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ON AN APPLICATION OF THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE TO THE PLANE PROBLEM OF NON-HOMOGENEOUS THEORY OF ELASTICITY

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Abstract. The first and the second boundary value problems of the theory of elasticity for the non-homogeneous elastic plane with rectilinear cuts are studied. Body's non-homogeneity is characterized by a variable Poisson coefficient (the shift modulus is a constant value) which is a continuous periodic function of the point coordinates of particular type. Using the method of the theory of a complex variable these problems are reduced to the problems of conjugation. Solutions are written out explicitly.

Under the conditions of plane stress we study the boundary value problem for non-homogeneous elastic plane with rectilinear cuts. The non-homogeneity of the body is characterized by a variable Poisson coefficient of the type

$$\sigma = A_0 + B_0 \cos \alpha y \quad (\mu = \text{const}), \tag{1}$$

where A_0, B_0, α are real constants, Γ is the shift modulus and σ is the Poisson coefficient; note that $A_0 > |B_0|$.

1. Complex Representation of a General Solution of Equations of the Non-Homogeneous Plane Theory of Elasticity. As it is known (see [1]. p. 186), the fundamental equations of the plane theory of elasticity for the non-homogeneous body which are

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representable in the general case in a complex form can be written as follows:

$$\frac{\partial}{\partial \overline{z}}\Big[\frac{\mu}{\varkappa-1}\Big(\frac{\partial U}{\partial z}+\frac{\partial \overline{U}}{\partial \overline{z}}\Big)\Big]+\frac{\partial}{\partial z}\Big(\mu\frac{\partial U}{\partial \overline{z}}\Big)=0, \tag{2}$$

$$\begin{gathered}X_x+Y_y=\frac{4\mu}{\varkappa-1}\Big(\frac{\partial U}{\partial z}+\frac{\partial \overline{U}}{\partial \overline{z}}\Big)=4\frac{\partial^2 F}{\partial z\partial \overline{z}}\\ Y_y-X_x-2iX_y=-4\mu\frac{\partial U}{\partial \overline{z}}=4\frac{\partial^2 F}{\partial \overline{z}}\end{gathered} \tag{3}$$

$$\varepsilon=\frac{i}{2}\Big(\frac{\overline{U}}{\partial \overline{z}}-\frac{\partial U}{\partial z}\Big)$$

where $z=x+iy$ is a complex variable, $\frac{\partial}{\partial z}=\frac{1}{2}\Big(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\Big)$, $\frac{\partial}{\partial \overline{z}}=\frac{1}{2}\Big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\Big)$, X_x, Y_y, X_y are stress components, $U=u+iv$ is a complex displacement, F is a stress function, ε is a body revolution, and

$$\varkappa=\frac{3-\sigma}{1+\sigma}. \tag{4}$$

Taking into account that in the case under consideration $\mu=\text{const}$, we can write equation (2) as

$$\frac{\partial}{\partial \overline{z}}\Big[\frac{1}{\varkappa-1}\Big(\frac{\partial U}{\partial z}+\frac{\partial \overline{U}}{\partial \overline{z}}\Big)+\frac{\partial U}{\partial z}\Big]=0,$$

whence

$$\frac{1}{\varkappa-1}\Big(\frac{\partial U}{\partial z}+\frac{\partial \overline{U}}{\partial \overline{z}}\Big)+\frac{\partial U}{\partial z}=\phi(z), \tag{5}$$

where $\phi(z)$ is an arbitrary analytic function in the region S occupied by a non-homogeneous elastic body.

We combine formula (5) and that obtained after passing to conjugate values and find that

$$\frac{1}{\varkappa-1}\Big(\frac{\partial U}{\partial z}+\frac{\partial \overline{U}}{\partial \overline{z}}\Big)=\frac{1}{\varkappa+1}(\phi(z)+\overline{\phi(z)}). \tag{6}$$

