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SOME NOTES ON GOODNESS-OF-FIT TESTS AND INNOVATION MARTINGALES

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Abstract. Chi-square test for testing parametric hypotheses in terms of the innovation process is constructed. A new method allowing to obtain innovation martingales as well as a way of their obtaining in non-regular case are proposed. Some priorities of this approach are discussed. One result simplifying transformations appearing in our method is given.

To the memory of Prof. Revaz Chitashvili

0. Introduction

The aim pursued in this paper is to give some clarifications of the applications of the innovation martingales theory to some problems of mathematical statistics, namely to applications of this theory to goodness-of-fit tests. In [2]-[4] the problem of testing the parametric hypotheses about distribution of the independent identically distributed m-dimensional random vectors, $X_1, X_2, \dots, X_n$ with an absolutely continuous distribution function depending on the d-dimensional unknown parameter θ was considered, i.e. we wish to check the hypothesis that $F \in \mathcal{F}$, where $\mathcal{F} = \{F(\cdot,\theta), \theta \in \Theta \subset R^d\}$ is a regular family of absolutely continuous distribution functions with density $f(\cdot,\theta)$ (see [5], Chapter 1).

For convenience, let $m=1$ and consider the parametric empirical process $U_n(x,\tilde{\theta}_n) = \sqrt{n}(F_n(x) - F(x,\tilde{\theta}_n))$, where $F_n(.)$ is an empirical distribution function corresponding to the sample $X_1, X_2, \dots, X_n$ and $\tilde{\theta}_n$ is a "projecting" estimator of the true value of the parameter θ (see [6]). Let $\tilde{\theta}_n$ be a maximum likelihood estimator, which is the particular case of the "projecting" estimators, $t = F(x,\theta)$ and $\tilde{u}_n(t) = U_n(x,\tilde{\theta}_n)$. It is well known that under conditions (see [2] and bibliography here) on the functional

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space $L_2[0,1]$ (for the proof on $D[0,1]$ see [7]), $\tilde{u}_n \to u$ in distribution, as $n \to \infty$, where

$$u(t) = v(t) - g^*(t,\theta)\int_0^1 g'(s,\theta)dv(s) \equiv \Pi v(t),$$

with $g'(t,\theta) = \frac{\partial}{\partial\theta}\log f(x,\theta)$. Here v is a Brownian bridge and the superscript denotes the conjugation. However the transformation Π of the process v into the process u is a projection (see[6]), and we cannot invert it as a transformation of the process u, i.e. we cannot use this transformation for constructing asymptotically distribution free goodness-of-fit tests. In [2]-[4] another invertable transformation Q of the process was considered, such that $Qv = Qu = Qb$, and the distribution of the process $Qu(\cdot)$ does not depend on the family $\mathcal{F}$. Here b denotes the standard Wiener process. This transformation has the form

$$Qu(t) = u(t) - \int_0^t g'^*(s)C^{-1}(s)\int_s^1 g'(z)du(z)ds \equiv w(t), \tag{1}$$

where

$$g'^*(t) = (1, g'^*(t,\theta)), \quad C(t) = \int_t^1 g(z)g'^*(z)dz.$$

To construct goodness-of-fit tests, we need the main result from [2]. Consider the process $w_n(t) = Q(\sqrt{n}F_n(x))$. Then the proposition bellow is valid.

Proposition. If the matrix $C(t)$ is finite and non-singular for all $0 \le t \le T < 1$, then on $L_2[0,1]$ $w_n \to_D w$ as $n \to \infty$, where w is a Wiener process.

The aim of the present paper is on the first hand to clarify the application of this *Proposition* to the construction of asymptotically distribution free goodness-of-fit tests, namely, to construct chi-square type test which exactly coincides with Nikulin's modified chi-square test (see [8]). This problem will be considered in Section 1 of the paper. Note that the analogous procedures may be performed for the general point processes (see [10]). If we construct a corresponding chi-square test in this case, then it will be easy to verify that this procedure provides the test which coincides with Hjort's chi-square test (see [11]).

On the other hand, we will clarify the situation when the matrix $C(t)$ is a singular for some t. As is shown in [9], the limiting process in such situations is still a Wiener's process, if we take any generalised inverse of the matrix $C(t)$. But what kind of the Wiener process can we obtain? This question will be answered in Section 2 of our paper, where a new method, a "Step-by-step" innovation, is proposed to obtain innovation martingales.

In Section 3 we will discuss about some preferences of our approach, which enables us to construct innovation processes even in the case of an infinite-dimensional parameter case.

Finally, in Section 4 further investigation enables us to simplify the calculations which arise while the empirical analogues of the innovation martingales are considered. Some simple examples of this kind are given. However, this section is of the other interest too. The specialists in the orthogonal polinomials may find here a new representation (in our opinion) for Lagguerre's polinomials.

