Topological completeness of polymodal provability logic GLP

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Gödel's incompleteness theorem

A theory T is gödelian, if

- natural numbers, $+$ and $\cdot$ are definable in T;
- T proves some obvious properties of these operations;
- the set of axioms of T is computable.

$$\mathrm{Con}(T) = \text{«}T \text{ is consistent»}$$

Gödel (1931): If a gödelian theory T is consistent, then $\mathrm{Con}(T)$ is true but unprovable in T.

Lindenbaum algebras

Lindenbaum algebra of a theory T:
$\mathcal{L}_T = \{\text{sentences of } T\}/\sim_T$, where

$$\varphi \sim_T \psi \iff T \vdash (\varphi \leftrightarrow \psi)$$

$\mathcal{L}_T$ is a boolean algebra with operations $\wedge$, $\vee$, $\neg$.
$\mathbf{1}$ = the set of provable sentences of T
$\mathbf{0}$ = the set of refutable sentences of T

For consistent gödelian T all such algebras are countable atomless, hence pairwise isomorphic.

Kripke, Pour-El: even computably isomorphic

Provability algebras

Emerged in 1970s: Macintyre/Simmons, Magari, Smoryński, ...

Consistency operator $\Diamond : \mathcal{L}_T \to \mathcal{L}_T$

$$\varphi \longmapsto \mathrm{Con}(T + \varphi).$$

$(\mathcal{L}_T, \Diamond)$ = *provability algebra of T*
$\Box\varphi = \neg\Diamond\neg\varphi =$ «φ is provable in T»

Characteristic of $(M, \Diamond)$:
$ch(M) = \min\{k : \Diamond^k \mathbf{1} = \mathbf{0}\}$;
$ch(M) = \infty$, if no such k exists.

Remark. If $\mathbb{N} \vDash T$, then $ch(\mathcal{L}_T) = \infty$.

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Identities of provability algebras

K. Gödel (33), M.H. Löb (55): Algebra $(\mathcal{L}_T, \Diamond)$ satisfies the following set of identities *GL*:

- boolean identities
- $\Diamond \mathbf{0} = \mathbf{0}$
- $\Diamond(\varphi \vee \psi) = (\Diamond\varphi \vee \Diamond\psi)$
- $\Diamond\varphi = \Diamond(\varphi \wedge \neg\Diamond\varphi)$ (Löb's identity)

GL-algebras = Magari algebras = diagonalizable algebras

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GL-algebras = Magari algebras = diagonalizable algebras

Provability logic

Let $\mathcal{A} = (A, \Diamond)$ be a boolean algebra with an operator $\Diamond$, and $\varphi(\vec{x})$ a term.

Def. Denote

- $\mathcal{A} \vDash \varphi$ if $\mathcal{A} \vDash \forall \vec{x}\, (\varphi(\vec{x}) = \mathbf{1})$;
- The logic of $\mathcal{A}$ is $Log(\mathcal{A}) = \{\varphi : \mathcal{A} \vDash \varphi\}$.

R. Solovay (76): If $ch(\mathcal{L}_T) = \infty$, then $Log(\mathcal{L}_T, \Diamond) = GL$.

GL is nice as a modal logic (decidable, Kripke complete, fmp, Craig, cut-free calculus, ...)

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n-consistency

Def. A gödelian theory T is n-consistent, if every provable Σ_n^0-sentence of T is true.

$n\text{-Con}(T) =$ « T is n-consistent »

n-consistency operator $\langle n \rangle : \mathcal{L}_T \to \mathcal{L}_T$

$$\varphi \longmapsto n\text{-Con}(T + \varphi).$$

$[n] = \neg\langle n \rangle\neg$ (n-provability)

The algebra of n*-provability*

$\mathcal{M}_T = (\mathcal{L}_T; \langle 0\rangle, \langle 1\rangle, \ldots)$.

The following identities GLP hold in $\mathcal{M}_T$:

- GL, for all $\langle n\rangle$;
- $\langle n+1\rangle\varphi \to \langle n\rangle\varphi$;
- $\langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$.

G. Japaridze (86): If $\mathbb{N} \models T$, then $Log(\mathcal{M}_T) = GLP$.

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GLP_n is GLP in the language with n operators. $GLP_1 = GL$.

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The significance of GLP

GLP is

- Useful for proof theory:
  - Ordinal notations and consistency proof for PA;
  - Independent combinatorial assertion;
  - Characterization of provably total computable functions of PA.