Substituting (6) into (5) and using formula

$$\frac{1}{\varkappa+1}=\frac{1+\sigma}{4}=\frac{1+A_0+B_0\cos\alpha y}{4}=A+B\cos(\overline{a}_0z+a_0\overline{z})$$

$$\Big(A=\frac{A_0+1}{4},\quad B=\frac{B_0}{4},\quad a_0=i\frac{\alpha}{2}\Big),$$

derived from formulas (1) and (4), after simple calculations we obtain the following formulas:

$$\begin{aligned}2\mu\frac{\partial U}{\partial z}=(1-A)\phi(z)-A\overline{\phi(z)}-B\overline{\phi(z)}\cos(\overline{a}_0z+a_0\overline{z})-&\\ -B\phi(z)\cos(\overline{a}_0z+a_0\overline{z}).&\end{aligned} \tag{7}$$

The general solution of equation (7) is given by the formula

$$2\mu U = U_0 - \overline{\psi(z)}, \tag{8}$$

where $\psi(z)$ is an arbitrary analytic in S function and U_0 is a particular solution of equation (7) which, evidently, can be represented in the form

$$\begin{aligned} U_0 = (1-A)\varphi(z) - Az\overline{\phi(z)} + \frac{B}{a_0}\overline{\phi(z)}\sin(\overline{a}_0 z + a_0\overline{z}) - \\ -BG'(z)\cos(\overline{a}_0 z + a_0\overline{z}) + Ba_0 G(z)\sin(\overline{a}_0 z + a_0\overline{z}), \end{aligned} \tag{9}$$

where $\varphi'(z) = \phi(z)$ and $G(z)$ is the analytic function satisfying the equation

$$G''(z) + a_0^2 G(z) = \phi(z). \tag{10}$$

Thus, on the basis of formulas (8) and (9) we can represent the general solution of equations of the plane theory of elasticity in displacements relating to the non-homogeneous case of type (1), when the body forces are absent, as follows:

$$\begin{aligned} 2\mu U = (1-A)\varphi(z) - Az\overline{\phi(z)} - \overline{\psi(z)} + \frac{B}{a_0}\overline{\phi(z)}\sin(\overline{a}_0 z + a_0\overline{z}) - \\ -BG'(z)\cos(\overline{a}_0 z + a_0\overline{z}) + Ba_0 G(z)\sin(\overline{a}_0 z + a_0\overline{z}). \end{aligned} \tag{11}$$

To derive formulas for stress and rotation components, we'll use the formulas (3). If we substitute in these formulas the expression (11), then for U we obtain

$$\frac{1}{2}(X_x + Y_y) = [A + B\cos(\overline{a}_0 z + a_0\overline{z})](\phi(z) + \overline{\phi(z)}), \tag{12}$$

$$\begin{aligned} \frac{1}{2}(Y_y - X_x - 2iX_y) = Az\overline{\phi'(z)} + \overline{\Psi(z)} - B\overline{\phi(z)}\cos(\overline{a}_0 z + a_0\overline{z}) - \\ -\frac{B}{a_0}\overline{\phi'(z)}\sin(\overline{a}_0 z + a_0\overline{z}) - Ba_0 G'(z)\sin(\overline{a}_0 z + a_0\overline{z}) - \\ -Ba_0^2 G(z)\cos(\overline{a}_0 z + a_0\overline{z})(\Psi(z) = \psi'(z)), \end{aligned} \tag{13}$$

$$\varepsilon = \frac{i}{2\mu}(\overline{\phi(z)} - \phi(z)). \tag{14}$$

By virtue of formulas (3) and (11) we can easily calculate the following expression:

$$\begin{aligned} f(x,y) = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = A\varphi(z) + Az\overline{\phi(z)} + \overline{\psi(z)} - \\ -\frac{B}{a_0}\overline{\phi(z)}\sin(\overline{a}_0 z + a_0\overline{z}) + BG'(z)\cos(\overline{a}_0 z + a_0\overline{z}) - \\ -Ba_0 G(z)\sin(\overline{a}_0 z + a_0\overline{z}). \end{aligned} \tag{15}$$