1. Construction of the Chi-square Test by an Innovation Martingale

For construction of our chi-square statistic we need one auxiliary lemma. Let $\xi^*_{(k+1)} = (\xi_0, \ldots, \xi_{k-1}, \xi_k) = (\xi^*_{(k)}, \xi_k)$ be a normally distributed random vector with zero mean and the covariance matrix

$$R_{(k+1)} = (E[\xi_r \xi_j])_{0 \le r,j \le k} \quad (k \ge 0).$$

Define random variables $\eta_0 = \xi_0, \eta_{k+1} = \xi_{k+1} - E[\xi_{k+1} | \xi_{(k+1)}]$, and let $\rho^*_{(k+1)} = (E[\xi_0 \xi_{k+1}], \ldots, E[\xi_k \xi_{k+1}])$ for all $k \ge 0$. It is well known (see, e.g., [12]) that if $\operatorname{rank} R_{(k+1)} = s \le k+1$, then the quadratic form $\xi^*_{(k+1)} R^-_{(k+1)}) \xi_{(k+1)}$, where $R^-_{(k+1)}$ is any generalised inverse of the matrix $R_{(k+1)}$, has the chi-square distribution with s degrees of freedom. However, there exists another representation of this statistic in terms of random variables $\eta_0, \eta_1, \ldots, \eta_k$. This representation exactly coincides with the general definition of the chi-square random variable as a sum of squares of independent standard normal random variables. This representation is given in the following

Lemma 1. *Let $\xi_{(k+1)}, R_{(k+1)}$ and $\eta_j, 0 \le j \le k$ be defined as above. Then*

$$\xi^*_{(k+1)} R^-_{(k+1)}) \xi_{(k+1)} = \sum_{j=0}^{k} \frac{\eta_j^2}{E[\eta_j^2]}.$$

Proof. Since the random vector $\xi_{(k+1)}$ is normally distributed, it is known that $E[\xi_{r+1} | \xi^*_{(r+1)}] = \rho^*_{(r+1)} R^-_{(r+1)} \xi_{(r+1)}$. So $\eta_{r+1} = \xi_{r+1} - \xi^*_{(r+1)} R^-_{(r+1)} \rho_{(r+1)}$ with

$$E[\eta_{r+1}] = 0, \quad E[\eta^2_{r+1}] = E[\xi^2_{r+1}] - \rho^*_{(r+1)} R^-_{(r+1)} \rho_{(r+1)},$$

for all $r \ge 0$. We may write the recurrent formula for generalised inverse matrix (see, e.g.,[12]):

$$R^-_{(r+1)} = \begin{cases} \dfrac{1}{E[\eta_0^2]}, & r = 0; \\ \dfrac{1}{E[\eta_r^2]} \cdot \begin{pmatrix} R^-_{(r)} \rho_{(r)} \rho^*_{(r)} R^-_{(r)} + R^-_{(r)} E[\eta_r^2] & -R^-_{(r)} \rho_{(r)} \\ -\rho^*_{(r)} R^-_{(r)} & 1 \end{pmatrix}, & r \ge 1. \end{cases}$$

Consider now the increment of the quadratic form $\xi^*_{(r+1)} R^-_{(r+1)}) \xi_{(r+1)}$:

$$\xi^*_{(r+1)} R^-_{(r+1)} \xi_{(r+1)} - \xi^*_{(r)} R^-_{(r)} \xi_{(r)} = (\xi^*_{(r)}, \xi_r) \cdot \frac{1}{E[\eta_r^2]} \times$$

$$\times \begin{pmatrix} R^-_{(r)} \rho_{(r)} \rho^*_{(r)} R^-_{(r)} + R^-_{(r)} E[\eta_r^2] & -R^-_{(r)} \rho_{(r)} \\ -\rho^*_{(r)} R^-_{(r)} & 1 \end{pmatrix} \cdot (\xi^*_{(r)}, \xi_r)^* - \xi^*_{(r)} R^-_{(r)} \xi_{(r)} =$$

$$= \frac{1}{E[\eta_r^2]} \cdot (-\eta_r \rho^*_{(r)} R^-_{(r)} + E[\eta_r^2] \cdot \delta^*_{(r)} R^-_{(r)}, \eta_r) \times (\xi^*_{(r)}, \xi_r)^* - \xi^*_{(r)} R^-_{(r)} \xi_{(r)} =$$

$$= \frac{1}{E[\eta_r^2]} \cdot (\eta_r (\xi_r - \rho^*_{(r)} R^-_{(r)} \xi_{(r)}) + E[\eta_r^2] \cdot \xi^*_{(r)} R^-_{(r)} \xi_{(r)}) - \xi^*_{(r)} R^-_{(r)} \xi_{(r)} = \frac{\eta_r^2}{E[\eta_r^2]}.$$

