Duality for completely distributive spaces

Dirk Hofmann

In [2] we introduced the notion of a *topological theory* $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ – consisting of a monad $\mathbb{T} = (T, e, m)$, a quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$ and a map $\xi : T\mathsf{V} \to \mathsf{V}$ – as a possible “syntax” for Topology. In fact, this concept permits us to view several objects of topology (including topological spaces, of course) as generalised ordered sets and allows to carry order-theoretic notions and results into the realm of topology.

In particular we are interested in the well-known adjunction

$$\mathsf{Ord} \underset{\perp}{\rightleftarrows} \mathsf{CCD}^{\mathrm{op}}$$

between the category Ord of ordered sets and monotone maps and the dual of the category CCD of (constructively) completely distributive lattices and left and right adjoint monotone maps. This adjunction can be constructed by either sending an ordered set X to the set of all down-sets of X ($X \mapsto \mathsf{Ord}(X^{\mathrm{op}}, 2)$) or to the set of all up-sets of X ($X \mapsto \mathsf{Ord}(X, 2)$). Certainly, the dual adjunction between Top and Frm can be seen as an extension of $\mathsf{Ord} \rightleftarrows \mathsf{CCD}^{\mathrm{op}}$; however, this is only really true for the second construction. The first one does not even seem to make sense for topological spaces since it is not clear what X^{op} means now. But our study of “spaces as categories” required such a notion anyway, and since [1] we have a candidate which so far proved to be useful. Therefore we ask here about the construction $X \mapsto \mathsf{Top}(X^{\mathrm{op}}, 2)$, and the answer leads to the adjunction

$$\mathsf{Top} \underset{\perp}{\rightleftarrows} \mathsf{CDTop}^{\mathrm{op}},$$

between topological and (suitably defined) completely distributive spaces which seems to be even closer to the Ord-case as the “usual” dual adjunction with frames. Based on the work [3, 4] of R. Rosebrugh and R.J. Wood, in this talk we present a general duality theory for completely distributive spaces

References

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[4] ——, *Split structures*, Theory Appl. Categ., 13 (2004), No. 12, 172–183.