Write out one more formula obtained from (12) and (13):

$$\begin{aligned} Y_y - iX_y &= A\phi(z) + A\overline{\phi(z)} + Az\overline{\phi'(z)} + \overline{\Psi(z)} + \\ &+ B\phi(z)\cos(\overline{a}_0 z + a_0\overline{z}) - \frac{B}{a_0}\overline{\phi'(z)}\sin(\overline{a}_0 z + a_0\overline{z}) - \\ &- Ba_0 G'(z)\sin(\overline{a}_0 z + a_0\overline{z}) - Ba_0^2 G(z)\cos(\overline{a}_0 z + a_0\overline{z}), \end{aligned} \tag{16}$$

and also formula

$$\begin{aligned} 2\mu\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big) &= (1-A)\phi(z) - A\overline{\phi(z)} - Az\overline{\phi'(z)} - \overline{\Psi(z)} + \\ &+ \frac{B}{a_0}\overline{\phi'(z)}\sin(\overline{a}_0 z + a_0\overline{z}) - B\phi(z)\cos(\overline{a}_0 z + a_0\overline{z}) + \\ &+ Ba_0 G'(z)\sin(\overline{a}_0 z + a_0\overline{z}) + Ba_0^2 G(z)\cos(a_0\overline{z} + a_0\overline{z}). \end{aligned} \tag{17}$$

Let the region S' occupied by an elastic body be an entire plane, cut along n segments $L_k = a_k b_k$ $(k = 1, 2, \dots, n)$ of the ox axis. Denote a family of these segments by L.

Proposition. *If the principal vector (X, Y) of external forces applied to the boundary L is finite, while forces and rotations are bounded at infinity, then, just as in the case of a homogeneous plane (see [2], p.124), the functions $G(z)$ and $\Psi(z)$ are holomorphic in S', including the point at infinity, and for large $|z|$ they can be written in the form*

$$\begin{aligned} G(z) &= \Gamma + \frac{\gamma}{z} + 0(\frac{1}{z^2}), \\ \Psi(z) &= \Gamma' + \frac{\gamma'}{z} + 0(\frac{1}{z^2}), \end{aligned} \tag{18}$$

where Γ, Γ', γ, γ' are constant quantities which can be expressed in terms of principal stresses and rotations at infinity and also by the values X and Y.

Proof. Consider a circumference L_R, taking the origin as a center, of radius R which is sufficiently large for all the contours L_k to be inside L_R.

Formula (12) shows (under the condition that the stress components are single-valued) that the real part of the function $\phi(z) = G''(z) + a_0^2 G(z)$ is single-valued, whereas the imaginary one is multi-valued, and for $\phi(z) = G''(z) + a_0^2 G(z)$ we have (see [2], p.119) $\phi(z) = G''(z) + a_0^2 G(z) = a_0^2 D \ln z$ + a single-valued function. This implies that the following representation is valid: $G(z) = D \ln z$ + a single-valued function, $\phi(z) = a_0^2 D \ln z$ + a single-valued function, where D is the real constant.

For the function $\varphi(z)$ we obtain (see [2], p.120) $\varphi(z) = \int_{z_0}^{z} \phi(z)dz = a_0^2 Dz \ln z + a_0^2 \gamma \ln z$ + a single-valued function.

Thus we have

$$G(z) = D \ln z + G^*(z), \tag{19}$$

$$\phi(z) = a_0^2 D \ln z + \phi^*(z), \tag{20}$$

$$\varphi(z) = a_0^2 Dz \ln z + a_0^2 \gamma \ln z + \varphi^*(z), \tag{21}$$

where γ is a constant (in general, complex), $G^*(z)$, $\phi^*(z)$ and $\varphi^*(z)$ are the functions, holomorphic outside L_R, except may be for the point at infinity.

We can see from (13), (19), (20) and conditions of single-valuedness of the left-hand side of (13) that Ψ is holomorphic, and for the function $\psi(z)$ we have (see [2], p.120):

$$\psi(z) = \gamma' \ln z + \psi^*(z). \tag{22}$$

Here γ' is a constant (in general, complex), and $\psi^*(z)$ is holomorphic outside L_R, except may be for the point at infinity.

Substituting in formula (11) the above-obtained expressions for $\varphi(z)$, $\psi(z)$, $\phi(z)$ and $G(z)$, we arrive at

$$2\mu[u + iv]_{L_R} = 2\pi i[Da_0^2 z + (1 - A)a_0^2\gamma + \overline{\gamma}'], \tag{23}$$

where the symbol $[\ ]_{L_R}$ denotes an increment of the expression in brackets, as the result of a circuit counter-clockwise.