Summing the last equation for $r = 0, 1, \dots, k$, we obtain the assertion. ■

For constructing a chi-square goodness-of-fit test by the process w_n, consider the partition of the real line $-\infty = x_0 < x_1 < \cdots < x_{N-1} < x_N = +\infty$ and investigate the discrete analogue of increments of the process w_n:

$$\Delta w_n(x_j) = \Delta U_n(x_j) -$$
$$-\Big(\frac{\partial}{\partial\theta}\Delta F(x_j,\theta)\Big)^* A^{-1}(x_j)\sum_{i=j}^{N}\frac{\partial}{\partial\theta}\ln\Delta F(x_i,\theta)\Delta U_n(x_i),$$

where

$$A(x_j) = \sum_{i=j}^{N}\frac{\partial}{\partial\theta}\ln\Delta F(x_i,\theta)(\frac{\partial}{\partial\theta}\ln\Delta F(x_i,\theta))^*\Delta F(x_i,\theta).$$

Clearly, since the vector of increments of w_n, according to **Proposition** given in introduction, tends in the distribution to the vector of increments of the Gaussian vector, the distribution of the quadratic form

$$\chi^2_{N,n} = \sum_{j=0}^{N}\frac{[\Delta w_n(x_j)]^2}{E[\Delta w_n(x_j)]^2}$$

by **Lemma 1** above tends to the chi-square distribution with $(N - \operatorname{rank} A)$ degrees of freedom.

2. Step-By-Step Innovation

Consideration of this Section we begin with the representation of the example. Let a family $\mathcal{F}$ of the distribution functions F be a family of the Laplas distribution:

$$\mathcal{F} = \left\{ F(x,\theta) : F(x,\theta) = \left\{\begin{array}{ll} \frac{1}{2}e^{x-\theta}, & x \le \theta; \\ 1 - \frac{1}{2}e^{\theta-x}, & x > \theta \end{array}\right. , \ \theta > 0 \right\}.$$

In this case $g'^*(t) = (1, \operatorname{sign}(x-\theta))$, and

$$C(t) = \left\{\begin{array}{ll} \begin{pmatrix} 1-F(x,\theta) & -F(x,\theta) \\ -F(x,\theta) & 1-F(x,\theta)\end{pmatrix}, & x \le \theta \\ (1-F(x,\theta))\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}, & x > \theta \end{array}\right.$$

where $t = F(x,\theta)$ again and we see that if $t > \frac{1}{2}$, then matrix $C(t)$ is singular. However, for all generalised inverses of the matrix $C(t)$ we obtain the same transformation

Q:

$$W(t) = Qb(t) = b(t) - \int_0^{t\wedge\frac{1}{2}} \frac{b(\frac{1}{2}) - b(s)}{\frac{1}{2} - s} ds -$$
$$- \int_{\frac{1}{2}}^{t} \frac{b(1) - b(s)}{1 - s} ds \cdot I(t \geq \frac{1}{2}). \tag{2}$$

Everywhere $a \wedge b$ denotes $\min(a, b)$. By the direct calculation of the covariates of the process W, one can easily show that W is the standard Wiener process.

This example shows that the condition of nonsingularity of the $C(t)$ matrix in Proposition given in the introduction may be weakened or omitted at all. In this Section we'll discuss this problem. In [2], for constructing the transformation Q the limit process $\{u(t), \mathcal{H}_t^u\}_{0\leq t\leq 1}$ was considered as semimartingale, where

$$\mathcal{H}_t^u = \mathcal{H}_t^b V \sigma\bigg(\int_0^1 g'^*(s)db(s)\bigg) = \mathcal{H}_t^b V \sigma\bigg(\int_t^1 g'^*(s)db(s)\bigg)$$

and $Qu(t)$ was obtained from the Doob-Meyer decomposition of this semimartingale, which in fact is the innovation of the process $b(t)$ also. To investigate our approach, let us introduce the following notation. Let $h_0(t), h_1(t), \ldots, h_d(t), \ldots$ be a sequence of the real functions on $[0, 1]$ (maybe $h_0(t) \equiv 1$ and $h_i(t) = g_i'(t), 1 \leq i \leq d$), and let $H_k^*(t) = (h_0(t), h_1(t), \ldots, h_{k-1}(t))$ be a k-dimensional vector constructed by the first k component of the system $h_0(t), h_1(t), \ldots$. Consider the sequence of the σ-algebra's defined by

$$\mathcal{R}_t^0 = \mathcal{H}_t^b, \mathcal{R}_t^{k+1} = \mathcal{R}_t^k V \sigma\bigg(\int_0^1 h_k(s)db(s)\bigg), \quad k \geq 0, \tag{3}$$

and also the sequence of the innovation processes $W^0(t), W^1(t), \ldots, W^d(t), \ldots$ defined as follows:

$$W^0(t) = b(t), dW^k(t) = db(t) - E[db(t)|\mathcal{R}_{t-}^{k+1}]. \tag{4}$$

It is easy to see that

$$\mathcal{R}_t^k = \mathcal{R}_t^0 V \sigma\bigg(\int_t^1 H_k(s)db(s)\bigg)$$

So $\mathcal{R}_t^0 \subset \mathcal{R}_t^1 \subset \mathcal{R}_t^2 \subset \cdots$ and also

$$dW^{k+1}(t) = dW^k(t) - E[dW^k(t)|\mathcal{R}_{t-}^{k+1}], \quad \text{for} \quad k \geq 0. \tag{5}$$

The last representation leads us to the main idea of our approach: the idea of the step-by-step innovation. First it is reasonable to say that this approach gives nothing new but its usefulness will be explained in the Discussion following after two theorems.