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Set-theoretic interpretation (neighborhood semantics)

Let X be a nonempty set, $\mathcal{P}(X)$ the b.a. of subsets of X.

Consider any operator $\delta : \mathcal{P}(X) \to \mathcal{P}(X)$ and the structure $(\mathcal{P}(X), \delta)$.

Question: Can $(\mathcal{P}(X), \delta)$ be a GL-algebra and, if yes, when?

Def. Write $(X, \delta) \vDash \varphi$ if $(\mathcal{P}(X), \delta) \vDash \varphi$. Also let $Log(X, \delta) := Log(\mathcal{P}(X), \delta)$.

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Derived set operators

Let X be a topological space, $A \subseteq X$.
Derived set $d(A)$ of A is the set of limit points of A:

$$x \in d(A) \iff \forall U_x \text{ open } \exists y \neq x \; y \in U_x \cap A.$$

Fact. If $(X, \delta) \models GL$ then X naturally bears a topology τ for which $\delta = d_\tau$, that is, $\delta : A \longmapsto d_\tau(A)$, for each $A \subseteq X$.

In fact, we can define: A is τ-closed iff $\delta(A) \subseteq A$.
Equivalently, $c(A) = A \cup \delta(A)$ is the closure of A.

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Scattered spaces

Definition (Cantor): X is scattered if every nonempty $A \subseteq X$ has an isolated point.

Cantor-Bendixon sequence:

$$X_0 = X, \quad X_{\alpha+1} = d(X_\alpha), \quad X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha, \text{if } \lambda \text{ is a limit.}$$

Notice that all X_α are closed and $X_0 \supset X_1 \supset X_2 \supset \ldots$

Fact (Cantor): X is scattered $\iff \exists\alpha : X_\alpha = \varnothing$.

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Examples

- Left topology $\tau_{\prec}$ on a strict partial ordering $(X, \prec)$.
  $A \subseteq X$ is open iff $\forall x, y\ (y \prec x \in A \Rightarrow y \in A)$.

  Fact: $(X, \prec)$ is well-founded iff $(X, \tau_{\prec})$ is scattered.

- Ordinal Ω with the usual order topology generated by intervals (α, β), $[0, \beta)$, (α, Ω) such that $\alpha < \beta$.

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Löb's identity = scatteredness

Simmons 74, Esakia 81

Löb's identity: $\Diamond A = \Diamond(A \wedge \neg\Diamond A)$.

Topological reading:

$$d(A) = d(A \setminus d(A)) = d(\mathrm{iso}(A)),$$

where $\mathrm{iso}(A) = A \setminus d(A)$ is the set of isolated points of A.

Fact: The following are equivalent:

- X is scattered;
- $d(A) = d(\mathrm{iso}(A))$ for any $A \subseteq X$;
- $(X, d) \vDash GL$.

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Completeness theorems

Theorem (Esakia 81): There is a scattered X such that $Log(X, d) = GL$. In fact, X is the left topology on a countable well-founded partial ordering.

Theorem (Abashidze/Blass 87/91): Consider $\Omega \geq \omega^\omega$ with the order topology. Then $Log(\Omega, d) = GL$.

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Topological models for GLP

We consider poly-topological spaces $(X; \tau_0, \tau_1, \ldots)$ where modality $\langle n \rangle$ corresponds to the derived set operator d_n w.r.t. τ_n.

Definition: X is a *GLP*-space if

- τ_0 is scattered;
- For each $A \subseteq X$, $d_n(A)$ is τ_{n+1}-open;
- $\tau_n \subseteq \tau_{n+1}$.

Remark: In a *GLP*-space, all τ_n are scattered.

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Remark: In a *GLP*-space, all τ_n are scattered.

Basic example: Esakia space

Consider a bitopological space (Ω, τ_0, τ_1), where

- Ω is an ordinal;
- τ_0 is the left topology on Ω;
- τ_1 is the interval topology on Ω.

Fact (Esakia): (Ω, τ_0, τ_1) is a model of GLP_2, but not an exact one: linearity axiom holds for $\langle 0 \rangle$.

Next topology and generated GLP-space

Let (X, τ) be a scattered space.

Fact: There is the coarsest topology τ^+ on X such that $(X; \tau, \tau^+)$ is a GLP_2-space.

The next topology τ^+ is generated by τ and $\{d(A) : A \subseteq X\}$ (as a subbase).