It is seen from (23) that for the single-valuedness of displacements it is necessary and sufficient that

$$D = 0, \quad (1 - A)a_0^2\gamma + \overline{\gamma'} = 0. \tag{24}$$

in formulas (19)–(23).

Formula (15), applied to the closed contour L_R, gives (see [2], p.113):

$$\begin{aligned}[A\varphi(z) + Az\overline{\phi(z)} + \overline{\psi(z)} - \frac{B}{a_0}\overline{\phi(z)}\sin(\overline{a}_0 z + a_0\overline{z}) +\\ +BG'(z)\cos(\overline{a}_0 z + a_0\overline{z}) - Ba_0 G(z)\sin(\overline{a}_0 z + a_0\overline{z})]_{L_R} = i(X + iY),\end{aligned} \tag{25}$$

assuming this time the circuit around L_R is clockwise. Taking this fact into account, on the basis of formulas (19)–(22), (24) and (25) we obtain

$$-2\pi(Aa_0^2\gamma - \overline{\gamma'}) = X + iY. \tag{26}$$

From (24) and (26) follows

$$\gamma = -\frac{X + iY}{2\pi a_0^2}, \quad \gamma' = \frac{(1 - A)(X - iY)}{2\pi}. \tag{27}$$

Thus, by formulas (21), (22) and (27) we have

$$\begin{aligned}\varphi(z) &= -\frac{X + iY}{2\pi}\ln z + \varphi^*(z),\\ \psi(z) &= \frac{(1 - A)(X - iY)}{2\pi}\ln z + \psi^*(z).\end{aligned} \tag{28}$$

By the Laurent theorem, the functions $\varphi^*(z)$ and $\psi^*(z)$ may be represented outside L_R by the series

$$\varphi^*(z) = \sum_{-\infty}^{\infty} a_n z^n, \quad \text{and} \quad \psi^*(z) = \sum_{-\infty}^{\infty} a_n' z^n; \tag{29}$$

note that these series converge uniformly in every finite region lying outside L_R.

If we substitute in (12) and (13) the expressions (28) and replace $\varphi^*(z)$ and $\psi^*(z)$ by the series (29), we analogously as in the case of homogeneous elastic plane (see [2], p.123) under the condition of bounded stress components for $|z|$, tending to ∞, have

$$a_n = \overline{a}_n = a'_n = 0 \quad (n \geq 2). \tag{30}$$

Thus, if conditions (30) are fulfilled, then, obviously, X_x, Y_y, X_y are bounded, and for the functions $G(z)$ and $\Psi(z)$ the representation of the type (18) is valid, where γ, γ' are defined by formulas (27), whereas Γ and Γ', by formulas (12)–(14).

Indeed, it follows from these formulas that for $|z|$, tending to ∞, we have

$$\lim(X_x + Y_y) = 4a_0^2[A + B\lim\cos(\overline{a}_0 z + a_0\overline{z})]\operatorname{Re}\Gamma, \tag{31}$$

$$\lim(Y_y - X_x - 2iX_y) = 2\overline{\Gamma'} - 4Ba_0^2\operatorname{Re}\Gamma\lim\cos(\overline{a}_0 z + a_0\overline{z}), \tag{32}$$

$$\lim\varepsilon = a_0^2\operatorname{Im}\Gamma/2\Gamma. \tag{33}$$

If we assume that equations (31)–(33) have solutions (that is, if we want for X_x and Y_y to have definite limits), then it is necessary and sufficient that either $B = 0$ or $\Gamma = 0$.

In case $B = 0$ (homogeneous plane), solutions of equations (31)–(33) are of the form [2]:

$$\operatorname{Re}\Gamma = \frac{1}{4}(N_1 + N_2), \quad \operatorname{Im}\Gamma = \frac{2\Gamma\varepsilon_\infty}{1+\varkappa}, \quad \Gamma' = -\frac{1}{2}(N_1 - N_2)e^{-2i\beta},$$

where N_1 and N_2 are the values of principal stresses at infinity, β is the angle between the principal axis corresponding to N_1 and the ox-axis, and $\varepsilon_\infty = \lim\varepsilon$.

