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II-CATEGORIES. IDENTITY 1-ARROW RETRACTS, CORETRACTS AND ISOMORPHIC 1-ARROWS

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Abstract. In any II-category, the identity 1-arrow retracts, coretracts and isomorphic 1-arrows are described.

II

Any second order category (shortly II-category [1]) may be expressed as a double category [2] with antidiscrete category structure on objects and v-arrows. 1-arrows of the given II-category will be h-arrows for the corresponding double category.

The aim of this paper is to describe the 1-arrows which are isomorphic to the identity 1-arrows, their retracts or coretracts.

In any II-category A, similarly to the definition of isomorphic objects,

Definition 1. The 1-arrows $a \xrightarrow{f} b$; and $c \xrightarrow{g} d$ are called isomorphic, if there exist the II-arrows

$$f \underset{\beta}{\overset{\alpha}{\rightleftarrows}} g$$

satisfying

$$\alpha\beta = 1_g, \quad \beta\alpha = 1_f.$$

In this case, the objects a, b, c, d are called connected by the isomorphism (f, g, α, β).

Any two objects a and b of the II-category A are connected by the isomorphism $(1_a, 1_b, 0_{ab}, 0_{ba})$.

1. Retracts and Coretracts of the Identity 1-Arrow.

In any II-category A

Definition 2. 1-arrow $a \xrightarrow{f} b$ is called an identity 1-arrow retract (coretract), if there exist II-arrows

$$1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b$$

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satisfying

$$f_* + f^* = 1_f; \quad (f_*\, f^* = 0_{ab}).$$

Remark 1.1. Using the interchange law of II-compositions [1]

$$f_* + f^* = f^* 0_{ba} + 0_{ba}\, f_* = f^* 0_{ba}\, f_* = 1_f.$$

The identity 1-arrow retracts and coretracts are described in Theorems 1.1, 1.2 and 1.3.

Theorem 1.1. *For any 1-arrows* $a \underset{g}{\overset{f}{\rightleftarrows}} b$, *the following conditions are equivalent.*

1. *There exist* II*-arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b \xrightarrow{g^*} g \xrightarrow{g_*} 1_a$ *such that*

$$f_* + f^* = 1_f, \quad f_* f^* 0_{ba} = 0_{ab} g_* g^*.$$

2. *There exist* II*-arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b$, $f \underset{\beta}{\overset{\alpha}{\rightleftarrows}} g$ *such that*

$$f_* + f^* = 1_f, \quad \beta\alpha = 1_f.$$

3. *There exist* II*-arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b$, $1_a \xrightarrow{\eta} gf$, $fg \xrightarrow{\epsilon} 1_b$ *such that*

$$f_* + f^* = 1_f, \quad (\epsilon + 1_f)(1_f + \eta) = 1_f.$$

4. *There exist* II*-arrows* $1_a \xrightarrow{\eta} gf \xrightarrow{\mu} f \xrightarrow{\phi} fg \xrightarrow{\epsilon} 1_b$ *such that*

$$\mu\eta 0_{ba} \epsilon\phi = 1_f, \quad (\epsilon + 1_f)(1_f + \eta) = 1_f.$$

5. *There exist* II*-arrows* $1_a \underset{\gamma}{\overset{\eta}{\rightleftarrows}} gf \underset{\omega}{\overset{\mu}{\rightleftarrows}} f$ *such that*

$$\mu\eta\gamma\omega = 1_f, \quad \gamma\eta = \gamma\omega\mu\eta.$$

6. *There exist* II*-arrows* $1_a \xrightarrow{\psi} fgf \xrightarrow{\sigma} 1_b$, $f \underset{\rho}{\overset{\lambda}{\rightleftarrows}} fgf$ *such that*

$$\rho\psi 0_{ba} \sigma\lambda = 1_f, \quad \sigma\psi = \sigma\lambda\rho\psi.$$

Proof. $1 \Rightarrow 2$. For the II-arrows $\alpha = g^* f_*$ and $\beta = f^* g_*$, the equalities $\beta\alpha = f^* g_* g^* f_* = f^* 0_{ba} f_* = 1_f$. hold.

$2 \Rightarrow 1$. For the II-arrows $g^* = \alpha f^* 0_{ba}$, $g_* = 0_{ba} f_* \beta$ the corresponding equalities are verified straightly.

$1 \Rightarrow 3$. For the II-arrows $\eta = g^* 0_{ab} + f^*$ and $\epsilon = f_* + 0_{ab} g_*$ the equality

$$\begin{aligned}(\epsilon + 1_f)(1_f + \eta) &= (f_* + 0_{ab} g_* + 1_f)(1_f + g^* 0_{ab} + f^*) = \\ &= f_* + 0_{ab} g_* g^* 0_{ab} + f^* = f_* + f_* f^* + f^* = 1_f\end{aligned}$$

holds.

$3 \Rightarrow 1$. The II-arrows $g^* = (1_g + f_*)\eta 0_{ba}$ and $g_* = 0_{ba}\epsilon(f^* + 1_g)$ satisfy

$$g_* + g^* = (1_g + f_*)\eta 0_{ba}\epsilon(f^* + 1_g) = (1_g + \epsilon)(1_g + f_* + f^* + 1_g)(\eta + 1_g) =$$

$$\begin{array}{ccccccccc}
b & \xrightarrow{g} & a & \xrightarrow{1_a} & a & & \xrightarrow{\quad 1_a \quad} & & a \\
 & \downarrow 1_g & & \downarrow 0_{aa} & & & \downarrow \eta & & \\
b & \xrightarrow{g} & a & \xrightarrow{1_a} & a & \xrightarrow{f} & b & \xrightarrow{g} & a \\
 & \downarrow 1_g & & \downarrow f^* & & \downarrow f_* & & \downarrow 1_g & \\
b & \xrightarrow{g} & a & \xrightarrow{f} & b & \xrightarrow{1_b} & b & \xrightarrow{g} & a \\
 & & \downarrow \epsilon & & & \downarrow 0_{bb} & & \downarrow 1_g & \\
b & & \xrightarrow{\quad 1_b \quad} & & b & \xrightarrow{1_b} & b & \xrightarrow{g} & a
\end{array}$$

$= (1_g + \epsilon)(\eta + 1_g) = 1_g$.

$g_* g^* = 0_{ba}\epsilon(f^* + 1_g + f_*)\eta 0_{ba} =$

$$\begin{array}{ccccccc}
b & \xrightarrow{1_b} & b & \xrightarrow{1_b} & b & \xrightarrow{1_b} & b \\
 & \downarrow 0_{ba} & & \downarrow 0_{ba} & & \downarrow 0_{ba} & \\
a & \xrightarrow{1_a} & a & \xrightarrow{1_a} & a & \xrightarrow{1_a} & a \\
 & & \downarrow \eta & & & \downarrow 0_{aa} & \\
a & \xrightarrow{f} & b & \xrightarrow{g} & a & \xrightarrow{1_a} & a \\
 & \downarrow f_* & & \downarrow 1_g & & \downarrow f^* & \\
b & \xrightarrow{1_b} & b & \xrightarrow{g} & a & \xrightarrow{f} & b \\
 & \downarrow 0_{bb} & & & \downarrow \varepsilon & & \\
b & \xrightarrow{1_b} & b & \xrightarrow{1_b} & b & \xrightarrow{1_b} & b \\
 & \downarrow 0_{ba} & & \downarrow 0_{ba} & & \downarrow 0_{ba} & \\
a & \xrightarrow{1_a} & a & \xrightarrow{1_a} & a & \xrightarrow{1_a} & a
\end{array}$$

$= 0_{ba} f_*(\epsilon + 1_f)(1_f + \eta) f^* 0_{ba} = 0_{ba} f_* f^* 0_{ba}$.