Thus, any (X, τ) generates a GLP-space $(X; \tau_0, \tau_1, \ldots)$ with $\tau_0 = \tau$ and $\tau_{n+1} = \tau_n^+$, for each n.

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Completeness for GLP_2

GLP_2 is complete w.r.t. GLP_2-spaces generated from the left topology on a well-founded partial ordering (with Guram Bezhanishvili and Thomas Icard).

Theorem: There is a countable GLP_2-space X such that $Log(X, d_0, d_1) = GLP_2$.

In fact, X has the form $(X; \tau_{\prec}, \tau^{+}_{\prec})$ where $(X, \prec)$ is a well-founded partial ordering.

Aside: This seems to be the first example of a finitely axiomatizable logic that is topologically complete but not Kripke complete.

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Difficulties

Difficulties for three or more operators.

Fact. If (X, τ) is hausdorff and first-countable (i.e. if each point has a countable neighborhood base), then (X, τ^+) is discrete.

Proof: Each $a \in X$ is a unique limit of a countable sequence $A = \{a_n\}$. Hence, $\{a\} = d(A)$ is open.

Ordinal GLP-spaces

Let τ_0 be the left topology on an ordinal Ω. It generates a *GLP*-space $(\Omega; \tau_0, \tau_1, \dots)$. What are these topologies?
Let θ_n denote the first limit point of τ_n.

	name	θ_n	$d_n(A)$
τ_0	left	1	$\{\alpha : A \cap \alpha \neq \varnothing\}$
τ_1	order	ω	$\{\alpha \in Lim : A \cap \alpha$ is unbounded in $\alpha\}$
τ_2	club	ω_1	$\{\alpha : cf(\alpha) > \omega$ and $A \cap \alpha$ is stationary in $\alpha\}$
τ_3	Mahlo	θ_3	$\dots \; \dots$

Remarks: 1) Set theorists call d_2 Mahlo operation.

2) θ_3 is the so-called *doubly reflecting cardinal*, its existence is not provable in ZFC (equiconsistent with the existence of weakly compact cardinals). Studied by Magidor, Shelah and others.

Questions

Corollary: It is consistent with ZFC that (Ω, τ_3) is discrete.

Questions:

- Is there a *GLP*-space for which all τ_n are non-discrete?
- Is *GLP* topologically complete?

Topological completeness

GLP is complete w.r.t. (countable, hausdorff) GLP-spaces.

Theorem (B., Gabelaia 10): There is a countable hausdorff GLP-space X such that $Log(X) = GLP$.

In fact, X is ε_0 equipped with topologies refining the order topology, where $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$.

Remark: If GLP is complete w.r.t. a GLP-space X, then all topologies of X have Cantor-Bendixon rank $\geq \varepsilon_0$.

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Remark: If *GLP* is complete w.r.t. a *GLP*-space X, then all topologies of X have Cantor-Bendixon rank $\geq \varepsilon_0$.

Some ideas of proof

- We are going to define a suitable class of scattered spaces, called *maximal*, which are well-behaved w.r.t. the operation $\tau \mapsto \tau^+$.
- We sketch how to build a non-discrete *GLP*-space using maximal spaces.
- Then we mention necessary modifications and some other ingredients needed for a completeness proof.

Rank function

Let X be a scattered space.
Let $d^{\alpha}(X)$ denote the α-th term in the Cantor–Bendixon sequence.
Let the *rank function* $\rho : X \to On$ be defined by

$$\rho(x) := \min\{\alpha : x \notin d^{\alpha+1}(X)\}.$$

$\rho(X) := \min\{\alpha : d^{\alpha}(X) = \varnothing\}$ is the *rank of* X.

Examples:

- $\rho_{<}(\alpha) = \alpha$, for the left topology;
- $r(\alpha) = \beta$, if $\alpha = \gamma + \omega^{\beta}$, and $r(0) = 0$, for the order topology.

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d-*maps*

Def. A function $f : X \to Y$ is a d-map, if

- f is open;
- f is continuous;
- f is pointwise discrete, i.e., $f^{-1}(a)$ is discrete, for each $a \in Y$.

Properties:

- $f^{-1}(d_Y(A)) = d_X(f^{-1}(A))$, for any $A \subseteq Y$;
- $f^{-1} : (\mathcal{P}(Y), d_Y) \to (\mathcal{P}(X), d_X)$ is a homomorphism;
- If f is onto, then $Log(X) \subseteq Log(Y)$.

d-maps and rank

Fact. Let On be the space of ordinals taken with the *left* topology.