In case $\operatorname{Re}\Gamma = 0$ (non-homogeneous plane of type (1)) we have

$$\operatorname{Im}\Gamma = \frac{2\Gamma\varepsilon_\infty}{a_0^2}, \quad N_1 = -N_2, \quad \Gamma' = -N_1 e^{-2i\beta}. \quad \blacksquare \tag{34}$$

2. The First Fundamental Problem for a Plane with Linear Cuts. Let on L there be given Y_y^+, X_Y^+, Y_y^-, X_y^- where the signs $(+)$ and $(-)$ denote boundary values at the upper and lower cut ends, respectively.

Assume that the stress values $\operatorname{Re}\Gamma$ and Γ' are given at infinity. Since $\operatorname{Re}\Gamma = 0$, and $\operatorname{Im}\Gamma$ has no influence on the stress distribution, we suppose that $\Gamma = 0$.

On the basis of (16), the boundary conditions of the problem take the form:

$$\begin{aligned}(A+B)\phi^+(t) - Ba_0^2G^+(t) + A\overline{\phi^+(t)} + At\overline{\phi^{+\prime}(t)} + \overline{\Psi^+(t)} &= Y_y^+ - iX_y^+ \text{ on } L\\ (A+B)\phi^-(t) - Ba_0^2G^-(t) + A\overline{\phi^-(t)} + At\overline{\phi^{-\prime}(t)} + \overline{\Psi^-(t)} &= Y_y^- - iX_y^- \text{ on } L.\end{aligned} \tag{35}$$

Denote

$$\mathcal{K}(z) = (A+B)\phi(z) - Ba_0^2G(z), \tag{36}$$

$$\Omega(z) = A\overline{\phi}(z) + Az\overline{\phi}'(z) + \overline{\Psi}(z), \tag{37}$$

By $\overline{F}(z)$ we denote the function obtained from $F(z)$ as follows: $\overline{F}(z) = \overline{F(\overline{z})}$.

Owing to (36) and (37), the boundary conditions (35) can be written as

$$\begin{aligned} \mathcal{K}^+(t) + \Omega^-(t) &= Y_y^+ - iX_y^+ \quad \text{on} \quad L, \\ \mathcal{K}^-(t) + \Omega^+(t) &= Y_y^- - iX_y^- \quad \text{on} \quad L. \end{aligned} \tag{38}$$

The functions $\mathcal{K}z$, $\Omega(z)$ will be assumed to be piecewise holomorphic in S' (see [2], p.382), Definition in §106) which for large $|z|$ on the basis of formulas (18), (27), (36) and (37) take the form

$$\begin{aligned} \mathcal{K}(z) &= -\frac{A(X+iY)}{2\pi}\frac{1}{z} + O(\frac{1}{z^2}), \\ \Omega(z) &= \overline{\Gamma}' + \frac{(1-A)(X-iY)}{2\pi}\frac{1}{z} + O(\frac{1}{z^2}). \end{aligned} \tag{39}$$

It easily follows from (38) that

$$\begin{aligned} [\mathcal{K}(t) + \Omega(t)]^+ + [\mathcal{K}(t) + \Omega(t)]^- &= 2P(t) \quad \text{on} \quad L, \\ [\mathcal{K}(t) - \Omega(t)]^+ - [\mathcal{K}(t) - \Omega(t)]^- &= 2Q(t) \quad \text{on} \quad L, \end{aligned} \tag{40}$$

where $P(t)$, $Q(t)$ are the given on L functions:

$$\begin{aligned} P(t) &= \frac{1}{2}(Y_y^- + Y_y^-) - \frac{i}{2}(X_y^+ + X_y^-), \\ Q(t) &= \frac{1}{2}(Y_y^- - Y_y^-) - \frac{i}{2}(X_y^+ - X_y^-), \end{aligned}$$

satisfying the Hölder condition on L.

