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RELATIONSHIP OF NON-ABELIAN TENSOR PRODUCTS AND NON-ABELIAN HOMOLOGY OF GROUPS WITH WHITEHEAD'S GAMMA FUNCTOR

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Abstract. Brown and Loday's exact sequence about the relationship of Whitehead's gamma functor with the non-abelian tensor product and the exterior product of groups is generalized for precrossed modules and is extended using their non-abelian left derived functors and non-abelian homology of groups.

Sufficient conditions of finite generation of the non-abelian tensor product of groups and sufficient conditions of finiteness of the non-abelian homology of groups with coefficients in groups are established.

Introduction

Let $\mu : M \to P$ and $\nu : N \to P$ be crossed modules over a group P and let M and N act on each other via P. Then R. Brown and J.-L. Loday get the following result [5]: there exists an exact sequence of groups

$$\Gamma(M \times_P N/ < M, N >) \xrightarrow{\Psi} M \otimes N \to M \wedge^P N \to 1,$$

where Γ is Whitehead's universal quadratic gamma functor [20] (see definition below), $M \times_P N = \{(m, n) \in M \times N \mid \mu(m) = \nu(n)\}$, $< M, N >= Im((\lambda, \lambda') : M \otimes N \to M \times_P N, (\lambda, \lambda')(m \otimes n) = (m \cdot^n m^{-1}, ^m nn^{-1}))$, $M \otimes N$ and $M \wedge^P N$ are non-abelian tensor and non-abelian exterior products of M and N respectively, $\Psi(\gamma(m, n)) = m \otimes n$. Also Ψ has central image.

The non-abelian tensor product of groups was introduced by R. Brown and J.-L. Loday [3-5] following works of A. Lue [15] and R.K. Dennis [6]. It arises in applications to homotopy theory of a generalized Van Kampen theorem. It was defined for any

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groups A and B which act on themselves by conjugation (${}^{x}y = xyx^{-1}$) and each of which acts on the other, but the theory got development when actions satisfy the following compatibility conditions:

$$ {}^{({}^{a}b)}a' = {}^{aba^{-1}}a', \qquad {}^{({}^{b}a)}b' = {}^{bab^{-1}}b' \tag{1} $$

for all $a, a' \in A$ and $b, b' \in B$.

Guin generalized the tensor product of Brown and Loday for $\mu : M \to P$, $\nu : N \to P$ precrossed modules [8] in the sense that the groups A and B acting on each other compatibly form in a natural way a pair of crossed modules for which Guin's tensor product coincides with Brown and Loday's tensor product. We always denote Guin's non-abelian tensor product of precrossed modules by $M \overset{G}{\otimes} N$ (see definition below).

Using the non-abelian tensor product Guin [8] defined non-abelian homology of a group G with coefficients in G-crossed module A in low dimensions $H_0(G, A)$, $H_1(G, A)$ and which have important applications to K-theory of non-commutative local rings.

The definition of non-abelian homology of a group G with coefficients in any group A in any dimensions $H_n(G, A)$, $n \geq 0$ necessitated another generalization of the non-abelian tensor product of Brown and Loday (see [11]).

Let A and B be arbitrary groups which act on themselves by conjugation and each of which acts on the other. The non-abelian tensor product $A \otimes B$ of the groups A and B [11] is the group generated by the symbols $a \otimes b$, ($a \in A$, $b \in B$) subject to the relations

$$ \begin{aligned} aa' \otimes b &= ({}^{a}a' \otimes {}^{a}b)(a \otimes b) \\ a \otimes bb' &= (a \otimes b)({}^{b}a \otimes {}^{b}b') \\ (a \otimes b)(a' \otimes b') &= ({}^{[a,b]}a' \otimes {}^{[a,b]}b')(a \otimes b) \\ (a' \otimes b')(a \otimes b) &= (a \otimes b)({}^{[b,a]}a' \otimes {}^{[b,a]}b') \end{aligned} $$

for all $a, a' \in A$ and $b, b' \in B$,where $[a, b] = aba^{-1}b^{-1} \in A * B$. Note that when the actions are compatible then this tensor product coincides the Brown-loday tensor product.

As the non-abelian left derived functors of the non-abelian tensor product of groups a non-abelian homology of groups is constructed with coefficients in any group [11] which generalizes the classical Eilenberg- MacLane homology theory of groups [1] and extends Guin's non-abelian homology [8]. For non-commutative local rings the relationship of Milnor's algebraic K-functor K_2 with the symbol group Sym is given in terms of a long exact sequence of non-abelian homology of groups [11] extending Guin's six terms exact sequence [8]. In fact, let R be a non-commutative local ring such that $R/\operatorname{Rad} R \neq \mathbb{F}_2$, then one has the following exact sequence of groups

$$ \begin{aligned} \cdots \longrightarrow H_3(R^*, [R^*, R^*]) \longrightarrow H_2(R^*, U(R), [R^*, R^*]) \longrightarrow H_2(R, U(R)) \longrightarrow \\ \longrightarrow H_2(R^*, [R^*, R^*]) \longrightarrow H_1(R^*, U(R), [R^*, R^*]) \longrightarrow H_1(R, U(R)) \longrightarrow \\ \longrightarrow H_1(R^*, [R^*, R^*]) \longrightarrow K_2(R) \longrightarrow \operatorname{Sym}(R) \longrightarrow [R^*, R^*]/[R^*, [R^*, R^*]] \longrightarrow 1. \end{aligned} $$

In Section 1 we continue to study the abovementioned generalized non-abelian tensor product of groups. It is shown the natural relation of the non-abelian tensor product of groups with any actions with the non-abelian tensor product of groups with compatible actions.

Sufficient conditions are given for the non-abelian tensor product of groups to be finitely generated. As the consequence of this fact we get that $GL_n(\mathbb{Z}) \otimes GL_n(\mathbb{Z})$, $E_n(\mathbb{Z}) \otimes E_n(\mathbb{Z})$ and $St_n(\mathbb{Z}) \otimes St_n(\mathbb{Z})$, $n \geq 5$ are finitely generated groups, where $GL_n(\mathbb{Z})$, $E_n(\mathbb{Z})$ and $St_n(\mathbb{Z})$ are the linear group of $n \times n$-matrices over $\mathbb{Z}$, the elementary group of $n \times n$-matrices over $\mathbb{Z}$ and the non-stable Steinberg group of $\mathbb{Z}$ respectively.

It is proved that the non-abelian tensor product of groups $M \otimes N$, of precrossed modules $\mu : M \to P$, $\nu : N \to P$ over the same group P is also a precrossed module over P.

We find nontrivial generalizations (Theorem 10 and Theorem 14) of the abovementioned theorem Brown and Loday about relationship of Whitehead's gamma functor with the non-abelian tensor product and the exterior product of groups for precrossed modules. A long exact sequence of their left derived functors and non-abelian homology of groups extends the natural epimorphism $M \otimes N \to M \wedge^P N$ and this is another application of non-abelian homology of groups.

Guin's homology with coefficients in precrossed modules ${}^G H_n(M, N)$, $n \geq 0$ is constructed using the non-abelian left derived functors of Guin's tensor product. It is easy to see that when (M, μ) and (N, ν) are crossed modules then ${}^G H_0(M, N) = H_0(M, N)$ and ${}^G H_1(M, N) = H_1(M, N)$.

Using the left derived functors of Γ, $- \otimes N$ and $\wedge^P N$ we obtain long exact sequences extending Brown and Loday's abovementioned three terms right exact sequence about relationship of Whitehead's gamma functor with non-abelian tensor product and non-abelian exterior product of crossed modules and making this relationship more precise.

Section 2 is concerned with finiteness of the non-abelian homology of groups. Using the loop functor $\mathbb{G}$ from the category of reduced complex to the category of simplicial groups [13] we construct a new simplicial projective resolution of a group G which give the possibility to prove that if G and A are finite groups, A acts trivially on G and G acts on A such that $G \otimes A$ is a finite group (see sufficient conditions in [12]) then $H_n(G, A)$, $n \geq 0$ is also a finite group.

1. Non-abelian Tensor Products of Precrossed Modules

First we recall some definitions from [12]

Definition 1. Let A and B be groups, which act on each other. The Comp-subgroup of A with B, denoted by $CompA(B)$, is the normal subgroup of A generated by the elements

$$ {}^{(a}b)a' \left({}^{aba^{-1}}a'\right)^{-1} $$

where $a, a' \in A$, $b, b' \in B$.

Definition 2. It will be said that A acts on B perfectly if the action of A on B induces an action of A on $\operatorname{Comp} B(A)$, and that A, B are groups with perfect actions if under these actions A acts on B perfectly and B acts on A perfectly.

Proposition 3. *Let A and B be groups. If they act on each other perfectly and the following conditions hold:*

$$^{a'}bb^{-1} \in \operatorname{Comp} B(A), \quad ^{b'}aa^{-1} \in \operatorname{Comp} A(B)$$

for each $a \in A$, $b \in B$, $a' \in \operatorname{Comp} A(B)$, $b' \in \operatorname{Comp} B(A)$, then there exists an isomorphism

$$\overline{A} \otimes \overline{B} \approx (A \otimes B)/H,$$

where H is a normal subgroup of $A \otimes B$ generated by $a' \otimes b'$, $a' \in \operatorname{Comp} A(B)$ or $b' \in \operatorname{Comp} B(A)$, and the groups $\overline{A} = A/\operatorname{Comp} A(B)$, $\overline{B} = B/\operatorname{Comp} B(A)$ act on each other compatibly by the induced actions.

Proof. It is easy to see that the actions of the groups A and B induce compatible actions of the groups $\overline{A} = A/\operatorname{Comp} A(B)$ and $\overline{B} = B/\operatorname{Comp} B(A)$.