$1 \Rightarrow 4$. The II-arrows

$$\eta = g^* 0_{ab} + f^*, \quad \mu = 0_{ab} g_* + 1_f$$
$$\epsilon = f_* + 0_{ab} g_* \quad \text{and} \quad \phi = 1_f + g^* 0_{ab}$$

satisfy

$$\mu\eta 0_{ba}\epsilon\phi = f_* + 0_{ab} g_* g^* 0_{ab} g_* g^* + f^* = 1_f,$$
$$(\epsilon + 1_f)(1_f + \eta) = f_* + f_* f^* + f^* = 1_f.$$

$4 \Rightarrow 1$. For the II-arrows

$$f^* = \mu\eta, \quad f_* = \epsilon\phi,$$
$$g^* = (1_g + \epsilon\phi)\eta 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\epsilon(\mu\eta + 1_g)$$

equalities:

$$
\begin{gathered}
f_* + f^* = f^* 0_{ba} f_* = \mu\eta 0_{ba} \epsilon\phi = 1_f \\
g_* g^* = 0_{ba}\epsilon(\mu\eta + 1_g)(1_g + \epsilon\phi)\eta 0_{ba} = 0_{ba}(\epsilon + \epsilon)(\mu + 1_g + \phi)(\eta + \eta)0_{ba} = \\
= 0_{ba}\epsilon\phi(\epsilon + 1_f)(1_f + \eta)\mu\eta 0_{ba} = 0_{ba}\epsilon\phi\mu\eta 0_{ba} = 0_{ba} f_* f^* 0_{ba}
\end{gathered}
$$

hold.

$1 \Rightarrow 5$. For the II-arrows

$$
\begin{gathered}
\eta = g^* 0_{ab} + f^*, \quad \mu = 0_{ab} g_* + 1_f \\
\gamma = g_* + 0_{ba} f_* \quad \text{and} \quad \omega = g^* + 1_f,
\end{gathered}
$$

equalities

$$
\begin{gathered}
\mu\eta\gamma\omega = f^* g_* g^* 0_{ab} g_* g^* f^* = (f^* 0_{ba} f_*)^2 = 1_f, \\
\gamma\omega\mu\eta = (0_{ba} f_* f^*)^3 = 0_{ba} f_* f^* = \gamma\eta,
\end{gathered}
$$

hold.

$5 \Rightarrow 1$. For the II-arrows

$$
\begin{gathered}
f^* = \mu\eta, \quad f_* = 0_{ab}\gamma\omega, \\
g^* = (1_g + 0_{ab}\gamma\omega)\eta 0_{ba} \quad \text{and} \quad g_* = \gamma(1_g + \mu\eta 0_{ba}),
\end{gathered}
$$

equalities

$$
\begin{gathered}
f^* 0_{ba} f_* = \mu\eta\gamma\omega = 1_f, \\
g_* g^* = \gamma(1_g + \mu\eta\gamma\omega)\eta 0_{ba} = \gamma\eta 0_{ba} = \gamma\omega\mu\eta 0_{ba} = 0_{ba} f_* f^* 0_{ba},
\end{gathered}
$$

hold.

$1 \Rightarrow 6$. The II-arrows

$$
\begin{gathered}
\psi = f^* + g^* 0_{ab} + f^* \quad \sigma = f_* + 0_{ab} g_* + f_*, \\
\lambda = f^* 0_{ba} + g^* + 1_f \quad \text{and} \quad \rho = f_* + 0_{ab} g_* + 1_f,
\end{gathered}
$$

satisfy

$$
\begin{gathered}
\rho\psi 0_{ba}\sigma\lambda = f_* f^* 0_{ba} + 0_{ab} g_* g^* + 1_f = f_* f^* 0_{ba} + f_* f^* 0_{ba} + 1_f = 1_f, \\
\sigma\psi = f_* f^* + 0_{ab} g_* g^* 0_{ab} + f_* f^* = 0_{ba} f_* f^* \\
\sigma\lambda\rho\psi = f_* f^* g_* g^* f_* f^* 0_{ba} f_* f^* g_* g^* 0_{ab} = 0_{ba} f_* f^*.
\end{gathered}
$$

$6 \Rightarrow 1$. For the II-arrows

$$
\begin{gathered}
f^* = \rho\psi, \quad f_* = \sigma\lambda, \\
g^* = (0_{ba}\sigma\lambda + 1_g + \sigma\lambda)\psi 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\sigma(\rho\psi \ + 1_g + \rho\psi 0_{ba})
\end{gathered}
$$

the equalities

$$
f^* 0_{ba} f_* = \rho\psi 0_{ba}\ \sigma\lambda = 1_f, \quad f_* f^* = \sigma\lambda\rho\psi = \sigma\psi = 0_{ab} g_* g^* 0_{ab}
$$

are true. ■

Theorem 1.2. *The following conditions are equivalent.*

1. *There exist* II-*arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b \xrightarrow{g^*} g \xrightarrow{g_*} 1_b$ *such that*

$$f_* + f^* = 1_f,$$
$$g_* + g^* = 1_g,$$
$$f_* f^* 0_{ba} = 0_{ab} g_* g^*.$$

2. *There exist* II-*arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b, f \underset{\beta}{\overset{\alpha}{\rightleftarrows}} g$ *such that*

$$f_* + f^* = 1_f,$$
$$\alpha\beta = 1_g$$
$$\beta\alpha = 1_f.$$

3. *There exist* II-*arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b,\ 1_a \xrightarrow{\eta} gf,\ \ fg \xrightarrow{\epsilon} 1_b$ *such that*

$$f^* 0_{ba} f_* = 1_f,$$
$$(1_g + \epsilon)(\eta + 1_g) = 1_g,$$
$$(\epsilon + 1_f)(1_f + \eta) = 1_f.$$

4. *There exist* II-*arrows* $1_a \xrightarrow{\eta} gf \xrightarrow{\mu} f \xrightarrow{\phi} fg \xrightarrow{\epsilon} 1_b$ *such that*

$$\phi + \mu = 1_{fgf}$$
$$(1_g + \epsilon)(\eta + 1_g) = 1_g,$$
$$(\epsilon + 1_f)(1_f + \eta) = 1_f.$$

4.1. *There exist* II-*arrows* $1_a \xrightarrow{\eta} gf \xrightarrow{\mu} f \xrightarrow{\phi} fg \xrightarrow{\epsilon} 1_b$ *such that*

$$\mu\eta 0_{ba} \epsilon\phi = 1_f,$$
$$(1_g + \epsilon)(\eta + 1_g) = 1_g,$$
$$(\epsilon + 1_f)(1_f + \eta) = 1_f.$$

5. *There exist* II-*arrows* $1_a \underset{\gamma}{\overset{\eta}{\rightleftarrows}} gf \underset{\omega}{\overset{\mu}{\rightleftarrows}} f$ *such that*

$$\mu\eta\gamma\omega = 1_f,$$
$$\gamma\eta = \gamma\omega\mu\eta,$$
$$(1_g + 0_{ab}\gamma\omega)\eta\gamma(1_g + \mu\eta 0_{ba}) = 1_g.$$

6. *There exist* II-*arrows* $1_a \xrightarrow{\psi} fgf \xrightarrow{\sigma} 1_b$, $f \underset{\rho}{\overset{\lambda}{\rightleftarrows}} fgf$ *such that*

$$\rho\psi 0_{ba} \sigma\lambda = 1_f,$$
$$(0_{ba}\sigma\lambda + 1_g + \sigma\lambda)\psi 0_{ba}\sigma(\rho\psi + 1_g + \rho\psi 0_{ba}) = 1_g$$
$$\sigma\psi = \sigma\lambda\rho\psi.$$

Proof.