Theorem 1. *If* $W^k(t), \mathcal{R}^k{}_{t\geq 0}, k=0,1,2,\dots$ *is the sequence of the processes defined by* (3) *and* (4), *then for all* $k \geq 0$,

$$W^{k+1}(t) = W^k(t) - \int_0^t \frac{l_k(s)}{\int_s^1 l_k^2(z)dz} \cdot \int_s^1 l_k(z)dW^k(z)ds, \tag{6}$$

where $l_0(s) \equiv h_0(s)$,

$$l_k(s) = h_k(s) - H_k^*(s)C_k^{-1}(s)\int_s^1 H_k(z)h_k(z)dz, k \geq 1. \tag{7}$$

Here $C_k(s) = \int_s^1 H_k(z)H_k^*(z)dz$.

Proof. For $k=0$ the proof is trivial and we omit it. Consider the case $k \geq 1$. We have

$$E[dW^k(t)|\mathcal{R}_{t-}^{k+1}] = E[db(t)|\mathcal{R}_{t-}^{k+1}] - E[db(t)|\mathcal{R}_{t-}^{k}] = [H_{k+1}^*(t)C_{k+1}^{-1}(t) \times \\ \times \int_t^1 H_{k+1}(z)db(z) - H_k^*(t)C_k^{-1}(t)\int_t^1 H_k(z)db(z)]dt. \tag{8}$$

For the brevity we denote

$$\psi_k(t) \equiv \int_t^1 H_k(z)h_k(z)dz, \Delta_k(t) \equiv \int_t^1 h_k^2(z)dz - \psi_k^*(t)C_k^{-1}(t)\psi_k(t).$$

Then from (7) and from the recurrent formula for the inverse matrix, we have

$$E[dW^k(t)|\mathcal{R}_{t-}^{k+1}] = \Bigg[\frac{\big(H_k^*(t), h_k(t)\big)}{\Delta_k(t)} \times \\ \times \begin{pmatrix} C_k^{-1}(t)\psi_k(t)\psi_k^*(t)C_k^{-1}(t) + C_k^{-1}(t)\Delta_k(t) & -C_k^{-1}(t)\psi_k(t) \\ -\psi_k^*(t)C_k^{-1}(t) & 1 \end{pmatrix} \times \\ \times \int_t^1 \big(H_k(z), h_k(z)\big)^* db(z) - H_k^*(t)C_k^{-1}(t)\int_t^1 H_k(z)db(z)\Bigg]dt = \\ = \Bigg[\frac{1}{\Delta_k(t)} \cdot \big(H_k^*(t)C_k^{-1}(t)\Delta_k(t) - \psi_k^*(t)C_k^{-1}(t)l_k(t), l_k(t)\big) \times \\ \times \int_t^1 \big(H_k(z), h_k(z)\big)^* db(z) - H_k^*(t)C_k^{-1}(t)\int_t^1 H_k(z)db(z)\Bigg]dt = \\ = \frac{l_k(t)}{\Delta_k(t)} \cdot \Bigg[\int_t^1 h_k(z)db(z) - \psi_k^*(t)C_k^{-1}(t)\int_t^1 H_k(z)db(z)\Bigg]dt.$$

It is easy to verify that

$$\Delta_k(t) = \int_t^1 l_k^2(z)dz.$$

To prove the theorem it suffices to show using (7) that

$$\int_t^1 l_k(z)dW^k(z) = \int_t^1 h_k(z)db(z) - \psi_k^*(t)C_k^{-1}(t)\int_t^1 H_k(z)db(z). \tag{9}$$

By definitions (3) and (4), we have

$$\int_t^1 l_k(z)dW^k(z) = \int_t^1 h_k(z)db(z) + \int_t^1 \psi_k^*(s)C_k^{-1}(ds)\int_s^1 H_k(z)db(z) -$$
$$- \int_t^1 \psi_k^*(s)C_k^{-1}(s)H_k(s)db(s) - \int_t^1 h_k(s)H_k^*(s)C_k^{-1}(s)\int_s^1 H_k(z)db(z)ds.$$

Integration by parts of the second summand in the last equality gives (9). ■

Theorem 2. *The system of functions $h_0(t), h_1(t), \ldots, h_k(t)$ is linearly dependent for some $t \geq t_0$ iff $l_k(t) = 0$ for $t \geq t_0$.*