We have a natural homomorphism $f' : A \otimes B \longrightarrow \overline{A} \otimes \overline{B}$. It is clear that $f'(H) = 1$. This induces a homomorphism

$$f : (A \otimes B)/H \longrightarrow \overline{A} \otimes \overline{B}.$$

Define a map $g' : \overline{A} \times \overline{B} \longrightarrow (A \otimes B)/H$ by the formula $g'([a], [b]) = [a \otimes b]$. Since

$$aa' \otimes bb' = ({}^a a' \otimes {}^a b \cdot {}^a b')(a \otimes bb') = ({}^a a' \otimes {}^a b)({}^{({}^a b)}({}^a a') \otimes {}^{({}^a b)}({}^a b')) \cdot \\ \cdot (a \otimes b)({}^b a \otimes {}^b b'),$$

for every $a \in A$, $b \in B$, $a' \in \operatorname{Comp} A(B)$, $b' \in \operatorname{Comp} B(A)$, the map g' is defined correctly. It is easy to see that g' satisfies the conditions of crossed pairing. Therefore g' induces a homomorphism

$$g : \overline{A} \otimes \overline{B} \longrightarrow (A \otimes B)/H, \\ g([a] \otimes [b]) = [a \otimes b].$$

It is clear that fg and gf are identity homomorphisms. The proof is complete. ■

Theorem 4. *Let M and N be normal subgroups of some group G acting on each other by conjugation in G. If the groups $[M, N]$, $M \cap N$, $H_2(MN)$, $H_3(MN/M)$, $H_3(MN/N)$ are finitely generated, then $M \otimes N$ is a finitely generated group.*

Proof. From [5, Theorem 4.5] we have an exact sequence

$$\longrightarrow H_3(MN/M) \oplus H_3(MN/N) \longrightarrow \operatorname{Ker}([,] : M \wedge N \to [M, N]) \longrightarrow \\ H_2(MN) \longrightarrow \cdots, \tag{2}$$

where $M \wedge N$ is the quotient of $M \otimes N$ by the subgroup generated by the elements $x \otimes x$ with $x \in M \cap N$ [5] and the homomorphism $[,] : M \wedge N \to [M, N]$ is defined by $m \otimes n \mapsto [m, n]$.

Under the conditions of the theorem the exact sequence (2) implies that $\mathrm{Ker}([,] : M \wedge N \to [M,N])$ is finitely generated and hence $M \wedge N$ is a finitely generated group.

From [5, Theorem 2.12] we have the exact sequence

$$\Gamma((M \cap N)/[M,N]) \longrightarrow M \otimes N \longrightarrow M \wedge N \longrightarrow 1. \tag{3}$$

Now we have to show that if A is a finitely generated abelian group, then the group $\Gamma(A)$ is finitely generated. Let consider the natural epimorphism $F \to A$, where F is the free abelian group generated by the generators of A. From [20] the induced homomorphism $\gamma(F) \to \gamma(A)$ is surjective and $\gamma(F)$ is a finitely generated free abelian group, therefore $\gamma(A)$ is a finitely generated abelian group.

Since $(M \cap N)/[M,N]$ is a finitely generated abelian group, this implies that $\Gamma((M \cap N)/[M,N])$ is also finitely generated. From (3) one deduces that $M \otimes N$ is a finitely generated group. The proof is complete. ■

Corollary 5. *Let G be a finitely generated group such that its abelization is finite and $H_2(G)$ is finitely generated then the non-abelian tensor square $G \otimes G$ is finitely generated.*

Note that the linear group $GL_n(\mathbb{Z})$, the elementary group $E_n(\mathbb{Z})$ and the Steinberg group $St(\mathbb{Z})$, $n \geq 5$, where $\mathbb{Z}$ is the ring of integers, satisfy conditions of Corollary 5. For instance it will be shown for the elementary group.

It is known that the group $E_n(\mathbb{Z})$ is perfect and finitely generated for $n \geq 3$ [2,10]. The non-stable Steinberg group $St_n(\mathbb{Z})$ of the ring $\mathbb{Z}$ is the universal central extension of $E_n(\mathbb{Z})$ for $n \geq 5$ [16] and we have the exact sequence

$$0 \longrightarrow K_2(n,\mathbb{Z}) \longrightarrow St_n(\mathbb{Z}) \longrightarrow E_n(\mathbb{Z}) \longrightarrow 1,$$

where $K_2(n,\mathbb{Z})$ is the non-stable Milnor functor of the ring $\mathbb{Z}$. Therefore $H_2(E_n(\mathbb{Z})) = K_2(n,\mathbb{Z})$, $n \geq 5$. By the stability theorem we have the isomorphism $K_2(n,\mathbb{Z}) \xrightarrow{\approx} K_2(\mathbb{Z})$ for $n \geq 3$ [19]. By Milnor's theorem [16] the group $K_2(\mathbb{Z})$ is cyclic of order 2.

A precrossed module (M,μ) over P is a group homomorphism $\mu : M \to P$ together with an action of P on M, which satisfies the following condition

$$\mu({}^p m) = p\mu(m)p^{-1}, \quad p \in P, \quad m \in M.$$

If in addition

$${}^{\mu(m)}m' = mm'm^{-1}, \quad m, m' \in M,$$

then (M,μ) is a crossed module over P.

Proposition 6. *Let $\mu\colon M \to P$, $\nu\colon N \to P$ be precrossed modules over the group P. Let M and N act on each other via P and themselves by conjugation. Then*

(i) *The group P acts on the group $M \otimes N$ as follows ${}^p(m \otimes n) = {}^p m \otimes {}^p n$,*
(ii) *$\lambda : M \otimes N \to P$ defined by the formula $\lambda(m \otimes n) = [\mu(m), \nu(n)]$ with the actions of P on $M \otimes N$ as in (i) is a precrossed module over P.*

Proof. (i) We only have to show that every element $p \in P$ induces a homomorphism

$$\widetilde{p} : M \otimes N \longrightarrow M \otimes N,$$
$$\widetilde{p}(m \otimes n) = {}^{p}(m \otimes n) = {}^{p}m \otimes {}^{p}n.$$

It follows from the equalities

$$ {}^{p}m \cdot {}^{p}m' \otimes {}^{p}n = ({}^{({}^{p}m)}({}^{p}m') \otimes {}^{\mu({}^{p}m)}({}^{p}n))({}^{p}m \otimes {}^{p}n) = ({}^{pmp^{-1}}{}^{p}m' \otimes {}^{pmp^{-1}}{}^{p}n) \cdot $$
$$ ({}^{p}m \otimes {}^{p}n) = ({}^{pm}m' \otimes {}^{pm}n)({}^{p}m \otimes {}^{p}n),$$
$$ {}^{p}m \otimes {}^{p}n \cdot {}^{p}n' = ({}^{p}m \otimes {}^{p}n)({}^{\nu({}^{p}n)}({}^{p}m) \otimes {}^{({}^{p}n)}({}^{p}n')) = ({}^{p}m \otimes {}^{p}n)({}^{pnp^{-1}}{}^{p}m \otimes {}^{pnp^{-1}}{}^{p}n') $$
$$ = ({}^{p}m \otimes {}^{p}n)({}^{pn}m \otimes {}^{pn}n'),$$
$$ {}^{p}({}^{[m,n]}m') \otimes {}^{p}({}^{[m,n]}n') = {}^{[{}^{p}m,{}^{p}n]}({}^{p}m') \otimes {}^{[{}^{p}m,{}^{p}n]}({}^{p}n') = ({}^{p}m \otimes {}^{p}n) \cdot $$
$$ ({}^{p}m' \otimes {}^{p}n')({}^{p}m \otimes {}^{p}n)^{-1},$$
$$ {}^{p}({}^{[n,m]}m') \otimes {}^{p}({}^{[n,m]}n') = {}^{[{}^{p}n,{}^{p}m]}({}^{p}m') \otimes {}^{[{}^{p}n,{}^{p}m]}({}^{p}n') = ({}^{p}m \otimes {}^{p}n)^{-1} \cdot $$
$$ ({}^{p}m' \otimes {}^{p}n')({}^{p}m \otimes {}^{p}n).$$

(ii) Let define a map $\widetilde{\lambda} : M \times N \to P$, $\widetilde{\lambda}(m,n) = [\mu(m), \nu(n)]$.

We have

$$\widetilde{\lambda}({}^{m}m', {}^{m}n)\widetilde{\lambda}(m,n) = [\mu({}^{m}m'), \nu({}^{m}n)] \cdot [\mu(m), \nu(n)] = \mu(m)\mu(m')\mu(m)^{-1}\mu(m) \cdot$$
$$\nu(n)\mu(m)^{-1} \cdot \mu(m)\mu(m')^{-1}\mu(m)^{-1}\mu(m)\nu(n)^{-1}\mu(m)^{-1} \cdot \mu(m)\nu(n)\mu(m)^{-1}\nu(n)^{-1} =$$
$$\mu(m)\mu(m')\nu(n)\mu(m')^{-1}\mu(m)^{-1}\nu(n)^{-1} = \widetilde{\lambda}(mm', n).$$

The prove of the rest three conditions of the crossed pairing (for the definition see [11]) is similar. Therefore $\widetilde{\lambda}$ is a crossed pairing and induces a homomorphism $\lambda : M \otimes N \to P$ defined by the formula $\lambda(m \otimes n) = [\mu(m), \nu(n)]$.

Show that λ is precrossed module over P with the action of P on $M \otimes N$ defined in (i). In effect

$$\lambda({}^{p}(m \otimes n)) = [\mu({}^{p}m), \nu({}^{p}n)] = [p\mu(m)p^{-1}, p\nu(n)p^{-1}] = p[\mu(m), \nu(n)]p^{-1} =$$
$$p\lambda(m \otimes n)p^{-1}.$$

The proof is complete. ■

Now we recall Guin's definition of the non-abelian tensor product of precrossed modules. Let $\mu : M \to P$ and $\nu : N \to P$ be precrossed modules over the group P. Let M and N act on each other and themselves via the group P i.e.

$$ {}^{m}m' = {}^{\mu(m)}m', \qquad {}^{m}n = {}^{\mu(m)}n,$$
$$ {}^{n}m = {}^{\nu(n)}m, \qquad {}^{n}n' = {}^{\nu(n)}n',$$

for any $m, m' \in M$, $n, n' \in N$.