$1 \Rightarrow 2$. The II-arrows

$$\alpha = g^* f_* \quad \text{and} \quad \beta = f^* g_*$$

satisfy

$$\alpha\beta = 1_g, \quad \beta\alpha = f^* g_* g^* f_* = f^* 0_{ba} f_* = 1_f.$$

$2 \Rightarrow 1$. The II-arrows

$$g^* = \alpha f^* 0_{ba} \quad \text{and} \quad g_* = 0_{ba} f_* \beta$$

satisfy

$$g_* + g^* = \alpha f^* 0_{ba} f_* \beta = 1_g, \quad 0_{ab} g_* g^* = f_* \beta \alpha f^* 0_{ba} = f_* f^* 0_{ba}$$

$1 \Rightarrow 3$. For the II-arrows

$$\eta = g^* 0_{ab} + f^* \quad \text{and} \quad \epsilon = f_* + 0_{ab} g_*$$

equalities

$$\begin{gathered}(\epsilon + 1_f)(1_f + \eta) = (f_* + 0_{ab} g_* + 1_f)(1_f + g^* 0_{ab} + f^*) = \\ = f_* + 0_{ab} g_* g^* 0_{ab} + f^* = f_* + f_* f^* + f^* = 1_f. \\ (1_g + \epsilon)(\eta + 1_g) = (1_g + f_* + 0_{ab} g_*)(g^* 0_{ab} + f^* + 1_g) = \\ = g_* 0_{ab} + f_* f^* + 0_{ab} g^* = (g^* 0_{ab} + 0_{ab} g_*)(f_* f^* + 1_g) = 0_{ab} g_* g^* 0_{ab} + 1_g = 1_g.\end{gathered}$$

are true.

$3 \Rightarrow 1$. The II-arrows

$$g^* = (1_g + f_*)\eta 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\epsilon(f^* + 1_g),$$

satisfy

$$\begin{gathered}g_* + g^* = (1_g + f_*)\eta 0_{ba}\epsilon(f^* + 1_g) = \\ =(1_g + \epsilon)(1_g + f_* + f^* + 1_g)(\eta + 1_g) = (1_g + \epsilon)(\eta + 1_g) = 1_g. \\ g_* g^* = 0_{ba}\epsilon(f^* + 1_g + f_*)\eta 0_{ba} = 0_{ba} f_*(\epsilon + 1_f)(1_f + \eta) f^* 0_{ba} = 0_{ba} f_* f^* 0_{ba}.\end{gathered}$$

$1 \Rightarrow 4$. For the II-arrows

$$\eta = g^* 0_{ab} + f^*, \quad \mu = 0_{ab} g_* + 1_f, \quad \epsilon = f_* + 0_{ab} g_* \quad \text{and} \quad \phi = 1_f + g^* 0_{ab}$$

equalities

$$\begin{gathered}\phi + \mu = 1_f + g^* 0_{ab} + 0_{ab} g_* + 1_f = 1_f + g_* + g^* + 1_f = 1_{fgf}, \\ (1_g + \epsilon)(\eta + 1_g) = (1_g + f_* + 0_{ab} g_*)(g^* 0_{ab} + f^* + 1_g) = \\ =g_* 0_{ab} + f_* f^* + 0_{ab} g^* = (g^* 0_{ab} + 0_{ab} g_*)(f_* f^* + 1_g) = 0_{ab} g_* g^* 0_{ab} + 1_g = 1_g. \\ (\epsilon + 1_f)(1_f + \eta) = (f_* + 0_{ab} g_* + 1_f)(1_f + g^* 0_{ab} + f^*) = \\ = f_* + 0_{ab} g_* g^* 0_{ab} + f^* = f_* + f_* f^* + f^* = 1_f\end{gathered}$$

are true.

$4 \Rightarrow 1$. The II-arrows

$$f^* = \mu\eta, \quad f_* = \epsilon\phi, \quad g^* = (1_g + \epsilon\phi)\eta 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\epsilon(\mu\eta + 1_g)$$

satisfy

$$
\begin{aligned}
&f_* + f^* = f^* 0_{ba} f_* = \mu\eta 0_{ba} \epsilon\phi = (\epsilon + 1_f)(\phi + \mu)(1_f + \eta) = 1_f,\\
&g_* + g^* = (1_g + \epsilon\phi)\eta 0_{ba}\ \epsilon(\mu\eta + 1_g) =\\
&= (1_g + \epsilon)(1_g + \epsilon\phi + \mu\eta + 1_g)(\eta + 1_g) =
\end{aligned}
$$

$$
\begin{array}{ccccccccccc}
b & \xrightarrow{g} & a & \xrightarrow{\quad 1_a \quad} & a & & \xrightarrow{\qquad 1_a \qquad} & & a \\
 & \downarrow 1_g & & \downarrow 0_{aa} & & & \downarrow \eta & & \\
b & \xrightarrow{g} & a & \xrightarrow{\quad 1_a \quad} & a & \xrightarrow{\quad f \quad} & b & \xrightarrow{g} & a \\
 & \downarrow 1 & & \downarrow \eta & & \downarrow \phi & & \downarrow 1 & \\
b & \xrightarrow{g} & a & \xrightarrow{f}\ b\ \xrightarrow{g} & a & b \xrightarrow{g} a \xrightarrow{f} & b & \xrightarrow{g} & a \\
 & \downarrow 1 & & \downarrow \mu & & \downarrow \varepsilon & & \downarrow 1 & \\
b & \xrightarrow{g} & a & \xrightarrow{\quad f \quad} & b & \xrightarrow{\quad 1_b \quad} & b & \xrightarrow{g} & a \\
 & & \downarrow \varepsilon & & & \downarrow 0_{bb} & & \downarrow 1_g & \\
b & & \xrightarrow{\qquad 1_b \qquad} & & b & \xrightarrow{\quad 1_b \quad} & b & \xrightarrow{g} & a
\end{array}
$$

$$
\begin{aligned}
&= (1_g + \epsilon)(1_g + \mu\eta 0_{ba} \epsilon\phi + 1_g)(\eta + 1_g) = (1_g + \epsilon)(\eta + 1_g) = 1_g.\\
&\quad g_* g^* = 0_{ba}\epsilon(\mu\eta + 1_g)(1_g + \epsilon\phi)\eta 0_{ba} = 0_{ba}(\epsilon + \epsilon)(\mu + 1_g + \phi)(\eta + \eta)0_{ba} =\\
&= 0_{ba}\epsilon\phi(\epsilon + 1_f)(1_f + \eta)\mu\eta 0_{ba} = 0_{ba}\epsilon\phi\mu\eta 0_{ba} = 0_{ba} f_* f^* 0_{ba}
\end{aligned}
$$