Proof. Consider the function $l_k(t)$ as a linear transformation of the function $h_k(t)$:

$$l_k(t) = h_k(t) - H_k^*(t)C_k^{-1}(t)\int_t^1 H_k(z)h_k(z)dz, \quad k \geq 1. \tag{10}$$

The kernel of this transformation consists only of functions of the form $H_k^*(t)\delta_k$, where δ_k is any k-dimensional vector. Therefore, $l_k(t) = 0$ for $t \geq t_0$ iff $h_k(t) = H_k^*(t)\delta_k$ for all $t \geq t_0$, i.e., when the function $h_k(t)$ is a linear combination of the previous components $h_0(t), h_1(t), \ldots, h_{k-1}(t)$ and the system of the functions $h_0(t), h_1(t), \ldots, h_k(t)$ is linearly dependent for $t \geq t_0$. ■

3. Discussion

Let us begin our discussion with the fact that within the conditions of the **Proposition** from the introduction, on the d-th step of our approach we get the same transformation Q as in [2]. However, our approach has some priorities:

1) the processes W^{k+1} are the linear transformations of the previous process W^k and these transformations are the exact one-dimensional analogues of the transformation Q with the corresponding function l_k. It is not so bed situation from the computational point of view or for the recurrent study of the properties of these transformations;

2) consider $\tilde{H}_k(t) = B_k H_k(t)$ for any k-dimensional matrix B_k. We have

$$\tilde{l}_k(t) = h_k(t) - \tilde{H}_k^*(t)\Big(\int_t^1 \tilde{H}_k(s)\tilde{H}_k^*(s)ds\Big)^{-1} \times$$
$$\times \int_t^1 \tilde{H}_k(s)h_k(s)ds = h_k(t) - H_k^*(t)\Big(\int_t^1 H_k(s)H_k^*(s)ds\Big)^{-1} \equiv l_k(t),$$

i.e., l_k is an invariant for any linear transformation of H_k;

3) let there exist $t_0 \in [0,1]$ such that $l_k(t) = 0$ for all $t \geq t_0$. By **Theorem 2** we know that $h_k(t) = H_k^*(t)\delta_k$ in this case. Therefore, $\mathcal{R}_t^{k+1} = \mathcal{R}_t^k$, i.e., $\int_t^1 h_k(s)db(s)$,

gives no new information contained in $\mathcal{R}_t^k$ for $t \geq t_0$. Now it is clear that in such situations the innovation process for W^k has the form

$$dW^{k+1}(t) = \begin{cases} dW^k(t) - \dfrac{l_k(t)}{\int_t^1 l_k^2(z)dz} \cdot \displaystyle\int_t^1 l_k(z)dW^k(z)dt, & t < t_0 \\ dW^k(t), & t \geq t_0 \end{cases},$$

or equivalently,

$$W^{k+1}(t) = W^k(t) - \int_0^{t\wedge t_0} \frac{l_k(s)}{\int_s^1 l_k^2(z)dz} \cdot \int_s^1 l_k(z)dW^k(z)ds.$$

Recall the example of Laplace's distribution family. It is easy to see that $l_0(t) = h_0(t) \equiv 1$, $h_1(t) = 2 \cdot I(t \geq 1/2) - 1$ and

$$l_1(t) = -\frac{I(t < 1/2)}{1-t}.$$

Now we see that $t_0 = 1/2$. By the direct calculations we have

$$W^1(t) = b(t) - \int_0^t \frac{b(1) - b(z)}{1-z} dz$$

and

$$W^2(t) = W^1(t) - \int_0^{t\wedge 1/2} \frac{1}{(1-s)\cdot \int_s^{1/2}(1-z)^{-2}dz} \times$$
$$\times \int_s^{1/2} (1-z)^{-1} dW^1(z)ds = W^1(t) - \int_0^{t\wedge 1/2} \frac{1}{1-2s} \cdot \int_s^{1/2} \frac{1}{1-z} dW^1(z)ds.$$

Putting the above value of the process $W^1(t)$ in the last equality, we get for $W^2(t)$ the expression which exactly coincides with the innovation martingale for the Laplace distribution family given by formula (2).

4) Finally, in terms of the functions $l_k(t)$ we give very nice representation for the $|C_k(t)|$ determinant of the $C(t)$ matrix.