Definition 7 ([8]). Guin's non-abelian tensor product of precrossed modules (M, μ) and (N, ν) over P is the group $M \overset{G}{\otimes} N$ generated by the symbols $m \otimes n$ and defined by the relations

$$mm' \otimes n = ({}^{m}m' \otimes {}^{m}n)(m \otimes n),$$
$$m \otimes nn' = (m \otimes n)({}^{n}m \otimes {}^{n}n'),$$

for every $m, m' \in M$ and $n, n' \in N$.

Let $\mu : M \to P$ and $\nu : N \to P$ be precrossed modules over P. The associated crossed modules are

$$\mu' : M/M' \longrightarrow P, \qquad \nu' : N/N' \longrightarrow P,$$

where M' is a normal subgroup of M generated by the elements ${}^{\mu(m)}m'mm'^{-1}m^{-1}$, for $m, m' \in M$, N' is a normal subgroup of N generated by the elements ${}^{\nu(n)}n'nn'^{-1}n^{-1}$, for $n, n' \in N$, μ' and ν' are induced by μ and ν respectively.

One has the canonical epimorphism

$$M \overset{G}{\otimes} N \xrightarrow{f} M/M' \otimes N/N'.$$

It is easy to verify that $\operatorname{Ker} f$ is a normal subgroup generated by the elements $m \otimes n$, where $m \in M'$ or $n \in N'$.

Let consider the fibre product $M \times_p N = \{(m, n) \in M \times N : \mu(m) = \nu(n)\}$.

Definition 8. The (non-abelian) exterior product $M \wedge^P N$ is obtained from the tensor product $M \overset{G}{\otimes} N$ by imposing the additional relations $m \otimes n = 1$ for all $(m, n) \in M \times_p N$.

Now we recall the definition of Whitehead's Γ-functor [20].

Definition 9. Let A be an abelian group. Then $\Gamma(A)$ is the group with generators $\gamma(a)$, $a \in A$ and the following relations:

$$\gamma(a) = \gamma(a^{-1}),$$
$$\gamma(abc)\gamma(a)\gamma(b)\gamma(c) = \gamma(ab)\gamma(bc)\gamma(ac),$$

for every $a, b, c \in A$.

Theorem 10. *Let (M, μ) and (N, ν) be precrossed modules over the group P such that $\mu M \cap \nu N$ acts trivially on X, where X is the subgroup of $M \overset{G}{\otimes} N$ generated by the elements $m \otimes n$ such that $\mu(m) = \nu(n)$. Then there is an exact sequence of groups*

$$\Gamma((M \times_p N)^{ab}) \xrightarrow{\Psi} M \overset{G}{\otimes} N \to M \wedge^p N \to 1,$$

where $\Psi(\gamma(m, n)) = m \otimes n$. Also Ψ has central image.

Proof. Let $\widetilde{\Psi} : M \times_p N \to M \overset{G}{\otimes} N$ be the map $(m, n) \mapsto m \otimes n$; Therefore $X = \operatorname{Ker}(M \overset{G}{\otimes} N \to M \wedge^p N)$ is the subgroup of $M \overset{G}{\otimes} N$ generated by $\operatorname{Im} \widetilde{\Psi}$.

Let consider a homomorphism $\lambda : M \overset{G}{\otimes} N \to P$, $\lambda(m \otimes n) = [\mu(m), \nu(n)]$. From [8, Proposition 2.4] it is known that $(M \overset{G}{\otimes} N, \lambda)$ is crossed module over P.

We have to show that X is central. Since $\lambda(m \otimes n) = [\mu(m), \nu(n)] = 1$ for $(m, n) \in M \times_p N$, one has $X \subseteq \operatorname{Ker} \lambda$. But $\operatorname{Ker} \lambda$ is central in $M \overset{G}{\otimes} N$ and therefore X is also central.

Let $(m,n),(u,v) \in M \times_p N$. Since $\mu(m) = \nu(n)$, $m \otimes n$, $u \otimes v \in Z(M \overset{G}{\otimes} N)$ and $\mu M \cap \nu N$ acts trivially on X, one has

$$mu \otimes nv = ({}^m u \otimes {}^m n {}^m v)(m \otimes nv) = ({}^m u \otimes {}^m n)({}^{({}^m n)}({}^m u) \otimes {}^{({}^m n)}({}^m v)) \cdot$$
$$(m \otimes n)({}^n m \otimes {}^v n) = {}^m\{(u \otimes n)(m \otimes v)\}(m \otimes n)(u \otimes v).$$

Hence $(u \otimes n)(m \otimes v) \in X$ and

$$mu \otimes nv = (u \otimes n)(m \otimes v)(m \otimes n)(u \otimes v), \tag{4}$$

Let $(m,n),(m_1,n_1),(m_2,n_2) \in M \times_p N$. Then by (4) one has
$m[m_1,m_2] \otimes n[n_1,n_2] = (m_1 m_2 m_1^{-1} m_2^{-1} \otimes n)(m \otimes n_1 n_2 n_1^{-1} n_2^{-1})(m \otimes n) \cdot$
$(m_1 m_2 m_1^{-1} m_2^{-1} \otimes n_1 n_2 n_1^{-1} n_2^{-1}) = {}^{m_1 m_2 m_1^{-1}}(m_2^{-1} \otimes n) \cdot {}^{m_1 m_2}(m_1^{-1} \otimes n) \cdot$
${}^{m_1}(m_2 \otimes n)(m_1 \otimes n)(m \otimes n_1) \cdot {}^{n_1}(m \otimes n_2) \cdot {}^{n_1 n_2}(m \otimes n_1^{-1}) \cdot {}^{n_1 n_2 n_1^{-1}}(m \otimes n_2^{-1})$
$(m \otimes n)(m_2 m_1^{-1} m_2^{-1} \otimes n_1)(m_1 \otimes n_2 n_1^{-1} n_2^{-1})(m_1 \otimes n_1)$
$(m_2 m_1^{-1} m_2^{-1} \otimes n_2 n_1^{-1} n_2^{-1}) = \{(m_1 \otimes n)(m \otimes n_1)\} \cdot {}^{m_1}\{(m_2 \otimes n)(m \otimes n_2)\} \cdot$
${}^{m_1 m_2}\left\{(m_1^{-1} \otimes n)(m \otimes n_1^{-1})\right\} \cdot {}^{m_1 m_2 m_1^{-1}}\left\{(m_2^{-1} \otimes n)(m \otimes n_2^{-1})\right\}(m \otimes n)$
$\{(m_2 \otimes n_1)(m_1 \otimes n_2)\} \cdot {}^{m_2}\left\{(m_1^{-1} \otimes n_1)(m_1 \otimes n_1^{-1})\right\} \cdot$
${}^{m_2 m_1^{-1}}\left\{(m_2^{-1} \otimes n_1) \cdot (m_1 \otimes n_2^{-1})\right\}(m_1 \otimes n_1)(m_1^{-1} m_2^{-1} \otimes n_2)(m_2 \otimes n_1^{-1} n_2^{-1}) \cdot$
$(m_2 \otimes n_2)(m_1^{-1} m_2^{-1} \otimes n_1^{-1} n_2^{-1}) = \{(m_1 \otimes n)(m \otimes n_1)\}\left\{(m_1^{-1} \otimes n)(m \otimes n_1^{-1})\right\} \cdot$
$\{(m_2 \otimes n)(m \otimes n_2)\}\left\{(m_2^{-1} \otimes n)(m \otimes n_2^{-1})\right\}(m \otimes n)\{(m_2 \otimes n_1)(m_1 \otimes n_2)\} \cdot$
$\left\{(m_2^{-1} \otimes n_1) \cdot (m_1 \otimes n_2^{-1})\right\}\left\{(m_1^{-1} \otimes n_1)(m_1 \otimes n_1^{-1})\right\}(m_1 \otimes n_1) \cdot$
$\left\{(m_1^{-1} \otimes n_2)(m_2 \otimes n_1^{-1})\right\} \cdot {}^{m_1^{-1}}\left\{(m_2^{-1} \otimes n_2)(m_2 \otimes n_2^{-1})\right\}(m_2 \otimes n_2) \cdot$
$\left\{(m_2^{-1} \otimes n_1^{-1})(m_1^{-1} \otimes n_2^{-1})\right\}(m_1^{-1} \otimes n_1^{-1})(m_2^{-1} \otimes n_2^{-1}) = (m \otimes n)(m_1 \otimes n_1)^{-1} \cdot$
$(m_1 \otimes n_2)^{-1} \cdot (m_2 \otimes n_1)^{-1}(m_2 \otimes n_2)^{-1}(m_2 \otimes n_2)^{-1} \cdot (m_2 \otimes n_2)(m_2 \otimes n_1) \cdot$
$(m_1 \otimes n_2)(m_1 \otimes n_1)(m_2 \otimes n_2) = (m \otimes n)$.
This proves that $\widetilde{\Psi}$ induces a map on $(M \times_p N)^{ab}$.