$1 \Rightarrow 5$. The II-arrows

$$\eta = g^* 0_{ab} + f^*, \quad \mu = 0_{ab} g_* + 1_f, \quad \gamma = g_* + 0_{ba} f_*, \quad \omega = g^* + 1_f,$$

satisfy

$$
\begin{aligned}
&\mu\eta\gamma\omega = f^* g_* g^* 0_{ab} g_* g^* f^* = (f^* 0_{ba} f_*)^2 = 1_f,\\
&(1_g + 0_{ab}\gamma\omega)\eta\gamma(1_g + \mu\eta 0_{ba}) = g^* 0_{ab} g_* + 0_{ab} g_* g^* f_* f^* 0_{ba} f_* f^* g_* g^* =\\
&= 1_g + (0_{ab} g_* g^*)^3 = 1_g,\\
&\gamma\omega\mu\eta = (0_{ba} f_* f^*)^3 = 0_{ba} f_* f^* = \gamma\eta.
\end{aligned}
$$

$5 \Rightarrow 1$. For The II-arrows

$$f^* = \mu\eta, \quad f_* = 0_{ab}.\gamma\omega, g^* = (1_g + 0_{ab}.\gamma\omega)\eta 0_{ba} \quad \text{and} \quad g_* = \gamma(1_g + \mu\eta 0_{ba}),$$

equalities

$$
\begin{gathered}
f^* 0_{ba} f_* = \mu\eta\gamma\omega = 1_f,\\
g^* 0_{ab} g_* = (1_g + 0_{ab}.\gamma\omega)\eta\gamma(1_g + \mu\eta 0_{ba}) = 1_g,\\
g_* g^* = \gamma(1_g + \mu\eta\gamma\omega)\eta 0_{ba} = \gamma\eta 0_{ba} = \gamma\omega\mu\eta 0_{ba} = 0_{ba} f_* f^* 0_{ba}
\end{gathered}
$$

are true.

$1 \Rightarrow 6$. For the II-arrows

$$
\begin{gathered}
\psi = f^* + g^* 0_{ab} + f^*, \quad \sigma = f_* + 0_{ab} g_* + \ f_*,\\
\lambda = f^* 0_{ba} + g^* + 1_f \quad \text{and} \quad \rho = f_* + 0_{ab} g_* + 1_f,
\end{gathered}
$$

equalities

$$\rho\psi = f^*g_*g^*0_{ab}, \qquad \sigma\lambda = 0_{ab}g_*g^*f_*,$$
$$\psi 0_{ba}\sigma = (f_* + f^* + 1_g)(1_f + g_* + g^*) + 1_f = 1_{fgf}$$

imply

$$\begin{aligned}
&\rho\psi 0_{ba}\sigma\lambda = f_*f^*0_{ba} + 0_{ab}g_*g^* + 1_f = f_*f^*0_{ba} + 1_f = 1_f,\\
&(0_{ba}\sigma\lambda + 1_g + \sigma\lambda)\psi 0_{ba}\sigma(\rho\psi + 1_g + \rho\psi 0_{ba}) =\\
&= (g_*g^*f_*f^*g_*g^*0_{ab} + 1_g + 0_{ab}g_*g^*f_*f^*g_*g^*) = (g_*g^*0_{ab} + 1_g + 0_{ab}g_*g^*) = 1_g\\
&\sigma\lambda\rho\psi = 0_{ab}g_*g^*f_*f^*g_*g^*0_{ab} = 0_{ab}g_*g^*0_{ab} = \sigma\psi;
\end{aligned}$$

$6 \Rightarrow 1$. The II-arrows

$$f^* = \rho\psi, \quad f_* = \sigma\lambda,$$
$$g^* = (0_{ba}\sigma\lambda + 1_g + \sigma\lambda)\psi 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\sigma(\rho\psi + 1_g + \rho\psi 0_{ba})$$

satisfy

$$f^*0_{ba}f_* = \rho\psi 0_{ba}\sigma\lambda = 1_f,$$
$$g^*0_{ab}g_* = (0_{ba}\sigma\lambda + 1_g + \sigma\lambda)\psi 0_{ba}\sigma(\ \rho\psi + 1_g + \rho\psi 0_{ba}) = 1_g,$$
$$f_*f^* = \sigma\lambda\rho\psi = \sigma\psi = 0_{ab}g_*g^*0_{ab}. \quad \blacksquare$$

Remark 1.2. By $(1 \Leftrightarrow 4)$, the 1-endomorphism

$$a \xrightarrow{f} a$$

is the identity retract if and only if it is selfadjoint [1] and

$$\phi + \mu = 1_{fgf} \quad (\text{or} \quad \mu\eta 0_{ba}\epsilon\phi = 1_f).$$

Theorem 1.3. *The following conditions are equivalent:*

1. *There exist* II-*arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b \xrightarrow{g^*} g \xrightarrow{g_*} 1_b$ *such that*

$$f_*f^* = 0_{ab},$$
$$g_*g^* = 0_{ba}$$

2. *There exist* II-*arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b, f \underset{\beta}{\overset{\alpha}{\rightleftarrows}} g$ *such that*

$$f_*f^* = 0_{ba}$$
$$\beta\alpha = f^*0_{ba}f_*$$

3. *There exist* II-*arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b,\ 1_a \xrightarrow{\eta} gf,\ \ fg \xrightarrow{\epsilon} 1_b$ *such that*

$$f_*f^* = 0_{ab},$$
$$(\epsilon + 1_f)(1_f + \eta) = f^*0_{ba}f_*$$

4.*There exist* II-*arrows* $1_a \xrightarrow{\eta} gf \xrightarrow{\mu} f \xrightarrow{\phi} fg \xrightarrow{\epsilon} 1_b$ *such that*

$$\epsilon\phi\mu\eta = 0_{ab},$$
$$(\epsilon + 1_f)(1_f + \eta) = \mu\eta 0_{ba}\epsilon\phi$$

5. *There exist* II-*arrows* $1_a \underset{\gamma}{\overset{\eta}{\rightleftarrows}} gf \underset{\omega}{\overset{\mu}{\rightleftarrows}} f$ *such that*

$$\gamma\omega\mu\eta = 0_{aa}$$
$$\gamma(1_g + \mu\eta\gamma\omega)\eta = 0_{aa}$$

6. *There exist* II-*arrows* $1_a \xrightarrow{\psi} fgf \xrightarrow{\sigma} 1_b$, $f \underset{\rho}{\overset{\lambda}{\rightleftarrows}} fgf$ *such that*

$$\sigma\lambda\rho\psi = 0_{ab},$$
$$\sigma(\rho\psi 0_{ba}\sigma\lambda + 1_g + \rho\psi 0_{ba}\lambda)\psi = 0_{ab}$$

Proof. The implications $(1 \Leftrightarrow 2)$, $(1 \Rightarrow 3)$, $(1 \Rightarrow 4)$, $(1 \Rightarrow 5)$ and $(1 \Rightarrow 6)$ are straightforward.