Corollary. *If* $\int_{t_1}^1 h_i^2(s)ds \neq 0$, $i = 1, 2, \ldots, k-1$, *for some* $0 \leq t_1 < 1$, *then*

$$-\left(\ln |C_k(t)|\right)' = H_k^*(t) C_k^{-1}(t) H_k(t) = \sum_{i=0}^{k-1} \frac{1}{\int_t^1 l_i^2(z)dz} \cdot l_i^2(t).$$

Proof. The first equality is an easy consequence of Jackobi's equality (see [1]): that if the matrix $X(t)$ of dimension $N \times N$ is the solution of the differential equation

$$\frac{dX(t)}{dt} = A(t)X(t), X(0) = I$$

for some $N \times N$ matrix A, for which there exists the integral $\int_0^{t_1} ||A(s)||ds$, where

$$||A(s)|| = \sum_{i,j=1}^{N} |a_{ij}|$$

for some $0 \le t \le t_1$, then

$$|X(t)| = \exp\left(\int_0^t SpA(s)ds\right).$$

Take $X_k(t) = C_k^{-1}(t)C_k(0)$. Then for this matrix we have the equation

$$\frac{dX(t)}{dt} = A(t)X(t), \quad X(0) = I$$

with

$$A(t) = -C_k^{-1}(t)C_k'(t) = C_k^{-1}(t)H_k(t)H_k^*(t),$$

and from Jackobi's equality we have

$$|C_k(t)| = |C_k(0)| \exp\left(-\int_0^t H_k^*(s)C_k^{-1}(s)H_k(s)ds\right).$$

To prove the second equality, consider the fraction

$$\frac{|C_{i+1}(t)|}{|C_i(t)|}$$

and apply the same teqnique as in the proof of **Theorem 1.** We get

$$\frac{|C_{i+1}(t)|}{|C_i(t)|} = \frac{|C_{i+1}(0)|}{|C_i(0)|} \times$$
$$\times \exp\int_0^t \big(H_i^*(s)C_i^{-1}(s)H_i(s) - H_{i+1}^*C_{i+1}^{-1}(s)H_{i+1}(s)\big)ds =$$
$$= \frac{|C_{i+1}(0)|}{|C_i(0)|} \cdot \exp\left\{-\int_0^t \frac{l_i(s)}{\Delta_i(s)} \cdot \big(h_i(s) - \psi_i^*(s)C^{-1}(s)H_i(s)\big)ds\right\} =$$
$$= \frac{|C_{i+1}(0)|}{|C_i(0)|} \cdot \exp\left(-\int_0^t \frac{l_i^2(s)}{\int_s^1 l_i^2(z)dz}ds\right) = \frac{|C_{i+1}(0)|}{|C_i(0)|} \cdot \frac{\int_t^1 l_i^2(s)ds}{\int_0^1 l_i^2(s)ds},$$

or equivalently,

$$-\Big(\ln|C_{i+1}(t)|\Big)' + \Big(\ln|C_i(t)|\Big)' = \frac{l_i^2(t)}{\int_t^1 l_i^2(s)ds}.$$

Summing the last equality from one to $(k-1)$ and recalling that $l_0(t) = h_0(t)$ we get the assertion. ■

4. Further Investigation.

Recall that from **Theorem 1** we have the recurrent formula for a new innovation process (see (6)) in which a new innovation W^{k+1} depends not only on the old one but also on the function l_k, which depends in general on k. This makes difficult the statistical application of our approach because it is not easy to calculate an empirical analogue of the innovation processes. In this section we intend to investigate a problem whether we may overcome this difficulty or not. In other words, we are going to study the question: does there exist a sequence of a functions $h_0(t),\dots,h_k(t),\dots$, such that the corresponding functions $l_0(t),\dots,l_k(t),\dots$ be the same for all k and equal to some given function $l(t)$ (for example, $l(t)\equiv 1$). The answer to this question gives the following theorem.

Let

$$L_k(z)=\sum_0^k \frac{(-1)^j C_k^j}{j!}\cdot z^j$$

be Laguerre's k-th polynomial.

Theorem 3. *If a sequence of the functions $h_0(t),\dots,h_k(t),\dots$ is such that*

$$h_k(t)=l(t)L_k\Big\{-\ln\frac{\int_t^1 l^2(s)ds}{\int_0^1 l^2(s)ds}\Big\},\quad k\ge 0$$

for some fixed integrable function $l(t)$ defined on the unit interval, then we have $l_k(t)\equiv l(t)$ for all $k\ge 0$.

Proof. It is easy to show that the transformation (10) has the inverse of the form

$$h_k(t)=l_k(t)-H_k^*(t)\int_0^t C_k^{-1}(s)H_k(s)l_k(s)ds,\quad k\ge 0,$$

and so it suffices to show that

$$h_k(t)=l(t)-H_k^*(t)\int_0^t C_k^{-1}(s)H_k(s)l ds,$$

for all $k\ge 0$. If we substitute here the expression for the function h_k given in the condition of the theorem, then can see that for proving the theorem it suffices to show that there exists the following equality for Laguerre's polynomials:

$$L_k(z)=1-P_k^*(z)\int_0^z C_{P_k}^{-1}(x)P_k(x)e^{-x}dx,\quad k\ge 0,\tag{11}$$

where $P_0(z)=1$, $P_k^*(z)=(L_0(z),\dots,L_{k-1}(z))$ and

$$C_{P_k}(z)=\int_z^{+\infty} P_k(x)P_k^*(x)e^{-x}dx,\quad k\ge 1.\quad \blacksquare$$

To prove equation (11) we need some auxiliary lemmas.