We now start the proof that $\widetilde{\Psi}$ induces a homomorphism on $\Gamma((M \times_p N)^{ab})$. Let $(m,n) \in M \times_p N$, then

$$m^{-1} \otimes n^{-1} = ({}^{m^{-1}} m \otimes^{m^{-1}} (n^{-1}))^{-1} = {}^{m^{-1}}({}^{n^{-1}} m \otimes^{n^{-1}} n) = {}^{m^{-1} n^{-1}}(m \otimes n) = m \otimes n.$$

Therefore $\widetilde{\Psi}[(m,n)] = \widetilde{\Psi}[(m^{-1},n^{-1})]$ for $(m,n) \in M \times_p N$.
Let $(m,n),(m_1,n_1),(m_2,n_2) \in M \times_p N$. Then by (4)
$mm_1m_2 \otimes nn_1n_2 = (m_1m_2 \otimes n)(m \otimes n_1n_2)(m \otimes n)(m_1m_2 \otimes n_1n_2) =$
${}^{m_1}(m_2 \otimes n)(m_1 \otimes n)(m \otimes n_1) \cdot {}^{n_1}(m \otimes n_2)(m \otimes n)(m_1m_2 \otimes n_1n_2) =$
$\{(m_1 \otimes n)(m \otimes n_1)\}\{(m_2 \otimes n)(m \otimes n_2)\}(m \otimes n)(m_1m_2 \otimes n_1n_2),$
$mm_1 \otimes nn_1 = \{(m_1 \otimes n)(m \otimes n_1)\}(m \otimes n)(m_1 \otimes n_1),$
$mm_2 \otimes nn_2 = \{(m_2 \otimes n)(m \otimes n_2)\}(m \otimes n)(m_2 \otimes n_2).$
Hence

$$(mm_1m_2 \otimes nn_1n_2)(m \otimes n)(m_1 \otimes n_1)(m_2 \otimes n_2) = (mm_1 \otimes nn_1) \cdot$$
$$(m_1m_2 \otimes n_1n_2)(mm_2 \otimes nn_2),$$

i.e.

$$\widetilde{\Psi}[(mm_1m_2, nn_1n_2)]\widetilde{\Psi}[(m,n)]\widetilde{\Psi}[(m_1,n_1)]\widetilde{\Psi}[(m_2,n_2)] = \widetilde{\Psi}[(mm_1, nn_1)]\cdot \widetilde{\Psi}[(m_1m_2, n_1n_2)]\widetilde{\Psi}[(mm_2, nn_2)].$$

Thus $\widetilde{\Psi}$ induces a homomorphism $\Psi : \Gamma((M \times_p N)^{ab}) \to M \overset{G}{\otimes} N$ given by $\Psi(\gamma(m,n)) = m \otimes n$. The proof is complete. ■

Corollary 11. *Let (M,μ) and (N,ν) be precrossed P-modules and let $[\mu M, \nu N] = \mu M \cap \nu N$. Then there is an exact sequence of groups*

$$\Gamma((M \times_p N)^{ab}) \xrightarrow{\Psi} M \overset{G}{\otimes} N \to M \wedge^p N \to 1,$$

where $\Psi(\gamma(m,n)) = m \otimes n$. Also Ψ has central image.

Proof. We have only to show that $\mu M \cap \nu N$ acts trivially on $X = Ker(M \overset{G}{\otimes} N \to M \wedge^p N)$.

Let $p \in \mu M \cap \nu N$ and $l \in \operatorname{Ker}\lambda$, where $\lambda : M \overset{G}{\otimes} N \to P$, $\lambda(m \otimes n) = [\mu(m), \nu(n)]$. Then there exists $l' \in M \overset{G}{\otimes} N$ such that $\lambda(l') = p$. Since [8] $(M \overset{G}{\otimes} N, \lambda)$ is a crossed module over P and therefore l is in the center of $M \otimes N$, one gets

$${}^p l = {}^{\lambda(l')} l = l' l l'^{-1} = l.$$

Hence $\mu M \cap \nu N$ acts trivially on $Ker\lambda$. Since $X \subseteq \operatorname{Ker}\lambda$, the group $\mu M \cap \nu N$ acts trivially on X. Then from Theorem 10 follows the assertion. The proof is complete. ■

We have a non-trivial

Example 12. Let P be a perfect group i.e. $P = [P,P]$, A be any group and P act on A non trivially. Let $(M,\mu) = (N,\nu) = (A \bowtie P, f)$, where $A \bowtie P$ is a semi-direct product of A and P , f is the natural surjective homomorphism of $A \bowtie P$ on P. The action of P on $A \bowtie P$ is given by the formula: ${}^{p'}(a,p) = ({}^{p'}a, {}^{p'}p)$. It is easy to verify that $(M,\mu) = (N,\nu)$ is a precrossed module, not crossed module and $[\mu M, \nu N] = \mu M \cap \nu N$.

A similar result holds for our non-abelian tensor product of precrossed modules extending Brown and Loday's Theorem 2.12 [5] for crossed modules.

Lemma 13. *Let A and B be groups which act on each other and on themselves by conjugation.*

(i) *If $a \in A$ such that ${}^{(a\bar{b})}\bar{a} = {}^{a\bar{b}a^{-1}}\bar{a}$, for every $\bar{a} \in A$, $\bar{b} \in B$, then a induces the homomorphism*

$$\widetilde{a} : A \otimes B \longrightarrow A \otimes B,$$
$$\widetilde{a}(a' \otimes b') = {}^a(a' \otimes b') = {}^a a' \otimes {}^a b'.$$

(ii) *If $b \in B$ such that ${}^{(b\bar{a})}\bar{b} = {}^{b\bar{a}b^{-1}}\bar{b}$, for every $\bar{a} \in A$, $\bar{b} \in B$, then b induces the homomorphism*

$$\widetilde{b} : A \otimes B \longrightarrow A \otimes B,$$
$$\widetilde{b}(a' \otimes b') = {}^{b}(a' \otimes b') = {}^{b}a' \otimes {}^{b}b'.$$

(iii) *If $a, a' \in A$ such that ${}^{(a\bar{b})}\bar{a} = {}^{a\bar{b}a^{-1}}\bar{a}$, ${}^{(a'\bar{b})}\bar{a} = {}^{a'\bar{b}a'^{-1}}\bar{a}$, for every $\bar{a} \in A$, $\bar{b} \in B$ and if $b, b' \in B$ such that ${}^{(b\bar{a})}\bar{b} = {}^{b\bar{a}b^{-1}}\bar{b}$, ${}^{(b'\bar{a})}\bar{b} = {}^{b'\bar{a}b'^{-1}}\bar{b}$, for every $\bar{a} \in A$, $\bar{b} \in B$, then*

$$g \cdot {}^{b}a^{-1} \otimes b' = (a \otimes b) \cdot {}^{b'}(a \otimes b)^{-1},$$
$$a' \otimes {}^{a}bb^{-1} = {}^{a'}(a \otimes b)(a \otimes b)^{-1}.$$

Proof. **(i)** Follows from the equalities

$$\begin{aligned}
&{}^{a}(a'a'' \otimes b') = {}^{a}a' \cdot {}^{a}a'' \otimes {}^{a}b' = ({}^{({}^{a}a')a}a'' \otimes {}^{({}^{a}a')a}b')({}^{a}a' \otimes {}^{a}b') = \\
&= ({}^{aa'}a'' \otimes {}^{aa'}b')({}^{a}a' \otimes {}^{a}b') = {}^{a}({}^{a'}a'' \otimes {}^{a'}b') \cdot {}^{a}(a' \otimes b'), \\
&{}^{a}(a' \otimes b'b'') = {}^{a}a' \otimes {}^{a}b' \cdot {}^{a}b'' = ({}^{a}a' \otimes {}^{a}b')({}^{({}^{a}b')a}a' \otimes {}^{({}^{a}b')a}b'') = \\
&= ({}^{a}a' \otimes {}^{a}b')({}^{ab'}a' \otimes {}^{ab'}b'') = {}^{a}(a' \otimes b') \cdot {}^{a}({}^{b'}a' \otimes {}^{b'}b''), \\
&{}^{a}(a' \otimes b') \cdot {}^{a}(a'' \otimes b'') \cdot {}^{a}(a' \otimes b')^{-1} = ({}^{a}a' \otimes {}^{a}b')({}^{a}a'' \otimes {}^{a}b'')({}^{a}a' \otimes {}^{a}b')^{-1} = \\
&= {}^{[{}^{a}a', {}^{a}b']a}a'' \otimes {}^{[{}^{a}a', {}^{a}b']a}b'' = {}^{a[a',b']}a'' \otimes {}^{a[a',b']}b'' = {}^{a}({}^{[a',b']}a'' \otimes {}^{[a',b']}b''), \\
&{}^{a}(a' \otimes b')^{-1} \cdot {}^{a}(a'' \otimes b'') \cdot {}^{a}(a' \otimes b') = ({}^{a}a' \otimes {}^{a}b')^{-1}({}^{a}a'' \otimes {}^{a}b'')({}^{a}a' \otimes {}^{a}b') = \\
&= {}^{[{}^{a}b', {}^{a}a']a}a'' \otimes {}^{[{}^{a}b', {}^{a}a']a}b'' = {}^{a[b',a']}a'' \otimes {}^{a[b',a']}b'' = {}^{a}({}^{[b',a']}a'' \otimes {}^{[b',a']}b'').
\end{aligned}$$

(ii) Can be proved analogously.
(iii)

$$\begin{aligned}
&a \cdot {}^{b}a^{-1} \otimes b' = ({}^{ab}a^{-1} \otimes {}^{a}b')(a \otimes b') = ({}^{ab}a^{-1} \otimes {}^{a}b^{-1})({}^{({}^{a}b^{-1})ab}a^{-1} \otimes {}^{({}^{a}b^{-1})ab}(b'b)) \\
&(a \otimes n') = (a \otimes b)(a^{-1} \otimes {}^{a}b' \cdot {}^{a}b)(a \otimes b') = (a \otimes b)(a^{-1} \otimes {}^{a}b')({}^{({}^{a}b')}a^{-1} \otimes {}^{({}^{a}b')a}b) \\
&(a \otimes b') = (a \otimes b)(a \otimes b')^{-1}({}^{ab'}a^{-1} \otimes {}^{ab'}b)(a \otimes b') = (a \otimes b) \cdot {}^{b'}(a \otimes b)^{-1}.
\end{aligned}$$

The second part of **(iii)** is proved similarly. The proof is complete. ■

Theorem 14. *Let $\mu : M \to P$ and $\nu : N \to P$ be precrossed modules over the group P. Let M and N act on each other via P and on themselves by conjugation and the following conditions hold:*
${}^{\mu(m)}m' = mm'm^{-1}$ if $\exists\ n \in N$ such that $\mu(m) = \nu(n)$,
${}^{\nu(n)}n' = nn'n^{-1}$ if $\exists\ m \in M$ such that $\nu(n) = \mu(m)$.
Then there is an exact sequence of groups

$$\Gamma((M \times_p N)^{ab}) \xrightarrow{\Psi} M \otimes N \to M \wedge^p N \to 1,$$

where $\Psi(\gamma(m, n)) = m \otimes n$. Also Ψ has central image.

