for $\mu x \varphi(x, x_1, \ldots, x_n)$ is undefined.

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Notation: To simplify the notations instead of $[\varphi(x_1, \ldots, x_n)]_h$ with $h(x_i) = a_i$, $1 \leq i \leq n$, we will simply write $\varphi(a_1, \ldots, a_n)$.

Semantics of the modal μ-calculus

Let (W, R) be a descriptive frame, $\mathfrak{F} \subseteq \mathcal{P}(W)$ and h an arbitrary assignment, that is, a map from the propositional variables to $\mathcal{P}(W)$. We define the semantics for modal μ-formulas by the following inductive definition.

- $[\![\bot]\!]_h^{\mathfrak{F}} = \emptyset$,
- $[\![\top]\!]_h^{\mathfrak{F}} = W$,
- $[\![x]\!]_h^{\mathfrak{F}} = h(x)$, where x is a propositional variable,
- $[\![\varphi \wedge \psi]\!]_h^{\mathfrak{F}} = [\![\varphi]\!]_h^{\mathfrak{F}} \cap [\![\psi]\!]_h^{\mathfrak{F}}$,
- $[\![\varphi \vee \psi]\!]_h^{\mathfrak{F}} = [\![\varphi]\!]_h^{\mathfrak{F}} \cup [\![\psi]\!]_h^{\mathfrak{F}}$,
- $[\![\neg\varphi]\!]_h^{\mathfrak{F}} = W \setminus [\![\varphi]\!]_h^{\mathfrak{F}}$,
- $[\![\Diamond\varphi]\!]_h^{\mathfrak{F}} = \langle R \rangle [\![\varphi]\!]_h^{\mathfrak{F}}$,
- $[\![\Box\varphi]\!]_h^{\mathfrak{F}} = [R] [\![\varphi]\!]_h^{\mathfrak{F}}$,

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Let $\varphi(x, x_1, \ldots, x_n)$ be a modal μ-formula. A set $U \in \mathfrak{F}$ is called a pre-fixed point if $[\![\varphi(x, x_1, \ldots, x_n)]\!]^{\mathfrak{F}}_{h_x^U} \subseteq U$.

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Let $\varphi(x, x_1, \ldots, x_n)$ be positive in x, then

$$[\![\mu x \varphi(x, x_1, \ldots, x_n))]\!]^{\mathfrak{F}}_h = \bigcap \{U \in \mathfrak{F} : [\![\varphi(x, x_1, \ldots, x_n)]\!]^{\mathfrak{F}}_{h_x^U} \subseteq U\}.$$

We assume that $\bigcap \emptyset = W$.

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Definition. A descriptive frame (W, R) is called a descriptive μ-frame if for each clopen assignment h and for each modal μ-formula φ, the set $[\![\varphi]\!]_h^{\mathsf{Clop}(W)}$ is clopen.

Notations

Notation: To simplify the notations instead of $[\![\varphi(x_1, \ldots, x_n)]\!]_h^{\mathfrak{F}}$ with $h(x_i) = U_i$, $1 \leq i \leq n$, we will simply write $\varphi(U_1, \ldots, U_n)^{\mathfrak{F}}$. Moreover, we will skip the index $\mathfrak{F}$ if it is clear from the context.

Modal μ-algebras and descriptive μ-frames

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Ambler and Co. also extend this to the duality of the corresponding categories of modal μ-algebras and descriptive μ-frames.

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3. Every locally finite modal algebra is a modal μ-algebra. An algebra is locally finite if its every finitely generated subalgebra is finite.

Normal modal fixed point logics

The axiomatization of Kozen's system $\mathbf{K}^{\mu}$ consists of the following axioms and rules

propositional tautologies,	
If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$	(Modus Ponens),
If $\vdash \varphi$, then $\vdash \varphi[p/\psi]$	(Substitution),
If $\vdash \varphi$, then $\vdash \Box\varphi$	(Necessitation),
$\vdash \Box(p \to q) \to (\Box p \to \Box q)$	(**K**-axiom),
$\vdash \varphi[x/\mu x\varphi] \to \mu x\varphi$	(Fixed Point axiom),
If $\vdash \varphi[x/\psi] \to \psi$, then $\vdash \mu x\varphi \to \psi$	(Fixed Point rule),

where x is not a bound variable of φ and no free variable of ψ is bound in φ.

Normal modal fixed point logics

Let Φ be a set of modal μ-formulas. We write $\mathbf{K}^\mu + \Phi$ for the smallest set of formulas which contains both $\mathbf{K}^\mu$ and Φ and is closed under the Modus Ponens, Substitution, Necessitation and Fixed Point rules.