Lemma 2. *For all integers $0 \le s \le k, \quad 0 \le r \le k-s$ define*

$$a_k(s,r) \equiv \sum_{q=0}^{s} C_{q+r}^{q} \cdot C_{k-(q+r)}^{s-q}.$$

Then $a_k(s,r) = C_{k+1}^{s}$.

Proof. Note that $a_k(0,r) = 1$ and there we have nothing to prove. Consider the case where $s \ge 1$. Then the proof follows from the chain of equalities

$$a_k(s,r) = C_{s+r}^{s} + \sum_{q=0}^{s-1} C_{q+r}^{q} \cdot \left(C_{k+1-(q+r)}^{s-q} - C_{k-(q+r)}^{s-q-1} \right) =$$

$$= \sum_{q=0}^{s} C_{q+r}^{q} \cdot C_{k+1-(q+r)}^{s-q} - \sum_{q=0}^{s-1} C_{q+r}^{q} \cdot C_{k-(q+r)}^{s-q-1} =$$

$$= C_{k-r+1}^{s} + \sum_{q=1}^{s} \left(C_{q+r}^{q} - C_{q+r-1}^{q-1} \right) \cdot C_{k+1-(q+r)}^{s-q} = \sum_{q=0}^{s} C_{q+r-1}^{q} \cdot C_{k-(q+r-1)}^{s-q} =$$

$$a_k(s,r-1) = \cdots = a_k(s,0) = \sum_{q=0}^{s} C_{k-q}^{k-s} = C_{k+1}^{s}. \quad \blacksquare$$

Lemma 3. *Define $b(k,s) \equiv \sum_{r=0}^{k} (-1)^r C_k^r \cdot C_{s+r}^{r}$. Then for all integers $0 \le s < k$, $k \ge 1$, we have $b(k,s) \equiv 0$.*

Proof. Let in the chain bellow all binomial coefficients be well-defined. Then we have

$$b(k,s) = \sum_{r=0}^{k-1} (-1)^r C_{k-1}^{r} \cdot C_{s+r}^{r} + \sum_{r=1}^{k} (-1)^r C_{k-1}^{r-1} \cdot C_{s+r}^{r} =$$

$$= \sum_{r=0}^{k-1} (-1)^r C_{-1}^{r} k \cdot C_{s+r}^{r} - \sum_{r=0}^{k-1} (-1)^r C_{k-1}^{r} \cdot (C_{s+r}^{r} + C_{s+r}^{r+1} =$$

$$= - \sum_{r=0}^{k-1} (-1)^r C_{k-1}^{r} \cdot C_{s+r}^{r+1} = -b(k-1, s-1) =$$

$$= \cdots = (-1)^m b(k-m, s-m) = \cdots =$$

$$= (-1)^s b(k-s, 0) = (-1)^s \sum_{r=0}^{k-s} (-1)^r C_{k-s}^{r} \equiv 0.$$

For all $k \ge 1$ introduce the notation:

$$V_k^*(x) = (L_{k-1}(-x), \ldots, L_0(-x)),$$

$$M_k(x,z) = V_k^*(x) \cdot P_k(z) = \sum_{r=0}^{k-1} L_{k-1-j}(-x) \cdot L_j(z). \quad \blacksquare$$

Then the following lemma is valid

Lemma 4. *For all $k \geq 1$ we have*

$$M_k(x,z) = \sum_{r=0}^{k-1} C_k^{r+1} \cdot \frac{(x-z)^r}{r!}.$$

Proof. The proof follows from the chain of equalities and from **Lemma 2**:

$$M_k(x,z) = \sum_{j=0}^{k-1} \sum_{m=0}^{k-1-j} \sum_{r=0}^{j} (-1)^r C_j^r C_{k-1-j}^m C_{m+r}^m \cdot \frac{x^m z^r}{(m+r)!} =$$

$$= \sum_{m=0}^{k-1} \sum_{j=0}^{k-1} \sum_{r=0}^{j} (-1)^r C_j^r C_{k-1-j}^{m-r} C_m^r \cdot \frac{x^m z^r}{m!} =$$

$$= \sum_{m=0}^{k-1} \left[\sum_{r=0}^{m} (-1)^r C_m^r (x/z)^{m-r} \sum_{j=r}^{k-1-m+r} C_j^r C_{k-1-j}^r \right] \frac{z^m}{m!} =$$

$$= \sum_{m=0}^{k-1} \left[\sum_{r=0}^{m} (-1)^r C_m^r (x/z)^{m-r} \sum_{q=0}^{k-1-m} C_{q+r}^q C_{k-1-q-r}^{m-r} \right] \frac{z^m}{m!} =$$

$$= \sum_{m=0}^{k-1} \sum_{r=0}^{m} (-1)^r C_m^r (x/z)^{m-r} \cdot C_k^{m+1} \frac{z^m}{m!} =$$

$$= \sum_{m=0}^{k-1} (x/z-1)^m C_k^{m+1} \cdot \frac{z^m}{m!} = \sum_{m=0}^{k-1} C_k^{m+1} \frac{(x-z)^m}{m!}. \quad \blacksquare$$