Proof. Let $\widetilde{\Psi} : M \times_p N \to M \otimes N$ be the map given by $(m,n) \mapsto m \otimes n$. Then $X = \mathrm{Ker}(M \otimes N \to M \wedge^p N)$ is the subgroup of $M \otimes N$ generated by $\mathrm{Im}\,\widetilde{\Psi}$.

Now we have to show that X is central in $M \otimes N$. Let $m \otimes n$, $m' \otimes n' \in M \otimes N$ and $\mu(m) = \nu(n)$. Then $m \cdot {}^n m^{-1} = 1$, ${}^m n n^{-1} = 1$ and

$$(m \otimes n)(m' \otimes n')(m \otimes n)^{-1} = {}^{mnm^{-1}n^{-1}}m' \otimes {}^{mnm^{-1}n^{-1}}n' = {}^{m\cdot {}^n m^{-1}}m' \otimes {}^{{}^m n n^{-1}}n' = \\ = m' \otimes n'$$

i.e. we have $(m \otimes n)(m' \otimes n') = (m' \otimes n')(m \otimes n)$ and X is central.

If $(m,n) \in M \times_p N$ then m and n verify conditions of Lemma 13, therefore they act on $M \otimes N$ and trivially on X.

Let (m,n), $(u,v) \in M \times_p N$. Since $\mu(m) = \nu(n)$ and $m \otimes n$, $u \otimes v \in Z(M \otimes N)$,

$$mu \otimes nv = ({}^m u \otimes {}^m n {}^m v)(m \otimes nv) = {}^m\{(u \otimes n)(m \otimes v)\}(m \otimes n)(u \otimes v).$$

It follows that $(u \otimes n)(m \otimes v) \in X$ and

$$mu \otimes nv = (u \otimes n)(m \otimes v)(m \otimes n)(u \otimes v). \tag{5}$$

Suppose $(m,n),(m_1,n_1),(m_2,n_2) \in M \times_p N$. Then by using (5) one has

$$m[m_1,m_2] \otimes n[n_1,n_2] = m \otimes n.$$

This proves that $\widetilde{\Psi}$ induces a map defined on $(M \times_p N)^{ab}$.

We now start the proof that $\widetilde{\Psi}$ induces a homomorphism on $\Gamma((M \times_p N)^{ab})$. Let $(m,n) \in M \times_p N$, then

$$m^{-1} \otimes n^{-1} = ({}^{m^{-1}}m \otimes {}^{m^{-1}}(n^{-1}))^{-1} = {}^{m^{-1}}({}^{n^{-1}}m \otimes {}^{n^{-1}}n) = {}^{m^{-1}n^{-1}}(m \otimes n) = m \otimes n.$$

Hence $\widetilde{\Psi}[(m,n)] = \widetilde{\Psi}[(m^{-1},n^{-1})]$.

Let $(m,n),(m_1,n_1),(m_2,n_2) \in M \times_p N$. Then from (5)
$mm_1m_2 \otimes nn_1n_2 = (m_1m_2 \otimes n)(m \otimes n_1n_2)(m \otimes n)(m_1m_2 \otimes n_1n_2) =$
${}^{m_1}(m_2 \otimes n)(m_1 \otimes n)(m \otimes n_1) \cdot {}^{n_1}(m \otimes n_2)(m \otimes n)(m_1m_2 \otimes n_1n_2) =$
$\{(m_1 \otimes n)(m \otimes n_1)\}\{(m_2 \otimes n)(m \otimes n_2)\}(m \otimes n)(m_1m_2 \otimes n_1n_2)$,
$mm_1 \otimes nn_1 = \{(m_1 \otimes n)(m \otimes n_1)\}(m \otimes n)(m_1 \otimes n_1)$,
$mm_2 \otimes nn_2 = \{(m_2 \otimes n)(m \otimes n_2)\}(m \otimes n)(m_2 \otimes n_2)$.
Hence we have

$$(mm_1m_2 \otimes nn_1n_2)(m \otimes n)(m_1 \otimes n_1)(m_2 \otimes n_2) = (mm_1 \otimes nn_1)\cdot \\ (m_1m_2 \otimes n_1n_2)(mm_2 \otimes nn_2),$$

i.e.

$$\widetilde{\Psi}[(mm_1m_2,nn_1n_2)]\widetilde{\Psi}[(m,n)]\widetilde{\Psi}[(m_1,n_1)]\widetilde{\Psi}[(m_2,n_2)] = \widetilde{\Psi}[(mm_1,nn_1)]\cdot \\ \widetilde{\Psi}[(m_1m_2,n_1n_2)]\widetilde{\Psi}[(mm_2,nn_2)].$$

Thus $\widetilde{\Psi}$ induces a homomorphism $\Psi{:}\Gamma((M \times_p N)^{ab}) \to M \otimes N$, $\Psi(\gamma(m,n)) = m \otimes n$.

The proof is complete. ■

Example 15. Let A be metabelian (not abelian) group i.e. $[A,A] \subset Z(A)$ (center of A). Let $M = A$, $N = A$, $P = (A/Z(A)) \times A$, the actions of P on M and N be given: $\forall\ p = ([a], a')$, $m = a''$, $n = a'''$

$$^{([a],a')}a'' = {}^{a'}a'',$$
$$^{([a],a')}a''' = {}^{a'}a'''$$

and $\mu : M \to P$, $\mu(a) = ([a], 1)$, $\nu : N \to P$, $\nu(a) = (1, a)$. Then it is easy to verify that (M, μ) is a precrossed module and not a crossed module, (N, ν) is a crossed module and conditions (1) hold.

Consider $\mathrm{PCM}(P)$ the category of precrossed modules over the group P. The objects of this category are (M, μ) precrossed modules over P and morphisms $(M, \mu) \xrightarrow{f} (M', \mu')$ of the category are homomorphisms $M \xrightarrow{f} M'$ such that 1. $f({}^p m) = {}^p f(m)$, 2. $\mu' f = \mu$.

Now we construct a cotriple $\mathcal{F} = (F, \tau, \delta)$ in the category $\mathrm{PCM}(P)$ in the following manner. Let $F : \mathrm{PCM}(P) \to \mathrm{PCM}(P)$ be an endofunctor defined as follows: for $(M, \mu) \in \mathrm{ob}\,\mathrm{PCM}(P)$ the object $F(M, \mu) \equiv (F(M), \widetilde{\mu})$ is a free precrossed module $\widetilde{\mu} : F(M) \to P$ generated by the precrossed module (M, μ), where $F(M)$ is the free group generated by M, $\widetilde{\mu}$ is the homomorphism induced by μ and the action of P on $F(M)$ is given by the formula: ${}^p(|m_1|^{\epsilon_1}|m_2|^{\epsilon_2}\cdots|m_s|^{\epsilon_s}) = |{}^p m_1|^{\epsilon_1}|{}^p m_2|^{\epsilon_2}\cdots|{}^p m_s|^{\epsilon_s}$, for $p \in P$, $|m_1|^{\epsilon_1}|m_2|^{\epsilon_2}\cdots|m_s|^{\epsilon_s} \in F(M)$ and $\epsilon_i = \pm 1$. If $f : (M, \mu) \to (M', \mu')$ is a morphism of $\mathrm{PCM}(P)$, then $F(f) : (F(M), \widetilde{\mu}) \to (F(M'), \widetilde{\mu'})$ is the natural homomorphism induced by f.

Let $\tau : F \to 1_{\mathrm{PCM}(P)}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for $(M, \mu) \in \mathrm{ob}\,\mathrm{PCM}(P)$ by the injection $M \to F(M)$. It is easy to verify that the homomorphisms $\tau_{(M,\mu)}$ and $\delta_{(M,\mu)}$ are morphisms of the category $\mathrm{PCM}(P)$ and that $\mathcal{F} = (F, \tau, \delta)$ is a cotriple which we call the free cotriple in the category $\mathrm{PCM}(P)$ of precrossed modules over P.

It is easy to see that there exist simplicial kernels in the category $\mathrm{PCM}(P)$.

Consider the non-abelian left derived functors $L_n^{\mathcal{P}}(- \otimes N)$, $L_n^{\mathcal{P}}(- \wedge^P N)$, $n \geq 0$, of the functors $- \otimes N$, $- \wedge^P N : \mathrm{PCM}(P) \to \mathfrak{Gr}$ respectively relative to the projective class $\mathcal{P}$ [9,14] induced by the free cotriple $\mathcal{F}$.

Theorem 16. *Let $M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 1$ be an exact sequence of precrossed modules. Then we have the following exact sequence of groups*

$$M_1 \wedge^P N \xrightarrow{f'} M \wedge^P N \xrightarrow{g'} M_2 \wedge^P N \to 1,$$

where $f' = f \wedge^P 1_N$, $g' = g \wedge^P 1_N$.

Proof. According to the right exactness of the non-abelian tensor product of groups [11, Theorem 1] we have an exact sequence of groups

$$M_1 \otimes N \xrightarrow{\overline{f'}} M \otimes N \xrightarrow{\overline{g'}} M_2 \otimes N \to 1.$$

Let X be the subgroup of $M \otimes N$ generated by $m \otimes n$ such that $\mu(m) = \nu(n)$ and X_2 be the subgroup of $M_2 \otimes N$ generated by $m_2 \otimes n$ such that $\mu_2(m_2) = \nu(n)$. Then the restriction of $\overline{g'}$ on X is surjective on X_2.