Since (39) imply $\mathcal{K}(\infty) - \Omega(\infty) = -\overline{\Gamma}'$, general solutions of the boundary value problem (40), bounded at infinity, are given by the following formulas (see [2], p.441):

$$\mathcal{K}(z) = \frac{1}{2\pi i X(z)} \int_L \frac{X(t)P(t)dt}{t-z} + \frac{1}{2\pi i} \int_L \frac{Q(t)dt}{t-z} + \frac{P_n}{X(z)} - \frac{\overline{\Gamma}'}{2}, \tag{41}$$

$$\Omega(z) = \frac{1}{2\pi i X(z)} \int_L \frac{X(t)P(t)dt}{t-z} - \frac{1}{2\pi i} \int_L \frac{Q(t)dt}{t-z} + \frac{P_n}{X(z)} + \frac{\overline{\Gamma}'}{2}, \tag{42}$$

where

$$\begin{aligned} X(z) &= \sum_{k=1}^{n} (z-a_k)^{\frac{1}{2}}(z-b_k)^{\frac{1}{2}}, \\ P_n(z) &= C_0 z^n + C_1 z^{n-1} + \cdots + C_n. \end{aligned} \tag{43}$$

Under $X(t)$ is meant a value taken by the function $X(z)$ on the left of L.

According to (10) and (36), for $G(z)$ we obtain the ordinary differential equation of the second order with the constant coefficients

$$(A+B)G''(z) + Aa_0^2 G(z) = \mathcal{K}(z). \tag{44}$$

Under the conditions $A+B=\frac{A_0+B_0+1}{4}>0$, $A=\frac{A_0+1}{4}>0$, $a_0^2=-\frac{\alpha^2}{4}$, the solution of equation (44) takes the form ([3], ch. 2)

$$G(z)=\frac{1}{A+B}\Bigg\{e^{\frac{\lambda}{2}z}\Bigg[\int\limits_{z_0}^{z}\mathcal{K}(t)e^{-\frac{\lambda}{2}t}dt+E_1(z_0)\Bigg]-$$
$$-\frac{e^{-\frac{\lambda}{2}z}}{\lambda}\Bigg[\int\limits_{z_0}^{z}\mathcal{K}(t)e^{\frac{\lambda}{2}t}dt+E_2(z_0)\Bigg]\Bigg\},$$

where $\lambda=|\alpha|\sqrt{\frac{A}{A+B}}$, $E_1'(z)=\mathcal{K}(z)e^{-\frac{\lambda}{2}z}$, $E_2'(z)=\mathcal{K}(z)e^{\frac{\lambda}{2}z}$ and the function $\mathcal{K}(z)$ is given by formula (41).

It remains to define the polynomial $P_n(z)$. For the sake of definiteness $X(z)$ will be assumed to be a branch which for large $|z|$ has the form

$$X(z)=+z^n+a_{n-1}z^{n-1}+\cdots.$$

Using formula (41) and condition C_0, we can directly define the coefficient $\mathcal{K}(\infty)$=0:

$$C_0=\frac{1}{2}\overline{\Gamma}'. \tag{45}$$

The other coefficients can de defined analogously, as for the homogeneous plane (see [2], p.442) from the conditions of single-valuedness of displacements.

Consider some particular cases.

a) Let the cut ends be free from stresses (problem of tension of a plate weakened by a crack). Assume $n=1$ (one cut), $a_1=-a$ and $b_1=a$. Then by (41)–(43) and (45) we find that $C_1=C_2=\cdots=C_n=0$, $C_0=\frac{1}{2}\overline{\Gamma}'$, $\mathcal{K}(z)=\frac{\overline{\Gamma}'z}{2\sqrt{z^2-a^2}}-\frac{1}{2}\overline{\Gamma}'$, $\Omega(z)=\frac{\overline{\Gamma}'z}{2\sqrt{z^2-a^2}}+\frac{1}{2}\overline{\Gamma}'$.

b) If equal and contrariwise directed stresses $y_y^+=y_y^-=p(t)$ are applied to the cut end, then for $n=1$, $a_1=-a$, $b_1=a$ we have $\Gamma=\overline{\Gamma}'=0$, $P(t)=p(t)$, $Q(t)=0$, $C_0=C_1=\cdots=C_n=0$ and

$$\mathcal{K}(z)=\Omega(z)=\frac{1}{2\pi i\sqrt{z^2-a^2}}\int\limits_{-a}^{a}\frac{\sqrt{t^2-a^2}p(t)dt}{t-z}.$$