Lemma 5. *Let $M_0(x,z) \equiv 1$. Then for all $k \geq 0$ we have*

$$\int_x^{+\infty} M_k(x,z) P_k(z) e^{-z} dz = P_k(x) e^{-x}. \tag{12}$$

Proof. By the application of the previous lemma and the property of Laguerre's polynomials $L_i'(z) = -\sum_{j=0}^{i-1} L_j(z)$, we obtain

$$\int_x^{+\infty} M_k(x,z) P_k(z) e^{-z} dz = e^{-x} \int_0^{+\infty} P_k(x+z) e^{-z} dL_k(z) =$$

$$= P_k(x) e^{-x} - e^{-x} \int_0^{+\infty} (P_k'(x+z) - P_k(x+z)) \cdot L_k(z) e^{-z} dz =$$

$$= P_k(x) e^{-x} + e^{-x} \int_0^{+\infty} T_k(x+z) L_k(z) e^{-z} dz,$$

where

$$T_k(z) = \left(T_k^i(z)\right)_{0 \leq i \leq k-1}, \quad T_k^i(z) = \sum_{j=0}^{i} L_j(z).$$

Consider the i-th component of the vector $\int_0^{+\infty} T_k(x+z)L_k(z)e^{-z}dz$:

$$\int_0^{+\infty} T_k^i(x+z)L_k(z)e^{-z}dz = \sum_{j=0}^{i} \int_0^{+\infty} L_j(x+z)L_k(z)e^{-z}dz.$$

Since $j \le i < k$, by the orthogonality of Laguerre's polynomial each summand of the last sum equals to zero. Indeed, consider

$$\int_0^{+\infty} L_j(x+z)L_k(z)e^{-z}dz = \sum_{m=0}^{j}(-1)^m \cdot \frac{C_j^m}{m!} \cdot \int_0^{+\infty} (x+z)^m L_k(z)e^{-z}dz =$$
$$= \sum_{m=0}^{j}(-1)^m \cdot \frac{C_j^m}{m!} \cdot \sum_{r=0}^{m} C_m^r x^{m-r} \cdot \int_0^{+\infty} z^r L_k(z)e^{-z}dz.$$

Since the integers r, m, j, i, k satisfy the relation $r \le m \le j \le i < k$, by **Lemma 3** each integral in the last sum equals to zero. Indeed,

$$\int_0^{+\infty} z^r L_k(z)e^{-z}dz = \sum_{m=0}^{k}(-1)^m \cdot \frac{C_k^m}{m!} \cdot \int_0^{+\infty} z^{r+m}e^{-z}dz =$$
$$= r! \cdot \sum_{m=0}^{k}(-1)^m C_k^m C_{m+r}^m = 0.$$

Now we return to the proof of the theorem. Since $M_k(x,z) = V_k^*(x) \cdot P_k(z)$, we rewrite equality (12) of **Lemma 5** in the following form:

$$\int_0^{+\infty} V_k(x)P_k(z)P_k^*(z)e^{-z}dz = P_k(x)e^{-x}$$

or equivalently, $V_k(x) = C_{P_k}^{-1}(x)P_k(x)e^{-x}$ and consider

$$1 - P_k^*(x)\int_0^x C_{P_k}^{-1}(z)P_k(z)e^{-z}dz = 1 - \int_0^x V_k^*(z)dzP_k(x) =$$
$$= 1 - \int_0^x M_k(z,x)dz = 1 - \sum_{m=0}^{k-1}\frac{C_k^{m+1}}{m!} \cdot \int_0^x (z-x)^m dz =$$
$$= 1 - \sum_{m=0}^{k-1}(-1)^m \cdot \frac{C_k^{m+1}}{(m+1)!} \cdot x^{m+1} = \sum_{m=0}^{k}(-1)^m \frac{C_k^m}{m!}x^m = L_k(x).$$

This completes the proof of the theorem. ■

Let a sequence of the functions be the same as in the condition of **Theorem 3**. Then this theorem asserts that the new innovation process is of the form

$$W^{k+1}(t) = W^k(t) - \int_0^t \frac{l(s)}{\int_s^1 l^2(z)dz} \cdot \int_s^1 l(z)dW^k(z)ds. \tag{13}$$

For $l(t) \equiv 1$ we have

$$W^{k+1}(t) = W^k(t) - \int_0^t \frac{W^k(1) - W^k(s)}{1-s} ds.$$

Thus we see that the recurrent relation has very simple form.

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