$3 \Rightarrow 1$. The II-arrows

$$g^* = (1_g + f_*)\eta 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\epsilon(f^* + 1_g),$$

satisfy

$$g_*g^* = 0_{ba}\epsilon(f^* + 1_g + f_*)\eta 0_{ba} = 0_{ba}f_*(\epsilon + 1_f)(1_f + \eta)f^*0_{ba} = 0_{ba}.$$

$4 \Rightarrow 1$.The II-arrows

$$f^* = \mu\eta, \quad f_* = \epsilon\phi, \quad g^* = (1_g + \epsilon\phi)\eta 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\epsilon(\mu\eta + 1_g)$$

satisfy

$$f_*f^* = \epsilon\phi\mu\eta = 0_{ab},$$
$$g_*g^* = 0_{ba}\epsilon(\mu\eta + 1_g)(1_g + \epsilon\phi)\eta 0_{ba} = 0_{ba}(\epsilon + \epsilon)(\mu + 1_g + \phi)(\eta + \eta)0_{ba} =$$
$$= 0_{ba}\epsilon(\mu + 1_g)(\eta + 1_g + \epsilon)(1_g + \phi)\eta 0_{ba} = 0_{ba}\epsilon\phi(\epsilon + 1_f)(1_f + \eta)\mu\eta 0_{ba} =$$
$$= 0_{ba}\epsilon\phi\mu\eta 0_{ba} = 0_{ba}.$$

$5 \Rightarrow 1$. The II-arrows

$$f^* = \mu\eta, \;\; f_* = 0_{ab}.\gamma\omega, \;\; g^* = (1_g + 0_{ab}.\gamma\omega)\eta 0_{ba} \quad \text{and} \quad g_* = \gamma(1_g + \mu\eta 0_{ba}),$$

satisfy

$$f_*f^* = 0_{ab}.\gamma\omega\mu\eta = 0_{ab}$$
$$g_*g^* = \gamma(1_g + \mu\eta\gamma\omega)\eta 0_{ba} = 0_{ba}$$

$6 \Rightarrow 1$. The II-arrows

$$f^* = \rho\psi, \quad f_* = \sigma\lambda,$$
$$g^* = (0_{ba}\sigma\lambda + 1_g + \sigma\lambda)\psi 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\sigma(\rho\psi \; + 1_g + \rho\psi 0_{ba}),$$

satisfy

$$f_* f^* = \sigma\lambda\rho\psi = 0_{ab}.$$
$$g_* g^* = 0_{ba}\sigma(\rho\psi 0_{ba}\sigma\lambda + 1_g + \rho\psi 0_{ba}\sigma\lambda)\psi 0_{ba} = 0_{ba}. \quad \blacksquare$$

2. Isomorphism With the Identity 1-Arrow.

The following theorem describes 1-arrows which form isomorphism with the identity 1-arrow.

In any II-category A, for any 1-arrows $a \underset{g}{\overset{f}{\rightleftarrows}} b$

Theorem 2.1. *The following conditions are equivalent:*

1. *There exist* II*-arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b \xrightarrow{g^*} g \xrightarrow{g_*} 1_b$ *satisfying*

$$f^* 0_{ba} f_* = 1_f, \quad f_* f^* = 0_{ab},$$
$$g^* 0_{ab} g_* = 1_g, \quad g_* g^* = 0_{ba};$$

2. *There exist* II*-arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b, f \underset{\beta}{\overset{\alpha}{\rightleftarrows}} g$ *satisfying*

$$f^* 0_{ba} f_* = 1_f, \quad f_* f^* = 0_{ab},$$
$$\alpha\beta = 1_g, \quad \beta\alpha = 1_f.$$

3. *There exist* II*-arrows* $1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b,\ 1_a \xrightarrow{\eta} gf,\ fg \xrightarrow{\epsilon} 1_b$ *satisfying*

$$f^* 0_{ba} f_* = 1_f, \quad f_* f^* = 0_{ab},$$
$$(1_g + \epsilon)(\eta + 1_g) = 1_g, \quad (\epsilon + 1_f)(1_f + \eta) = 1_f.$$

4. *There exist* II*-arrows* $1_a \xrightarrow{\eta} gf \xrightarrow{\mu} f \xrightarrow{\phi} fg \xrightarrow{\epsilon} 1_b$ *satisfying*

$$\mu\eta 0_{ba}\epsilon\phi = 1_f, \quad \epsilon\phi\mu\eta = 0_{ab},$$
$$(1_g + \epsilon)(\eta + 1_g) = 1_g, \quad (\epsilon + 1_f)(1_f + \eta) = 1_f.$$

5. *There exist* II*-arrows* $1_a \underset{\gamma}{\overset{\eta}{\rightleftarrows}} gf \underset{\omega}{\overset{\mu}{\rightleftarrows}} f$ *satisfying*

$$\mu\eta\gamma\omega = 1_f, \quad \gamma\omega\mu\eta = 0_{aa}$$
$$(1_g + 0_{ab}\gamma\omega)\eta\gamma(1_g + \mu\eta 0_{ba}) = 1_g, \quad \gamma\eta = 0_{aa},$$

6. *There exist* II*-arrows* $1_a \xrightarrow{\psi} fgf \xrightarrow{\sigma} 1_b\ ,\ f \underset{\rho}{\overset{\lambda}{\rightleftarrows}} fgf$ *satisfying*

$$\rho\psi 0_{ba}\sigma\lambda = 1_f, \quad \sigma\lambda\rho\psi = 0_{ab},$$
$$(0_{ba}\sigma\lambda + 1_g + \sigma\lambda)\psi 0_{ba}\sigma(\rho\psi + 1_g + \rho\psi 0_{ba}) = 1_g, \quad \sigma\psi = 0_{ab}.$$

Proof. the implications ($1 \Leftrightarrow 2$), ($1 \Rightarrow 3$), ($1 \Rightarrow 4$), ($1 \Leftrightarrow 5$) and ($1 \Rightarrow 6$) are straightforward.

$3 \Rightarrow 1$. For the II-arrows $g^* = (1_g + f_*)\eta 0_{ba}$ and $g_* = 0_{ba}\epsilon(f^* + 1_g)$ equalities

$$\begin{aligned}
&g_* + g^* = (1_g + f_*)\eta 0_{ba}\epsilon(f^* + 1_g) = (1_g + \epsilon)(1_g + f_* + f^* + 1_g)(\eta + 1_g) = \\
&= (1_g + \epsilon)(\eta + 1_g) = 1_g. \\
&g_* g^* = 0_{ba}\epsilon(f^* + 1_g + f_*)\eta 0_{ba} = 0_{ba} f_*(\epsilon + 1_f)(1_f + \eta) f^* 0_{ba} = 0_{ba}
\end{aligned}$$

hold.

$4 \Rightarrow 1$. For the II-arrows

$$f^* = \mu\eta, \quad f_* = \epsilon\phi, \quad g^* = (1_g + \epsilon\phi)\eta 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\epsilon(\mu\eta + 1_g),$$

equalities

$$\begin{aligned}
&f_* + f^* = f^* 0_{ba} f_* = \mu\eta 0_{ba}\epsilon\phi = 1_f, \\
&f_* f^* = \epsilon\phi\mu\eta = 0_{ab}, \\
&g_* + g^* = (1_g + \epsilon\phi)\eta 0_{ba}\epsilon(\mu\eta + 1_g) = (1_g + \epsilon)(1_g + \epsilon\phi + \mu\eta + 1_g)(\eta + 1_g) = \\
&=(1_g + \epsilon)(1_g + \mu\eta 0_{ba}\epsilon\phi + 1_g)(\eta + 1_g) = (1_g + \epsilon)(\eta + 1_g) = 1_g. \\
&g_* g^* = 0_{ba}\epsilon(\mu\eta + 1_g)(1_g + \epsilon\phi)\eta 0_{ba} = 0_{ba}(\epsilon + \epsilon)(\mu + 1_g + \phi)(\eta + \eta)0_{ba} = \\
&=0_{ba}\epsilon(\mu + 1_g)(\eta + 1_g + \epsilon)(1_g + \phi)\eta 0_{ba} = 0_{ba}\epsilon\phi(\epsilon + 1_f)(1_f + \eta)\mu\eta 0_{ba} = 0_{ba}
\end{aligned}$$

hold.