We have a commutative diagram of groups

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & X & \longrightarrow & X_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \\
M_1 \otimes N & \xrightarrow{\overline{f'}} & M \otimes N & \xrightarrow{\overline{g'}} & M_2 \otimes N & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
M_1 \wedge^P N & \xrightarrow{f'} & M \wedge^P N & \xrightarrow{g'} & M_2 \wedge^P N & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & &
\end{array}.$$

Since columns and two top rows are exact, the bottom row is also exact. The proof is complete. ■

Theorem 17. *There exists a natural isomorphism*

$$L_0^{\mathcal{P}}(- \wedge^P N) \approx - \wedge^P N$$

Proof. Consider a $\mathcal{P}$-projective pseudosimplicial resolution of an object M of the category PCM(P)

$$\cdots \begin{array}{c}\rightarrow\\ \vdots\\ \rightarrow\end{array} X_n \quad \begin{array}{c}\xrightarrow{\partial_0^n}\\ \vdots\\ \xrightarrow[\partial_n^n]{}\end{array} \cdots \begin{array}{c}\rightarrow\\ \rightarrow\\ \rightarrow\end{array} X_1 \quad \begin{array}{c}\xrightarrow{\partial_0^1}\\ \Longrightarrow\\ \xrightarrow[\partial_1^1]{}\end{array} X_0 \xrightarrow{\partial_0^0} M.$$

Then the sequence of precrossed modules

$$\operatorname{Ker} \partial_0^1 \xrightarrow{\partial_1^1 \sigma} X_0 \xrightarrow{\partial_0^0} M \to 1$$

is exact, where $\sigma : \operatorname{Ker} \partial_0^1 \to X_1$ is the inclusion and from Theorem 16 it follows that the sequence

$$\operatorname{Ker} \partial_0^1 \wedge^P N \xrightarrow{\partial_1^1 \sigma \wedge^P 1_N} X_0 \wedge^P N \xrightarrow{\partial_0^0 \wedge^P 1_N} M \wedge^P N \to 1$$

is also exact.

We have the commutative diagram

$$\begin{array}{ccccc}
\operatorname{Ker} \partial_0^1 \wedge^P N & \xrightarrow{\partial_1^1 \sigma \wedge^P 1_N} & X_0 \wedge^P N & & \\
\downarrow f & & \| & & \\
\operatorname{Ker}(\partial_0^1 \wedge^P 1_N) & \xrightarrow{\partial_1^1 \wedge^P 1_N} & X_0 \wedge^P N & \xrightarrow{\partial_0^0 \wedge^P 1_N} & M \wedge^P N \longrightarrow 1,
\end{array}$$

where the homomorphism f induced by the inclusion σ is surjective by

Theorem 16. Thus the bottom sequence of this diagram is exact. The proof is complete. ■

Theorem 18. *Let $\mu : M \to P$, $\nu : N \to P$ be precrossed modules over the group P. Then we have the following exact sequence of groups*

$$\cdots \longrightarrow H_3(M,N) \longrightarrow L_2^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_1^{\mathcal{P}}(f,N) \longrightarrow$$
$$H_2(M,N) \longrightarrow L_1^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_0^{\mathcal{P}}(f,N) \longrightarrow M \otimes N \xrightarrow{f} ,$$
$$M \wedge^P N \longrightarrow 1$$

where

$$L_n^{\mathcal{P}}(f,N) = \pi_n(\operatorname{Ker} f_*), \quad n \geq 0,$$
$$\operatorname{Ker} f_* = \{\operatorname{Ker} f^n \mid f^n : F^n(M) \otimes N \to F^n(M) \wedge^P N,\ n \geq 1\}.$$

Proof. Let consider the cotriple $\mathcal{F}$- resolution of an object M of the category $\mathrm{PCM}(P)$

$$\cdots \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} F^n(M) \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} \cdots \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} F^2(M) \rightrightarrows F^1(M) \to M,$$

where $F^n(B) = F(F^{n-1}(B))$, $\partial_i^n = F^i \tau F^{n-i}$, $s_i^n = F^i \delta F^{n-i}$.

It is known [17] that the derived functors relative to the projective class induced by a cotriple are isomorphic to the derived functors relative to this cotriple, $L_n^{\mathcal{P}}(-) \approx L_n^{\mathcal{F}}(-)$.

We have the commutative diagram of groups

$$\begin{array}{ccccc}
\cdots \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & F^n(M) \otimes N & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} \cdots \rightrightarrows & F^1(M) \otimes N & \to & M \otimes N \\
& \downarrow f_n & & \downarrow f_1 & & \downarrow f \\
\cdots \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & F^n(M) \wedge^P N & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} \cdots \rightrightarrows & F^1(M) \wedge^P N & \to & M \wedge^P N
\end{array} .$$

and hence an exact sequence of groups

$$\cdots \longrightarrow L_2^{\mathcal{P}}(M \otimes N) \longrightarrow L_2^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_1^{\mathcal{P}}(f,N) \longrightarrow$$
$$L_1^{\mathcal{P}}(M \otimes N) \longrightarrow L_1^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_0^{\mathcal{P}}(f,N) \longrightarrow L_0^{\mathcal{P}}(M \otimes N) \xrightarrow{f}$$
$$L_0^{\mathcal{P}}(M \wedge^P N) \longrightarrow 1.$$

It is clearly that the derived functors $L_n^{\mathcal{P}}(- \otimes N)$ coincide with the derived functors of the non-abelian tensor product as a functor from the category $\mathcal{A}_N$ to the category $\mathfrak{Gr}$ relative to the projective class induced by the free cotriple. Hence $L_0^{\mathcal{P}}(M \otimes N) \approx M \otimes N$ and $L_n^{\mathcal{P}}(M \otimes N) = H_{n+1}(M,N)$, $n \geq 1$ [11] (see definition in Section 2). ■

Now we consider Guin's non-abelian tensor product $- \overset{G}{\otimes} N$ and its exterior product $- \wedge^P N$ as functors from the category $\mathrm{PCM}(P)$ to the category $\mathfrak{Gr}$ of groups and their non-abelian left derived functors relative to the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$.

Theorem 19. *Let $M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 1$ be an exact sequence of precrossed modules. Then we have the following exact sequence of groups*

$$M_1 \wedge^P N \xrightarrow{f'} M \wedge^P N \xrightarrow{g'} M_2 \wedge^P N \to 1,$$

(*the exterior product* $M \wedge^P N$ *is taken for the Guin's tensor product*) *where* $f' = f \wedge^P 1_N$, $g' = g \wedge^P 1_N$.

Proof. Similar to the proof of Theorem 16. ■

Theorem 20. *There exist natural isomorphisms*

$$L_0^{\mathcal{P}}(- \overset{G}{\otimes} N) \approx - \overset{G}{\otimes} N,$$
$$L_0^{\mathcal{P}}(- \wedge^P N) \approx - \wedge^P N,$$

where the exterior product $- \wedge^P N$ *is taken for the Guin's tensor product.*

Proof. Similar to the proof of Theorems 4[11] and Theorem 17. ■

Definition 21. Let (M, μ) be an object of the category PCM(P). Then we define Guin's homology ${}^G H_*(M, N)$ of a precrossed module (M, μ) with coefficients in any precrossed module (N, ν) over P as follows

$${}^G H_n(M, N) = L_{n-1}^{\mathcal{P}}(M \overset{G}{\otimes} N), \qquad n \geq 2,$$
$${}^G H_1(M, N) = \operatorname{Ker} \lambda', \qquad {}^G H_0(M, N) = \operatorname{Coker} \lambda',$$

where $\lambda' : M \overset{G}{\otimes} N \to N/N'$, $\lambda'(m \otimes n) = [{}^{\mu(m)}nn^{-1}]$ and N' is the normal subgroup generated by the elements ${}^{\nu(n)}n'nn'^{-1}n^{-1}$ for $n, n' \in N$.

Remark 22. It is clear that the groups ${}^G H_n(M, N)$ are abelian for $n \geq 2$. When (M, μ) and (N, ν) are crossed modules then ${}^G H_1(M, N) = H_1(M, N)$ is also abelian and ${}^G H_0(M, N) = H_0(M, N)$. We will show that $\operatorname{Im} \lambda'$ is a normal subgroup of N/N'.

In fact, let $[n'] \in \operatorname{Im} \lambda'$ and $[n] \in N/N'$, then we have

$[n][n'][n] =$
$= [n]\lambda'((m_1 \otimes n_1) \cdots (m_k \otimes n_k))[n]^{-1} = [n][{}^{\mu(m_1)}n_1 n_1^{-1} \ldots {}^{\mu(m_k)}n_k n_k^{-1}][n]^{-1} =$
$= [{}^{\nu(n)\mu(m_1)\nu(n)^{-1}}({}^{\nu(n)}n_1)({}^{\nu(n)}n_1)^{-1}] \cdots [{}^{\nu(n)\mu(m_k)\nu(n)^{-1}}({}^{\nu(n)}n_k)({}^{\nu(n)}n_k)^{-1}] =$
$= [{}^{\mu({}^{\nu(n)}m_1)}({}^{\nu(n)}n_1)({}^{\nu(n)}n_1)^{-1}] \cdots [{}^{\mu({}^{\nu(n)}m_k)}({}^{\nu(n)}n_k)({}^{\nu(n)}n_k)^{-1}] =$
$= \lambda'(({}^{\nu(n)}m_1 \otimes^{\nu(n)} n_1) \cdots ({}^{\nu(n)}m_k \otimes^{\nu(n)} n_k))$

Theorem 23. *Let* (M, μ), (N, ν) *be precrossed modules over the same group* P. *Then we have the following exact sequence of groups*

$$\cdots \longrightarrow {}^G H_3(M, N) \longrightarrow L_2^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_1^{\mathcal{P}}(f, N) \longrightarrow$$
$${}^G H_2(M, N) \longrightarrow L_1^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_0^{\mathcal{P}}(f, N) \longrightarrow M \overset{G}{\otimes} N \xrightarrow{f} ,$$
$$M \wedge^P N \longrightarrow 1$$

where

$$L_n^{\mathcal{P}}(f, N) = \pi_n(\operatorname{Ker} f_*), \quad n \geq 0$$
$$\operatorname{Ker} f_* = \{\operatorname{Ker} f^n \mid f^n : F^n(M) \overset{G}{\otimes} N \to F^n(M) \wedge^P N, \ n \geq 1\}.$$

Proof. Similar to the proof of the Theorem 18 (i). ■

Consider the category $\mathrm{CM}(P)$ of crossed modules over the group P. The objects of this category are (M,μ) crossed modules over P, and morphisms $(M,\mu) \xrightarrow{f} (M',\mu')$ of the category are the homomorphisms $M \xrightarrow{f} M'$ such that 1. $f({}^p m) = {}^p f(m)$, 2. $\mu' f = \mu$.