- $\rho : X \to On$ is a d-map;
- If $f : X \to On$ is a d-map, then $f = \rho$.

Corollary. If $f : X \to Y$ is a d-map, then $\rho_X = \rho_Y \circ f$.

Maximal spaces

Def. Let $f : X \to Y$ be a d-map.

- (X, τ) is *maximal w.r.t. f*, if τ is a maximal topology on X such that f is a d-map (equivalently, f is open).
- (X, τ) is *maximal*, if (X, τ) is maximal w.r.t. the rank function $\rho_\tau : X \to On$, that is,

$$\forall \sigma \, (\sigma \supsetneqq \tau \;\Rightarrow\; \exists x \, \rho_\sigma(x) \neq \rho_\tau(x)).$$

Fact. For every d-map $f : X \to Y$, the topology of X can be extended to a maximal one w.r.t. f.

Lifting lemma

Recall that τ^+ on X is generated by τ and $\{d(A) : A \subseteq X\}$.

Let X^+ denote the space (X, τ^+).

Lemma. Let $f : X \twoheadrightarrow Y$ be an onto d-map. If X is maximal, then $f : X^+ \twoheadrightarrow Y^+$ is a d-map.

Comment. In general, 'next topology' operation is non-monotonic: There is a space X such that X^+ is discrete while $(X')^+$ is not, where X' is some maximal extension of X.

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Rank function for the next topology

Let ρ^+ be the rank function of X^+.

Corollary. If X is maximal, then $\rho^+ = r \circ \rho$.

Proof. Let $\Omega := \rho(X)$ be the rank of X. Consider the d-map $\rho : X \twoheadrightarrow \Omega$ where Ω is taken with the left topology.

- By Lemma, $\rho : X^+ \twoheadrightarrow \Omega^+$ is a d-map.
- r is the rank function of Ω^+ (the order topology on Ω).
- Hence, $r \circ \rho$ is the rank function of X^+.

Comment. For an arbitrary scattered X we only have $\rho^+ \leq r \circ \rho$.

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Let ρ^+ be the rank function of X^+.

Corollary. If X is maximal, then $\rho^+ = r \circ \rho$.

Proof. Let $\Omega := \rho(X)$ be the rank of X. Consider the d-map $\rho : X \twoheadrightarrow \Omega$ where Ω is taken with the left topology.

- By Lemma, $\rho : X^+ \twoheadrightarrow \Omega^+$ is a d-map.
- r is the rank function of Ω^+ (the order topology on Ω).
- Hence, $r \circ \rho$ is the rank function of X^+.

Comment. For an arbitrary scattered X we only have $\rho^+ \leq r \circ \rho$.

ME spaces

Def. A GLP-space $(X, \tau_0, \tau_1, \ldots)$ is *ME* if

- τ_0 is maximal;
- for each n, τ_{n+1} is a maximal extension of τ_n^+.

Let ρ_n be the rank function of τ_n.

Lemma. $\rho_{n+1} = r \circ \rho_n$.

Proof. τ_{n+1} has the same rank function as τ_n^+, being its maximal extension, hence $\rho_{n+1} = \rho_n^+$. By the Corollary, $\rho_n^+ = r \circ \rho_n$.

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A non-discrete GLP-space

Take any scattered space (X, τ) whose rank Ω satisfies $\omega^\Omega = \Omega$. For example, $X = \varepsilon_0$ with the order topology.

Construct topologies $\tau_0 \subseteq \tau_1 \subseteq \tau_2 \subseteq \dots$ by:

$$\tau_0 = \tau'; \quad \tau_{n+1} = (\tau_n^+)',$$

where σ' means any maximal extension of σ.

Theorem.

1. $(X, \tau_0, \tau_1, \dots)$ is an ME *GLP*-space.
2. $\rho_n(X) = r^n(\rho_0(X)) = r^n(\Omega) = \Omega$, for each n.

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Ingredients of the completeness proof

- Weakening maximality to *limit maximality* condition. A larger class of LME-spaces is defined.
- A well-behaved subsystem J of GLP with finite Kripke models, J-models.
- Constructing for each finite J-model M a LME-space X together with a *weak* d*-map* $X \twoheadrightarrow M$.

Topological constructions

This is based on two topological constructions with LME-spaces:

- lifting;
- d-product.

d-product generalizes to arbitrary scattered spaces the operation of ordinal multiplication.

Thank you!