$6 \Rightarrow 1$. For the II-arrows

$$\begin{gathered}
f^* = \rho\psi, \quad f_* = \sigma\lambda, \\
g^* = (0_{ba}\sigma\lambda + 1_g + \sigma\lambda)\psi 0_{ba} \quad \text{and} \quad g_* = 0_{ba}\sigma(\rho\psi + 1_g + \rho\psi 0_{ba})
\end{gathered}$$

equalities

$$\begin{gathered}
f^* 0_{ba} f_* = \rho\psi 0_{ba}\sigma\lambda = 1_f, \\
f_* f^* = \sigma\lambda\rho\psi = 0_{ab}. \\
g^* 0_{ab} g_* = (0_{ba}\sigma\lambda + 1_g + \sigma\lambda)\psi 0_{ba}\sigma(\rho\psi + 1_g + \rho\psi 0_{ba}) = 1_g. \\
g_* g^* = 0_{ba}\sigma(\rho\psi 0_{ba}\sigma\lambda + 1_g + \rho\psi 0_{ba}\sigma\lambda)\psi 0_{ba} = 0_{ba}\sigma\psi 0_{ba} = 0_{ba}.
\end{gathered}$$

hold. ■

3. The Main Theorem.

In any II-category A for the II-arrows η, γ, θ, τ, ε, δ:

$$1_a \underset{\gamma}{\overset{\eta}{\rightleftarrows}} gf \underset{\tau}{\overset{\theta}{\rightleftarrows}} fg \underset{\delta}{\overset{\epsilon}{\rightleftarrows}} 1_b,$$

the following theorem holds.

Theorem 3.1. *The following conditions are equivalent:*
1). *There exist* Π-*arrows* γ, τ, δ *satisfying*

$$\gamma + 1_g + \delta = 1_{gfg}, \quad \gamma\tau\delta = 0_{ba};$$

2). *There exist* Π-*arrows* γ, τ, δ *satisfying*

$$\delta + 1_f + \gamma = 1_{fgf}, \quad \gamma\tau\delta = 0_{ba};$$

3). *There exist* Π-*arrows* ϵ, η, θ *satisfying*

$$\epsilon + 1_f + \eta = 1_{fgf}, \quad \epsilon\theta\eta = 0_{ab};$$

4). *There exist* Π-*arrows* η, ϵ, γ, δ, τ *satisfying*

$$\epsilon + 1_f + \eta = 1_{fgf}, \quad \gamma\tau\delta = 0_{ba}, \quad \eta 0_{ba}\epsilon = \tau, \quad \epsilon\delta = 0_{bb};$$

5). *There exist* Π-*arrows* η, γ, ϵ, δ, τ *satisfying*

$$\epsilon + 1_f + \eta = 1_{fgf}, \quad \gamma\tau\delta = 0_{ba}, \quad \eta 0_{ba}\epsilon = \tau, \quad \gamma\eta = 0_{aa};$$

6). *There exist* Π-*arrows* η, γ, ϵ, δ, τ *satisfying*

$$\epsilon + 1_f + \eta = 1_{fgf}, \quad \gamma\tau\delta = 0_{ba}, \quad \eta 0_{ba}\epsilon = \tau, \quad \eta\gamma = 1_{gf};$$

7). *There exist* Π-*arrows:* η, γ, ϵ, θ *satisfying*

$$(\epsilon + 1_f)(1_f + \eta) = 1_f, \quad \epsilon\theta\eta = 0_{ab}, \quad \eta\gamma = 1_{gf};$$

8). *There exist* Π-*arrows* η, γ, ϵ, δ, θ *satisfying*

$$\epsilon + 1_f = 1_f + \gamma, \quad \eta + 1_g = 1_g + \delta, \quad \epsilon\theta\eta = 0_{ab}, \quad \eta\gamma = 1_{gf};$$

9). *There exist* Π-*arrows* η, γ, ϵ, δ, τ *satisfying*

$$\epsilon + 1_f = 1_f + \gamma, \quad \eta + 1_g = 1_g + \delta, \quad \gamma\tau\delta = 0_{ba}, \quad \eta\gamma = 1_{gf};$$

10). *There exist* Π-*arrows* η, γ, ϵ, δ, τ *satisfying*

$$\epsilon + 1_f = 1_f + \gamma, \quad \eta + 1_g = 1_g + \delta, \quad \gamma\tau\delta = 1_{ba}, \quad \delta\epsilon = 1_g;$$

11). *There exist* Π-*arrows* η, ϵ, θ *satisfying*

$$\epsilon + 1_f + \eta = 1_{fgf}, \quad \epsilon\theta\eta = 0_{ab}.$$

For the proof some technical lemmas are needed.

Lemma 3.1. *For the* Π-*arrows* γ, η*; and* ϵ, *the equalities*

$$\epsilon + 1_{fgf} + \eta = 1_{fgfgf}, \quad \eta = 0_{aa}$$

imply:
1) $\epsilon + 1_f = 1_f + \gamma$,
2) $\eta\gamma = 1_{gf}$,
3) $(\epsilon + 1_f)(1_f + \eta) = 1_f$,
4) $(1_g + \epsilon)(\eta + 1_g) + 1_f = 1_{gf}$,
5) $\eta + 1_g + \epsilon + 1_f = 1_{gfgf}$

6) $\eta + 1_g + \epsilon = 1_{gf} + (1_g + \epsilon)(\eta + 1_g)$,
7) $\gamma + 1_{gf} = 1_g + \epsilon + 1_f = 1_{gf} + \gamma$,
8) $\gamma + 1_{gf} + \eta = \eta + 1_{gf} + \gamma = 1_{gfgf}$,
9) $\epsilon + 1_{fgf} = 1_{fgf} + \gamma$,
10) $\epsilon + 1_{fgf} + \eta = 1_{fgfgf}$.

Proof. 1) The equality $(1_f + \gamma)(\epsilon + 1_f + \eta) = 1_f + \gamma$ means that

$$\epsilon + 1_f = 1_f + \gamma$$

2) From (1) it follows $\epsilon + 1_f + \eta = 1_f + \eta\gamma$.
Transforming $1_f + \eta\gamma = 1_{fgf}$, the equalities

$$(\gamma + 1_{gf})(1_{gf} + \eta\gamma)(\eta + 1_{gf}) = \gamma\eta + 1_{gf} \quad \text{and} \quad \eta\gamma = 1_{gf}$$

are true.
3) Follows from (1) and (2).
4) $(1_g + \epsilon)(\eta + 1_g) + 1_f = (1_g + \epsilon + 1_f)(\eta + 1_{gf}) = (1_{gf} + \gamma)(\eta + 1_{gf}) = \eta\gamma = 1_{gf}$.
5) The equality $\eta + (1_g + \epsilon)(\eta + 1_g) = (\eta + 1_g + \epsilon)(\eta + 1_g)$ implies

$$(\eta + (1_g + \epsilon)(\eta + 1_g))(\gamma + 1_g) = (\eta + 1_g + \epsilon)(\eta + 1_g)(\gamma + 1_g),$$
$$\eta\gamma + (1_g + \epsilon)(\eta + 1_g) = \eta + 1_g + \epsilon,$$
$$1_{gf} + (1_g + \epsilon)(\eta + 1_g) + 1_f = \eta + 1_g + \epsilon + 1_f,$$
$$\eta + 1_g + \epsilon + 1_f = 1_{gfgf}.$$

