Subordinated Boolean Algebras and Pointfree Covers

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We describe a pointfree approach to covers in topology. A *cover* of a space X is a completely regular space Y together with a perfect irreducible surjection (*covering map*) $\pi : Y \twoheadrightarrow X$. For an arbitrary c.r. X (and sometimes just regular X), there exist covers which are minimal among certain special classes of spaces such as extremally disconnected, basically disconnected, and quasi-F spaces. In some cases, the mininal covers in these classes can be characterized as maximal covers when the covering map is required to have certain additional properties. In an attempt to address some of the open questions in this area, we describe Boolean algebras equipped with an auxiliary binary relation used by H. deVries [1] to extend Stone duality to arbitrary compact spaces. This is applied to the Booleanization of a frame to describe a pointfree version of covers given by V. V. Fedorchuck [2].

A *subordination* on a Boolean algebra B is a binary relation $\ll$ on B which satisfies the following 5 postulates:

S1 $a \ll b \Longrightarrow a \leq b$

S2 $a \ll b \Longrightarrow b' \ll a'$

S3 $a \ll b \Longrightarrow \exists\ c$ with $a \ll c$ and $c \ll b$

S4 $a \neq 0 \Longrightarrow \exists\ b \neq 0$ with $b \ll a$

S5 $a_1 \ll b$ and $a_2 \ll b \Longleftrightarrow a_1 \vee a_2 \ll b$

We use the frame of *subordinated ideals* to develop a pointfree version of the deVries duality. If B is a Boolean algebra with a subordination, we construct a unique compact regular frame L such that B is isomorphic to an order-dense subalgebra of the regular open algebra (Booleanization) $\boldsymbol{B}(L)$ of L and $a \ll b$ in B if and only if $\widehat{a} \prec \widehat{b}$ as elements of L. If you start with a c.r. frame M, and consider a *frame* subordination $\ll$ on $\boldsymbol{B}(M)$**,** in which axiom S4 is enhanced to recognize the frame as follows:

S4f $a = \vee_M \{b : b \ll a\}$ for all $a \in B(M)$,

the resulting compact regular frame is the pointfree analog Fedorchuk's perfect irreducible preimage. In this way the study of *frame covers* of M becomes the study of frame subordination relations on the algebras $B(M)$.

References (partial list)

[1] H. de Vries, Compact Spaces and Compactifications, Van Gorcum & Comp. N.V., Assen, 1962

[2] V. V. Fedorchuk, Perfect Irreducible Mappings and Generalized Proximities, Mat. Sbornik 76 (1968), 489-508.

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