Now we construct a cotriple $\mathcal{F}' = (F', \tau', \delta')$ in the category $\mathrm{CM}(P)$. Let $F' : \mathrm{CM}(P) \to \mathrm{CM}(P)$ be an endofunctor defined as follows: for $(M,\mu) \in \mathrm{ob}\,\mathrm{CM}(P)$ the object $F'(M,\mu)$ is the crossed module $(F(M)/H, \widetilde{\mu})$, where $F(M)$ is the free group generated by M and H is the normal subgroup generated by the elements $\widetilde{\mu(x)} x' x x'^{-1} x^{-1}$ for each $x, x' \in F(M)$. If $f : (M,\mu) \to (M',\mu')$ is a morphism of $\mathrm{CM}(P)$, then $F'(f) : (F(M)/H, \widetilde{\mu}) \to (F(M')/H', \widetilde{\mu'})$ is induced by the natural homomorphism $F(f)$.

Let (M,μ) be a crossed module. Then the natural epimorphism $F(M) \to M$ induces a morphism $(F(M)/H, \widetilde{\mu}) \xrightarrow{\tau'(M)} (M,\mu)$ of the category $\mathrm{CM}(P)$ and $\tau' : F' \to 1_{\mathrm{CM}(P)}$ satisfies the condition of natural transformation.

Let $\delta' : F' \to F'^2$ be the natural transformation induced for $(M,\mu) \in \mathrm{ob}\,\mathrm{CM}(P)$ by the composition of the injection $M \to F(M)$ and the natural epimorphism $F(M) \to F(M)/H$ (because any map of sets $f : M \to M'$ which satisfies conditions $f({}^p m) = {}^p f(m)$, $\mu' f = \mu$ induces a morphism $F(M)/H \to F(M')/H'$ of crossed modules).

It is easy to see that there exist simplicial kernels in the category $\mathrm{CM}(P)$.

From [8] it is known that $- \overset{G}{\otimes} N$ and $- \otimes N$ are the same functors on the category $\mathrm{CM}(P)$. Consider the non-abelian left derived functors $\mathbb{L}_n^{\mathcal{P}'}(- \otimes N)$ and $\mathbb{L}_n^{\mathcal{P}'}(- \wedge^P N)$, $n \geq 0$ of the functors $- \otimes N$ and $- \wedge^P N$ respectively relative to the projective class $\mathcal{P}'$ induced by the cotriple $\mathcal{F}'$. Then we have

Theorem 24. *There exist natural isomorphisms*

$$\mathbb{L}_0^{\mathcal{P}'}(- \otimes N) \approx - \otimes N,$$
$$\mathbb{L}_0^{\mathcal{P}'}(- \wedge^P N) \approx - \wedge^P N.$$

Proof. Consider a projective pseudosimplicial resolution of an object M of the category $\mathrm{CM}(P)$ relative to the projective class $\mathcal{P}'$ induced by the cotriple $\mathcal{F}'$

$$\cdots \overset{\to}{\underset{\to}{\overset{\to}{\vdots}}} X_n \quad \overset{\partial_0^n}{\underset{\partial_n^n}{\overset{\to}{\vdots}}} \cdots \overset{\to}{\underset{\to}{\to}} X_1 \quad \overset{\partial_0^1}{\underset{\partial_1^1}{\rightrightarrows}} X_0 \xrightarrow{\partial_0^0} M.$$

Since $\mathcal{P}'$-epimorphism is surjective, then

$$\mathrm{Ker}\,\partial_0^1 \xrightarrow{\partial_1^1 \sigma} X_0 \xrightarrow{\partial_0^0} M \to 1$$

is an exact sequence of crossed modules, where $\sigma : \mathrm{Ker}\,\partial_0^1 \to X_1$ is the inclusion. From Theorem 1[11] and Theorem 16 it follows that the sequences of groups

$$\mathrm{Ker}\,\partial_0^1 \otimes N \xrightarrow{\partial_1^1 \sigma \otimes 1_N} X_0 \otimes N \xrightarrow{\partial_0^0 \otimes 1_N} M \otimes N \to 1,$$

$$\operatorname{Ker}\partial_0^1 \wedge^P N \xrightarrow{\partial_1^1\sigma\wedge^P 1_N} X_0 \wedge^P N \xrightarrow{\partial_0^0\wedge^P 1_N} M \wedge^P N \to 1$$

are also exact.

We have the commutative diagram

$$\begin{array}{ccccccc} \operatorname{Ker}\partial_0^1 \otimes N & \xrightarrow{\partial_1^1\sigma\otimes 1_N} & X_0 \otimes N & & & & \\ \downarrow f & & \| & & & & \\ \operatorname{Ker}(\partial_0^1 \otimes 1_N) & \xrightarrow{\partial_1^1\otimes 1_N} & X_0 \otimes N & \xrightarrow{\partial_0^0\otimes 1_N} & M \otimes N & \longrightarrow & 1, \end{array}$$

where the homomorphism f induced by the inclusion σ is surjective by Theorem 1[11]. Thus the bottom sequence of this diagram is exact. The exactness of the sequence $\operatorname{Ker}\partial_0^1 \wedge^P N \xrightarrow{\partial_1^1\sigma\wedge^P 1_N} X_0 \wedge^P N \xrightarrow{\partial_0^0\wedge^P 1_N} M \wedge^P N \to 1$ is proved analogously. The proof is complete. ■

It is easy to see that $(- \times_P N)/ < -, N >$ is a covariant functor from the category $\mathrm{CM}(P)$ to the category $\mathfrak{Ab}$ of abelian groups, then $\Gamma(- \times_P N/ < -, N >)$ will be a functor from the category $\mathrm{CM}(P)$ to the category $\mathfrak{Ab}$. Consider the non-abelian left derived functors $\mathbb{L}_n^{\mathcal{P}'}\Gamma(-\times_P N/ < -, N >)$, $n \geq 0$ of the functor $\Gamma(-\times_P N)/ < -, N >$ relative to the projective class $\mathcal{P}'$ induced by the cotriple $\mathcal{F}'$.

Theorem 25. *Let $(M, \mu), (N, \nu) \in \operatorname{ob}\mathrm{CM}(P)$, then we have the following exact sequences of groups*

$$\begin{array}{c} \cdots \longrightarrow \mathbb{L}_2^{\mathcal{P}'}(M \wedge^P N) \longrightarrow \mathbb{L}_1^{\mathcal{P}'}(f, N) \longrightarrow \mathbb{L}_1^{\mathcal{P}'}(M \otimes N) \longrightarrow \\ \mathbb{L}_1^{\mathcal{P}'}(M \wedge^P N) \longrightarrow \mathbb{L}_0^{\mathcal{P}'}(f, N) \longrightarrow M \otimes N \xrightarrow{f} M \wedge^P N \longrightarrow 1 \end{array},$$

where

$$\begin{array}{c} \mathbb{L}_n^{\mathcal{P}'}(f, N) = \pi_n(\operatorname{Ker} f_*), \quad n \geq 0, \\ \operatorname{Ker} f_* = \{\operatorname{Ker} f^n \mid f^n : F'^n(M) \otimes N \longrightarrow F'^n(M) \wedge^P N, \quad n \geq 1\} \end{array}$$

and

$$\begin{array}{c} \cdots \longrightarrow \mathbb{L}_2^{\mathcal{P}'}(f, N) \longrightarrow \mathbb{L}_1^{\mathcal{P}'}(\psi, N) \longrightarrow \\ \mathbb{L}_1^{\mathcal{P}'}\Gamma(M \times_P N/ < M, N >) \longrightarrow \mathbb{L}_1^{\mathcal{P}'}(f, N) \longrightarrow \mathbb{L}_0^{\mathcal{P}'}(\psi, N) \\ \longrightarrow \mathbb{L}_0^{\mathcal{P}'}\Gamma(M \times_P N/ < M, N >) \longrightarrow \mathbb{L}_0^{\mathcal{P}'}(f, N) \longrightarrow 1 \end{array},$$

where

$$\begin{array}{c} \mathbb{L}_n^{\mathcal{P}'}(\psi, N) = \pi_n(\operatorname{Ker}\psi_*), \quad n \geq 0, \\ \operatorname{Ker}\psi_* = \{\operatorname{Ker}\psi^n \mid \psi^n : \Gamma(F'^n(M) \times_P N/ < F'^n(M), N >) \to F'^n(M) \otimes N, \ n \geq 1\}; \end{array}$$

Proof. Consider $\mathcal{F}'$ cotriple resolution of an object M of the category $\mathrm{CM}(P)$

$$\cdots \underset{\to}{\overset{\to}{\vdots}} F'^n(M) \underset{\to}{\overset{\to}{\vdots}} \cdots \underset{\to}{\overset{\to}{\rightrightarrows}} F'^2(M) \rightrightarrows F'^1(M) \to M.$$

Put $\Omega^n \equiv \Gamma(F'^n(M) \times_P N/ < F'^n(M), N >)$. Then from [5, Theorem 2.12] we have the comutative diagram

$$\begin{array}{ccccccc}
\cdots \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & \Omega^n & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} \cdots \rightrightarrows & \Omega^1 & \to & \Gamma(M \times_P N/ < M, N >) \\
& \downarrow \psi_n & & \downarrow \psi_1 & & \downarrow \psi \\
\cdots \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & F'^n(M) \otimes N & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} \cdots \rightrightarrows & F'^1(M) \otimes N & \to & M \otimes N \\
& \downarrow f_n & & \downarrow f_1 & & \downarrow f \\
\cdots \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & F'^n(M) \wedge^P N & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} \cdots \rightrightarrows & F'^1(M) \wedge^P N & \to & M \wedge^P N
\end{array} .$$

which has exact vertical rows. The proof is complete. ■

Remark 26. There exists the natural epimorphism

$$\mathbb{L}_0^{\mathcal{P}'}\Gamma(M \times_P N/ < M, N >) \longrightarrow \Gamma(M \times_P N/ < M, N >).$$

2. Finiteness of a Non-abelian Homology of Groups

The non-abelian homology of groups with coefficients in any group was introduced in [11]. The construction was made possible by using the left derived functors of the non-abelian tensor product of groups. Now we recall the definition of the non-abelian homology of groups from [11].