6) Straightforward.
7) (5) and (1) imply

$$(\gamma + 1_{gf})(\eta + 1_g + \epsilon + 1_f) = \gamma + 1_{gf},$$
$$1_g + \epsilon + 1_f = 1_{gf} + \gamma.$$

8) (6) and (5) imply

$$\gamma + 1_{gf} + \eta = 1_g + \epsilon + 1_f + \eta = 1_{gfgf} \text{ and } \eta + 1_{gf} + \gamma = \eta + 1_g + \epsilon + 1_f = 1_{gfgf},$$

respectively.
9) The equality $(\epsilon + 1_f + \eta)(1_{fgf} + \gamma) = 1_{gf} + \gamma$ means that $\epsilon + 1_{fgf} = 1_{fgfgf} + \gamma$.
10) It is a consequence of (9). ■

Lemma 3.2. *For the* II*-arrows* γ, η*; and* ϵ *equalities*

$$(\epsilon + 1_f)(1_f + \eta) = 1_f, \quad \eta\gamma = 1_{gf},$$

imply
1) $\epsilon + 1_f = 1_f + \gamma$,
2) $\epsilon + 1_f + \eta = 1_{fgf}$,
3) $1_f + \gamma\eta = 1_f$,
4) $\epsilon + 1_{fg} = 1_f + \gamma + 1_f = 1_{fg} + \epsilon$,
5) $(1_g + \epsilon)(\eta + 1_g) + 1_f = 1_{gf}$,
6) $\eta + 1_g + \epsilon + 1_f = 1_{gfgf}$
7) $\gamma + 1_{gf} = 1_g + \epsilon + 1_f$,

8) $1_{gf} + \eta = \eta + 1_{gf}$,
9) $\epsilon + 1_{fgf} + \eta = 1_{fgfgf}$,

Proof.
1).The equality: $(\epsilon + 1_f)(1_f + \eta)(1_f + \gamma) = 1_f + \gamma$ means that

$$\epsilon + 1_f = 1_f + \gamma.$$

2) From (1) it follows $\epsilon + 1_f + \eta = 1_f + \eta\gamma = 1_{fgf}$.
3) $1_f + \gamma\eta = (1_f + \gamma)(1_f + \eta) = (\epsilon + 1_f)(1_f + \eta) = 1_f$.
4) Straightforward.
5) Follows from (1).
6) The equality $\eta\gamma + (1_g + \epsilon)(\eta + 1_g) + 1_f = \eta\gamma + 1_{gf}$ implies

$$(\eta + 1_g + \epsilon + 1_f)(\gamma + \eta + 1_{gf}) = 1_{gfgf},$$
$$\eta + 1_g + \epsilon + 1_f = 1_{gfgf}.$$

7) Straightforward.
8) $\eta + 1_{gf} = \eta + \eta\gamma = (1_{gf} + \eta)(\eta + \gamma) = 1_{gf} + \eta$.

Lemma 3.3. *For the* II*-arrows* γ, ϵ *and* η, *the equalities* $\epsilon + 1_f = 1_f + \gamma$ *and* $\eta\gamma = 1_{gf}$, *imply*
1) $\epsilon + 1_f + \eta = 1_{fgf}$,
2) $\epsilon + 1_{fgf} = 1_{fgf} + \gamma$,
3) $\epsilon + 1_{fgf} + \eta = 1_{fgfgf}$,
4) $(\epsilon + 1_f)(1_f + \eta) + 1_{gf} = 1_{fgf}$,
5) $1_f + \gamma\eta + 1_{gf} = 1_{fgf}$,

Proof.
1) Straightforward.
2) The equality $(\epsilon + 1_{fgf} + \eta)(1_{fgf} + \gamma) = 1_{fgf} + \gamma$ means that $\epsilon + 1_{fgf} = 1_{fgf} + \gamma$.
3) Straightforward.
4) The equality $((\epsilon + 1_f)(1_f + \eta) + \eta)(1_f + \gamma) = (\epsilon + 1_f + \eta)(1_f + \eta\gamma)$ implies $(\epsilon + 1_f)(1_f + \eta) + 1_{gf} = 1_{fgf}$
5) $1_f + \gamma\eta + 1_{gf} = (1_f + \gamma)(1_f + \eta) + 1_{gf} = (\epsilon + 1_f)(1_f + \eta) + 1_{gf} = 1_{fgf}$. ■

Proof of Theorem 3.1.
(1) ⇒ (2). For the II-arrow $\eta = \tau\delta 0_{ab}$ the equality $\gamma\eta = 0_{aa}$ holds.
By **Lemma 3.1**

$$1_g + \delta + 1_f + \gamma = 1_{gfgf},$$
$$(0_{ab}\gamma\tau + 1_{fgf})(1_{fg} + \delta + 1_f + \gamma)(\delta + 1_{fgf}) = 0_{ab}\gamma\tau\delta + 1_{fgf},$$
$$\delta + 1_f + \gamma = 1_{fgf}.$$

(2) ⇒ (3). The II-arrows $\eta = \tau\delta 0_{ab}$. $\epsilon = 0_{ab}\,\gamma\tau$, $\theta = \delta 0_{ab}\gamma$ satisfy

$$\gamma\eta = 0_{aa}, \quad \epsilon\delta = 0_{bb} \quad \text{and} \quad \epsilon\theta\eta = 0_{ba}.$$

From **Lemma 3.1** it follows that: $\eta\gamma = 1_{gf}$ and $\epsilon + 1_f + \eta = 1_{fgf}$.

(3) $\Rightarrow$ (4). The II-arrows

$$\gamma = 0_{ba}\epsilon\theta, \quad \delta = \theta\eta 0_{ba} \quad \text{and} \quad \tau = \eta 0_{ba}\epsilon,$$

satisfy

$$\gamma\eta = 0_{aa}, \quad \epsilon\delta = 0_{bb}, \quad \gamma\tau\delta = 0_{ba}.$$

By **Lemma 3.1**

$$\eta\gamma = 1_{gf} \quad \text{and} \quad \epsilon + 1_f + \eta = 1_{fgf}.$$

(4) $\Rightarrow$ (5). Straightforward.

(5) $\Rightarrow$ (6). By **Lemma 3.1** it follows that $\eta\gamma = 1_{gf}$.

(6) $\Rightarrow$ (7). By the conditions: $\eta = \tau\delta 0_{ab}$, $\epsilon = 0_{ab}\gamma\tau$, $\gamma\eta = 0_{aa}$ and $\epsilon\delta = 0_{bb}$. Using Lemma 3.1 the equality: $(\epsilon + 1_f)(1_f + \eta) = 1_f$ holds. There exists $\theta = \delta 0_{ab}\gamma$ such that $\epsilon\theta\eta = 0_{ab}$.

(7) $\Rightarrow$ (8). From **Lemma 3.2.** it follows that: $\epsilon + 1_f = 1_f + \gamma$ and $\epsilon + 1_f + \eta = 1_{fgf}$. According to (3) $\Rightarrow$ (4) the equality $\epsilon\delta = 0_{bb}$ holds;
According to (1) $\Rightarrow$ (2) $\eta + 1_g + \epsilon = 1_{gfg}$; and by **Lemma 3.2** the equality $\eta + 1_g = 1_g + \delta$ holds.