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Let $L = \mathbf{K}^\mu + \Phi$ be a normal modal fixed point logic. A modal μ-algebra $(B, \Diamond)$ is called an L-algebra if it validates all the formulas in Φ.

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Let $L = \mathbf{K}^\mu + \Phi$ be a normal modal fixed point logic. A modal μ-algebra $(B, \Diamond)$ is called an L-algebra if it validates all the formulas in Φ. A descriptive μ-frame (W, R) is called an L-frame if (W, R) validates all the formulas in Φ with respect to clopen assignments.

Completeness

Theorem (Ambler and Co. 1995, ten Cate and Fontaine 2010).
Let L be a normal modal fixed point logic. Then

1. L is sound and complete with respect to the class of modal μ-L-algebras.
2. L is sound and complete with respect to the class of descriptive μ-L-frames.

Part II: Generalized fixed points

Comparing the semantics

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Lemma. Let (W, R) be a descriptive μ-frame dual to a locally finite modal algebra. Then for each formula φ and clopen assignment h, we have

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Lemma. Let (W, R) be a descriptive μ-frame dual to a complete modal algebra. Then for each modal μ-formula φ and each clopen assignment h, we have

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There exists, however, a descriptive μ-frame dual to a complete modal algebra and a closed assignment such that all the three semantics differ.

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Then $[\![\mu x \varphi]\!]_h^{\mathsf{Clop}(W)}$ is the least fixed point of $f_{\varphi,h}$.

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Then $[\![\mu x\varphi]\!]_{h}^{\mathsf{Clop}(W)}$ is a fixed point of the map $f_{\varphi,h}$. But, in general, is not the least fixed point.

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Then $[\![\mu x \varphi]\!]_h^{\mathsf{Clop}(W)}$, in general, is not a fixed point of $f_{\varphi,h}$.

Part III: Sahlqvist's theorem

The intersection lemma

Lemma (Esakia-Sambin-Vaccaro). Let (W, R) be a descriptive frame and $F \subseteq W$ a closed set. Then for each positive modal formula φ we have

$$\varphi(F) = \bigcap \{\varphi(U) : U \in \mathsf{Clop}(W), F \subseteq U\}.$$

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Sahlqvist's formulas

Definition. A formula $\varphi(p_1, \ldots, p_n)$ is called a Sahlqvist fixed point formula if it is obtained from formulas of the form $\neg\Box^m p_i$ ($m \in \omega$, $i \leq n$) and positive formulas (in the language with the μ-operator) by applying the operations $\vee$ and $\Box$.

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Theorem. Let (W, R) be a descriptive μ-frame, $w \in W$ and $\varphi(p_1, \ldots, p_n)$ a Sahlqvist fixed point formula. Then $w \in [\![\varphi]\!]_h^{\mathsf{Clop}(W)}$, for each clopen assignment h, implies $w \in [\![\varphi]\!]_f^{\mathsf{Clop}(W)}$, for each set-theoretic assignment f.

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That is, $(W, R, f), w \not\models \Box(\neg\Box p \vee \psi)$.

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Finally, by the intersection lemma,
$\psi(F) = \bigcap\{\psi(U) \in \mathsf{Clop}(W) : U \in \mathsf{Clop}(W), F \subseteq U\}$.

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The following are equivalent:

- The correspondent is true in a descriptive frame (W, R).
- The Sahlqvist formula is valid in (W, R) under clopen assignments.
- The Sahlqvist formula is valid in (W, R) under set-theoretic assignments.

The Sahlqvist theorem

Theorem. Every Sahlqvist modal fixed point logic is sound and complete under clopen assignments wrt a class of descriptive frames that is definable in the first-order logic with fixed points.

Conclusions and future work

- We looked into order-topological semantics of modal fixed point logics.
- Extended the Esakia-Sambin-Vaccaro Lemma and the proof of Sahlqvist's theorem to modal fixed point logics.
- Next step is to look into particular examples of Sahlqvist formulas and derive, from our general theory, some concrete (interesting) completeness results.

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We say that a modal fixed point formula φ is valid in a descriptive frame (W, R) if $[\![\varphi]\!]_h^{\mathsf{Clop}(W)} = W$, for each set-theoretic assignment h.

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However, the fixed point rule is not sound wrt to this semantics.

Question: Find an axiomatization of the valid modal fixed point formulas under this semantics.