This implies that for $p(t)=p_0=\text{const}$ [2],

$$\mathcal{K}(z)=\Omega(z)=\frac{p_0(z-\sqrt{z^2-a^2})}{2\sqrt{z^2-a^2}}.$$

3. The Second Fundamental Problem for a Plane with Linear Cuts. Let on L be given the displacements $u^{\pm}(t)$ and $v^{\pm}(t)$; moreover,

$$\begin{aligned}u^{\pm}(a_k)&=u(a_k), \quad v^{\pm}(a_k)=v(a_k),\\ u^{\pm}(b_k)&=u(b_k), \quad v^{\pm}(b_k)=v(b_k),\end{aligned} \tag{46}$$

where $u(a_k)$, $v(a_k)$, and $u(b_k)$, $v(b_k)$ denote displacements at the points a_k and b_k. The constants $\operatorname{Im}\Gamma$ and Γ', as well as the vector of external stresses applied to L are assumed to be given.

By (17), the boundary conditions can be written as follows (see [2], p. 443)

$$\begin{aligned} W^+(t) - \Omega^-(t) &= 2\Gamma(u^{+'}(t) + iv^{+'}(t)), \\ W^-(t) - \Omega^+(t) &= 2\Gamma(u^{-'}(t) + iv^{-'}(t)), \end{aligned} \tag{47}$$

where

$$W(z) = (1 - A - B)\phi(z) + Ba_0^2 G(z). \tag{48}$$

Whence

$$\begin{aligned} [W(t) - \Omega(t)]^+ + [W(t) - \Omega(t)]^- &= 2f(t) \quad \text{on} \quad L, \\ [W(t) - \Omega(t)]^+ - [W(t) + \Omega(t)]^- &= 2g(t) \quad \text{on} \quad L. \end{aligned} \tag{49}$$

Here $f(t)$ and $g(t)$ are the given on L functions satisfying the H conditions:

$$\begin{aligned} f(t) &= \mu\big[\big(u^{+'}(t) + u^{-'}(t)\big) + i(v^{+'}(t) + v^{-'}(t)\big)\big], \\ g(t) &= \mu\big[\big(u^{+'}(t) - u^{-'}(t)\big) + i(v^{+'}(t) - v^{-'}(t)\big)\big]. \end{aligned}$$

General solutions of the boundary value problem (49) are given by the following formulas [2]:

$$W(z) + \Omega(z) = \frac{1}{\pi i}\int_L \frac{g(t)dt}{t - z} + W(\infty) + \Omega(\infty), \tag{50}$$

$$W(z) - \Omega(z) = \frac{1}{\pi i X(z)}\int_L \frac{X(t)f(t)dt}{t - z} + \frac{2P_n(z)}{X(z)}, \tag{51}$$

where $X(z)$ is the function defined by formula (43), $P_n(z)$ is the polynomial of degree not higher than n, $X(t)$ is the value of the function $X(z)$ on the left of L, and

$$W(\infty) + \Omega(\infty) = (1 - 2A)a_0^2 \operatorname{Im}\Gamma + \overline{\Gamma}'.$$

The functions $W(z)$ and $\Omega(z)$ are defined by formulas (50) and (51), while functions $G(z)$ and $\phi(z)$ are defined by virtue of (10) and (48). Using formula (37), we can easily calculate the function Ψ.

The first two coefficients of the polynomial $P_n(z) = C_0 z^n + C_1 z^{n-1} + \cdots + C_n$ can be defined by formula (51), if we take into account the fact that owing to formulas (18), (27), (37) and (48), for large $|z|$ we must have

$$W(z) - \Omega(z) = (1 - A)a_0^2\Gamma - Aa_0^2\overline{\Gamma} - \overline{\Gamma}' - \frac{(1 - A)(X + iY)}{\pi}\frac{1}{z} + O(\frac{1}{z^2}).$$

To define the other coefficients of the polynomial $P_n(z)$ we easily obtain a system of $n-1$ linear equations of the type (see [2], p.444)

$$\int_{b_k}^{a_{k+1}} [W(t) - \Omega(t)]dt = 2\mu\{u(a_{k+1}) - u(b_k) + i[v(a_{k+1}) - v(b_k)]\},$$

where on the right-hand side there appear the same values as in formulas (46), and expression (51) in the left-hand side.

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