(8) $\Rightarrow$ (9). For $\gamma = 0_{ba}\epsilon\theta$, it follows

$$\gamma\eta = 0_{aa}, \quad \epsilon + 1_f + \eta = 1_{fgf}, \quad \gamma + 1_g + \delta = 1_{gfg},$$

and according to (1) $\Rightarrow$ (4), we obtain that $\epsilon\delta = 0_{bb}$;
There exists $\tau = \eta 0_{ba}\epsilon$ such that $\gamma\tau\delta = 0_{ba}$.

(9) $\Rightarrow$ (10). For the II-arrow $\gamma = 0_{ba}\epsilon\theta$, it follows

$$\eta = \tau\delta 0_{ab}, \quad \gamma\eta = 0_{aa}, \quad \epsilon + 1_f + \eta = 1_{fgf}.$$

By **Lemma 3.1** the equality $\eta + 1_g + \epsilon + 1_f = 1_{gfgf}$ holds. The equality $\eta\gamma + (1_g + \epsilon)(\eta + 1_g) + 1_f = \eta\gamma + 1_{gf}$, implies that

$$(1_{fgf} + 0_{ab}\gamma\tau)(\eta + 1_g + \epsilon + 1_f)(1_{gfg} + \delta) = 1_{gfgf} + 0_{ab}\gamma\tau\delta$$
$$\eta + 1_g + \epsilon = 1_{gfg}$$
$$1_g + \delta\epsilon = \eta + 1_g + \epsilon = 1_{gfg}$$
$$(0_{ab}\gamma\tau + 1_{fg})(1_g + \delta\epsilon)(\delta + 1_{fg}) = 0_{ab}\gamma\tau\delta + 1_{fg}.$$
$$\delta\epsilon = 1_{fg},$$

(10) $\Rightarrow$ (11). For the II-arrow $\epsilon = 0_{ab}\gamma\tau$, it follows

$$\epsilon\delta = 0_{bb}, \quad \delta + 1_f + \gamma = 1_{fgf}, \quad \eta + 1_g + \epsilon = 1_{gfg}$$

By **Lemma 3.1** the following equalities hold:

$$1_f + \gamma + 1_g + \delta = 1_{fgfg}, \quad \gamma + 1_g + \delta = 1_{gfg}, \quad 1_f + \eta = \delta + 1_f,$$
$$\epsilon + 1_{fgf} + \eta = 1_f + \gamma + 1_g + \delta + 1_f = 1_{fgfgf}.$$

(11) $\Rightarrow$ (1). There exist II-arrows $\gamma = 0_{ba}\epsilon\theta$, $\delta = \theta\eta 0_{ba}$, $\tau = \eta 0_{ba}\epsilon$, such that

$$\gamma\eta = 0_{aa}, \quad \epsilon\delta = 0_{bb}, \quad \gamma\tau\delta = 0_{ba}, \quad 1_{fgf} + \eta = \delta + 1_{fgf},$$
$$\delta + 1_f + \gamma = (1_{fgf} + \gamma)(\delta + 1_{fgf}) = 1_{fgf},$$

according to (1) $\Rightarrow$ (2), $\gamma + 1_g + \delta = 1_{gfg}$. ■

Remark 3.1. According to (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) , each condition of the **Theorem 3.1.** is equivalent to its dual and transposed, respectively.

Remark 3.2. For each condition of **Theorem 3.1** the following equalities hold:

$$\begin{gathered}
(\epsilon + 1_f)(1_f + \eta) = 1_f, \quad \epsilon + 1_f + \eta = 1_{fgf}, \quad \epsilon + 1_{fgf} + \eta = 1_{fgfgf}, \\
(1_g + \epsilon)(\eta + 1_g) = 1_g, \quad \eta + 1_g + \epsilon = 1_{gfg}, \quad \eta + 1_{gfg} + \epsilon = 1_{gfgfg}, \\
(1_g + \epsilon)(\eta + 1_g) = 1_g, \quad \eta + 1_g + \epsilon = 1_{gfg}, \quad \eta + 1_{gfg} + \epsilon = 1_{gfgfg}, \\
(\gamma + 1_g)(1_g + \delta) = 1_g, \quad \gamma + 1_g + \delta = 1_{gfg}, \quad \gamma + 1_{gfg} + \delta = 1_{gfgfg}, \\
1_f + \eta = \delta + 1_f, \quad \epsilon + 1_f = 1_f + \gamma, \quad \eta + 1_g = 1_g + \delta, \quad 1_g + \epsilon = \gamma + 1_g \\
\gamma\eta = 0_{aa}, \quad \epsilon\delta = 0_{bb}, \quad \gamma\tau\delta = 0_{ba}, \quad \eta 0_{ba}\epsilon = \tau, \quad \tau\theta = 1_{gf}, \\
\eta\gamma = 1_{gf}, \quad \delta\epsilon = 1_{fg}, \quad \epsilon\theta\eta = 0_{ab}, \quad \delta 1_{ab}\gamma = \theta, \quad \theta\tau = 1_{fg}.
\end{gathered}$$

Corollary 3.1. *For the* II*-arrows*

$$1_a \xrightarrow{f^*} f \xrightarrow{f_*} 1_b \xrightarrow{g^*} g \xrightarrow{g_*} 1_b,$$

the conditions:

1)

$$\begin{gathered}
f_* + 0_{ab}g_* + 1_f + g^*0_{ab} + f^* = 1_{fgf}, \\
f_*f^*g_*g^*\ f_*f^* = 0_{ab}.
\end{gathered}$$

2)

$$\begin{gathered}
f_* + 0_{ab}g_* + 1_f + g^*0_{ab} + f^* = 1_{fgf}, \\
f_*f^* = 0_{ab}, \\
g_*g^* = 0_{ba}, \\
g^* + f^*0_{ba}f_* + g_* = 0_{ba}f_* + 1_g + f^*0_{ba}
\end{gathered}$$

3)

$$\begin{gathered}
f^*g_*g^*\ f_* = 1_f, \\
f_*f^* = 0_{ab}, \\
g^*0_{ab}g_* + f^*0_{ba}f_* = 1_{gf.}
\end{gathered}$$

4)

$$\begin{gathered}
f_* + 0_{ab}g_* + 1_f = 1_f + g_* + 0_{ba} + f_*, \\
g^*0_{ab} + f^* + 1_g = 1_g + f^*0_{ba} + g^* \\
f_*f^*g_*g^*\ f_*f^* = 0_{ab}, \\
g^*0_{ab}g_* + f^*0_{ba}f_* = 1_{gf.}
\end{gathered}$$

5)

$$f_* + 0_{ab} g_* + 1_{fgf} + g^* 0_{ab} + f^* = 1_{fgfgf},$$
$$f_* f^* g_* g^* \; f_* f^* = 0_{ab},$$

6)

$$f \cong 1_x \cong g.$$

are equivalent.

Proof. After constructing II-arrows γ, η, δ, ε, θ and τ

$$\gamma = g_* + 0_{ba} f_*, \quad \eta = g^* 0_{ab} + f^*,$$
$$\delta = f^* 0_{ba} + g^*, \quad \epsilon = f_* + 0_{ab} g_*,$$
$$\theta = f^* g_* + g^* f_*, \quad \tau = g^* f_* + f^* g_*$$

the proof is straightly follows from the **Theorem 3.1**. ■

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