Let consider the category $\mathcal{A}_A$ of groups acting on a fixed group A and the group A acts on these groups. Morphisms of the category $\mathcal{A}_A$ are homomorphisms $\alpha : G \to H$, G, $H \in \operatorname{ob}\mathcal{A}_A$ of groups which preserve the actions in sense that: $\alpha({}^a g) = {}^a\alpha(g)$ and ${}^g a = {}^{\alpha(g)}a$, for all $a \in A$, $g \in G \in \operatorname{ob}\mathcal{A}_A$.

Let $F : \mathcal{A}_A \to \mathcal{A}_A$ be an endofunctor defined as follows: for $G \in \operatorname{ob}\mathcal{A}_A$ the object $F(G)$ is the free group generated by G with actions: ${}^{|g_1|^{\epsilon_1}\cdots|g_s|^{\epsilon_s}}a = {}^{g_1^{\epsilon_1}}(\dots({}^{g_s^{\epsilon_s}}a)\dots)$, and ${}^a(|g_1|^{\epsilon_1}\cdots|g_s|^{\epsilon_s}) = |{}^a g_1|^{\epsilon_1}\cdots|{}^a g_s|^{\epsilon_s}$ for $a \in A$, $|g_1|^{\epsilon_1}\cdots|g_s|^{\epsilon_s} \in F(G)$ and $\epsilon_i = \pm 1$; for $\alpha : G \to G'$ the morphism $F(\alpha)$ is the canonical homomorphism from $F(G)$ to $F(G')$ induced by α.

Let $\tau : F \to 1_{\mathcal{A}_A}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for every $G \in \operatorname{ob}\mathcal{A}_A$ by the injection $G \to F(G)$. We obtain a cotriple $\mathcal{F} = (F, \tau, \delta)$, which we call the free cotriple in the category $\mathcal{A}_A$.

The non-abelian tensor product of groups defines a covariant functor $- \otimes A$ from the category $\mathcal{A}_A$ to the category $\mathfrak{Gr}$ of groups. Consider the non-abelian left derived functors $L_n^{\mathcal{P}}(- \otimes A)$, $n \geq 0$, of the functor $- \otimes A$ relative to the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$.

Definition 27. Let G be an object of the category $\mathcal{A}_A$. Then we define

$$H_n(G, A) = L_{n-1}^{\mathcal{P}}(G \otimes A), \qquad n \geq 2,$$
$$H_1(G, A) = \operatorname{Ker}\lambda', \qquad H_0(G, A) = \operatorname{Coker}\lambda',$$

where $\lambda' : G \otimes A \to A/H$, $\lambda'(g \otimes a) = [{}^g a a^{-1}]$, and H is the normal subgroup generated by the elements ${}^{({}^a g)}a' \cdot {}^{aga^{-1}}a'^{-1}$, for each $a, a' \in A$, $g \in G$.

The present section is concerned solely with finiteness of homology of groups.

Let G be any group and A a finite group acting on each other. It is easy to see that $H_0(G, A)$ is a finite group.

Proposition 28. *Let G and A be finite groups which act on each other compatibly i.e. conditions (1) hold. Then $H_1(G, A)$ is a finite group.*

Proof. If the finite groups G and A act on each other compatibly then from [7] $G \otimes A$ is also finite. From the definition $H_1(G, A)$ is a normal subgroup of $G \otimes A$. Thus $H_1(G, A)$ is a finite group. The proof is complete. ■

Remark 29. If finite groups G and A act on each other such that one of the following sufficient conditions for the finiteness of the non-abelian tensor product of groups [12] holds:

(i) G and A act on each other half compatibly (see [12]);
(ii) A is soluble;
(iii) $[G, A]^n$ is abelian for some $n \geq 1$, where $[G, A]^n = [G, [G, A]^{n-1}]$ and $[G, A]$ is a normal subgroup of A generated by the elements ${}^g aa^{-1}$ for all $g \in G$, $a \in A$.

Then $H_1(G, A)$ is a finite group.

It is interesting to establish the finiteness of non-abelian homology in higher dimensions. Below we only concern the non-abelian homology of a group G with coefficients in a G-group A. From Definition 3[11] the homology of G with coefficient in G-group A is $H_*(G, A)$ with trivial action of A on G.

Now we construct a new simplicial $\mathcal{P}$-projective resolution of G in this case. For this we recall the construction of the loop functor $\mathbb{G}$ from the category of reduced complexes to the category of simplicial groups [13].

Definition 30. Let K be a reduced complex (i.e. K is simplicial set which has only one 0-simplex ϕ). Then we define a simplicial group $\mathbb{G}K$ as follows: the group of the n-simplices is a group which has

(i) one generator $\overline{\sigma}$ for every $n+1$-simplex $\sigma \in K_{n+1}$,
(ii) one relation $\overline{s_n^n \tau} = e_n$ for every n-simplex $\tau \in K_n$.

The face and degeneracy homomorphisms $\partial_i^n : \mathbb{G}_n K \to \mathbb{G}_{n-1} K$ and $s_i^n : \mathbb{G}_n K \to \mathbb{G}_{n+1} K$ are given by the formulas

$$\begin{aligned} \partial_i^n \overline{\sigma} &= \overline{\partial_i^n \sigma}, 0 \leq i < n, \\ \partial_n^n \overline{\sigma} &= \overline{\partial_n^n \sigma} \cdot \overline{\partial_{n+1}^n \sigma}^{-1}, \\ s_i^n \overline{\sigma} &= \overline{s_i^n \sigma}, 0 \leq i \leq n. \end{aligned}$$

Clearly the groups $\mathbb{G}_n K$ are free and from [13] it is known that $\pi_n(K) \approx \pi_{n-1}(\mathbb{G}K)$ for $n > 0$.

Let $\underline{G}$ be a category which has only one object O and $Hom(O, O) = G$, and consider the nerve $M_*(\underline{G})$ of the category $\underline{G}$. It is easy to see that $M_*(\underline{G})$ is a reduced complex. It is known that $\pi_1(M_*(\underline{G})) \approx G$ and $\pi_n(M_*(\underline{G})) = 0$, $n \neq 1$. Let consider the simplicial group $\mathbb{G}M_*(\underline{G})$. Then $\pi_0(\mathbb{G}M_*(\underline{G})) \approx \pi_1(M_*(\underline{G})) \approx G$ and

$\pi_{n-1}(\mathbb{G}M_*(\underline{G})) \approx \pi_n(M_*(\underline{G})) = 0$, $n > 1$. Hence $\mathbb{G}M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ is a $\mathcal{P}$-projective simplicial resolution in the category of groups, where ∂_0^0 is the natural epimorphism, $\mathcal{P}$ is the projective class in the category of groups induced by the free cotriple.

Remark 31. An epimorphism of groups $\alpha : G \to H$, $G, H \in A_A$, which preserves the actions, is a $\mathcal{P}$-epimorphism in the category $\mathcal{A}_A$ if A acts trivially on G and H, and any object of the projective class $\mathcal{P}$ in the category $\mathcal{A}_A$ induced by free cotriple (F, τ, δ) is a free group $F(X)$, where X is a set on which A acts such that ${}^a x \in X$ for every $x \in X$.

Thus, if we define the action of A on $\mathbb{G}_n M_*(\underline{G})$ trivially and the action of $\mathbb{G}_n M_*(\underline{G})$ on A as follows: ${}^m a = {}^{\partial_0^0 \partial_0^1 \cdots \partial_0^n m} a$, $m \in \mathbb{G}_n M_*(\underline{G})$, then from Remark $\mathbb{G}_n M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ is the $\mathcal{P}$- projective simplicial resolution in the category $\mathcal{A}_A$.

Theorem 32. *Let G and A be finite groups. A acts on G trivially and G acts on A such that the non-abelian tensor product $G \otimes A$ is finite* (*see Remark* 29*, conditions* (i-iii)).

If F is a finitely generated free group which acts on A and A acts on F trivially and α is an epimorphism $\alpha : F \to G$ preserving the actions, then $F \otimes A$ is a finite group.

Proof. Consider the exact sequence of groups

$$1 \to \operatorname{Ker}\alpha \to F \xrightarrow{\alpha} G \to 1$$

From the right exactness of the non-abelian tensor product [11, Theorem 1], we have the exact sequence of groups

$$\operatorname{Ker}\alpha \otimes A \to F \otimes A \xrightarrow{\alpha \otimes 1_A} G \otimes A \to 1. \tag{6}$$

The group $\operatorname{Ker}\alpha$ is the subgroup with finite index of the finitelly generated free group F and from [18] it follows that $\operatorname{Ker}\alpha$ is a finitely generated free group. Since $\operatorname{Ker}\alpha$ and A act on each other trivially, from [3] we have

$$\operatorname{Ker}\alpha \otimes A \approx \operatorname{Ker}\alpha^{ab} \otimes_{\mathbb{Z}} A^{ab} \approx \sum_m \mathbb{Z} \otimes_{\mathbb{Z}} A^{ab} \approx \sum_m A^{ab}.$$

Hence $\operatorname{Ker}\alpha \otimes A$ is a finite group. From (6) one gets that $F \otimes A$ is a finite group. The proof is complete. ■

Theorem 33. *Let G and A be finite groups. If A acts on G trivially and G acts on A such that the non-abelian tensor product $G \otimes A$ is finite, then $H_n(G, A)$, $n \geq 0$, is a finite group.*

Proof. Consider the $\mathcal{P}$-projective pseudosimplicial resolution $\mathbb{G}M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ in the category $\mathcal{A}_A$ for G constructed above. Every $\mathbb{G}_n M_*(\underline{G})$ is a finitely generated free group and since the face operators are epimorphisms, $\alpha : \mathbb{G}_n M_*(\underline{G}) \to G$, $\alpha = \partial_0^0 \partial_0^1 \cdots \partial_0^n$ is an epimorphism which preserve the actions. Thus from Theorem 32 $\mathbb{G}_n M_*(\underline{G}) \otimes A$ is a finite group. Clearly $H_n(G, A)$, $n > 1$, is a finite group. The finiteness of $H_1(G, A)$ and $H_0(G, A)$ is shown above. The proof is complete. ■

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