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THE DIFFERENCE SCHEMES OF HIGH ORDER ACCURACY FOR ELLIPTIC EQUATIONS WITH LOWER DERIVATIVES

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Abstract. The scale of a priori estimates of convergence rate of the considered difference schemes is obtained when the solution of the initial problem belongs to the Sobolev spaces.

In this paper we investigated the finite-difference schemes of high order accuracy approximating the boundary value problem

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\lambda_1\frac{\partial u}{\partial x_1}+\lambda_2\frac{\partial u}{\partial x_2}=-f(x),\ \ x\in\Omega,\ \ u(x)=0,\ x\in\Gamma \tag{1}$$

with solutions from $W_2^m(\Omega), m\geq 2$. Here $\Omega=\{x=(x_1,x_2):0\leq x_1\leq l_1, 0\leq x_2\leq l_2\}$ is a rectangle with boundary Γ, λ_1,λ_2 are constant coefficients.

Difference scheme of fourth order accuracy was considered in [1] for the problem (1) on the square mesh. The indicated scheme is a particular case of a scheme constructed and investigated earlier in [2] converging with rate $O(|h|^4)$ to the solution $u\in C^6(\bar\Omega)$.

In this paper the convergence estimates of the difference schemes for the problem (1) are established applying the known method [3] which in its turn uses the generalization of Bramble-Hilbert lemma [4]. They are as follovs

$$\|y-u\|_{W_2^s(\omega)}\leq M|h|^{m-s}\|u\|_{W_2^m(\Omega)}\ , \tag{2}$$

where the constant $M>0$ is independent on h and $u(x)$. Let's assume, that the sides l_1,l_2 of the rectangle Ω are commensurable. In $\bar\Omega=\Omega\cup\Gamma$ we define the following meshes: $\bar\omega_\alpha=\{x_\alpha=i_\alpha h: i_\alpha=0,1,\cdots,N_\alpha,\ h=l_\alpha/N_\alpha\}$, $\omega_\alpha=\bar\omega_\alpha\cap(0,l_\alpha)$, $\omega_\alpha^+=\bar\omega_\alpha\cap(0;l_\alpha]$, $\omega_{(\alpha)}=\omega_\alpha^+\times\omega_{3-\alpha}$, $\omega=\omega_1\times\omega_2$, $\bar\omega=\bar\omega_1\times\bar\omega_2$, $\omega^+=\omega_1^+\times\omega_2^+$, $\gamma=\bar\omega\backslash\omega$.

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We shall need the averaging operators:

$$T_1u(x)=\frac{1}{h^2}\int\limits_{x_1-h}^{x_1+h}(h-|x_1-\xi_1|)u(\xi_1,x_2)d\xi_1,\quad S_1^-u(x)=\frac{1}{h}\int\limits_{x_1-h}^{x_1}u(\xi_1,x_2)d\xi_1,$$

T_2 and S_2^- are defined similarly.

Let H be the space of discrete functions $y=y(x)$, defined on the mesh ω, with the inner product $(y,v)\equiv(y,v)_\omega$ and the norm $\|y\|\equiv\|y\|_\omega$, where

$$(y,v)_{\widetilde{\omega}}=\sum_{x\in\widetilde{\omega}}h^2y(x)v(x),\quad \|y\|_{\widetilde{\omega}}=(y,y)_{\widetilde{\omega}}^{1/2},\ \widetilde{\omega}\subseteq\bar{\omega}.$$

Let $y^{(-1_1)}(x)=y(x_1-h,x_2)$, $\ y^{(-1_2)}(x)=y(x_1,x_2-h)$.

Further, in the space H we define the operators $\Lambda_\alpha,\partial_\alpha$ $\ \alpha=1,2$ in the following way: $\Lambda_\alpha y=\overset{\circ}{y}_{\bar{x}_\alpha x_\alpha}$, $\partial_\alpha y=\overset{\circ}{y}_{\overset{\circ}{x}_\alpha}$, $y\in H$ and $\overset{\circ}{y}(x)=y(x)$ for $x\in\omega$, $\overset{\circ}{y}(x)=0$ for $x\in\gamma$.

Let

$$\Lambda=-\Lambda_1-\Lambda_2.$$

It is linear, self-adjoint and positive definite operator in the Hilbert space H, besides [5]

$$(8/l_1^2+8/l_2^2)E\le\Lambda\le 8/h^2E,\quad Ey\equiv y. \tag{3}$$

Define the following norms of discrete functions

$$\|y\|_{C(\omega)}=\max_{x\in\omega}|y(x)|,\ \|y\|^2_{W_2^1(\omega)}=\|y_{\bar{x}_1}\|^2_{(1)}+\|y_{\bar{x}_2}\|^2_{(2)},\ \|y\|_{(\alpha)}=\|y\|_{\omega_{(\alpha)}},$$

$$\|y\|^2_{W_2^2(\omega)}=\|y_{\bar{x}_1x_1}\|^2+\|y_{\bar{x}_2x_2}\|^2+2\|y_{\bar{x}_1\bar{x}_2}\|^2_{\omega^+}.$$

It's easy to verify that

$$\|y\|_{W_2^1(\omega)}=(\Lambda y,y)^{1/2},\ \ \|y\|_{W_2^2(\omega)}=\|\Lambda y\|. \tag{4}$$

The problem (1) is approximated by the difference scheme

$$L_hy\equiv(A+B+C)y=\varphi,\ x\in\omega,\ y\in H, \tag{5}$$

where

$$\varphi(x)=T_1T_2\Big(f+\frac{h^2}{12}\sum_{\alpha=1}^{2}\lambda_\alpha\frac{\partial f}{\partial x_\alpha}\Big), \tag{6}$$

$$A=-\sum_{\alpha=1}^{2}\Lambda_\alpha\big(E+\frac{h^2}{12}\Lambda_{3-\alpha}\big),\quad C=\sum_{\alpha=1}^{2}\lambda_\alpha\partial_\alpha\big(E+\frac{h^2}{6}\Lambda_{3-\alpha}\big),$$

$$B=-\frac{h^2}{12}(\lambda_1^2\Lambda_1+\lambda_2^2\Lambda_2+2\lambda_1\lambda_2\partial_1\partial_2).$$

The following proposition is valid.

Lemma 1. *L_h is a positive definite operator in the space H and the estimates*

$$L_h \geq 2/3\Lambda, \tag{7}$$

$$\|\Lambda y\| \leq c\|L_h y\|, \tag{8}$$

$$\|L_h^{-1} y\| \leq c\|\Lambda^{-1} y\|, \tag{9}$$

where $c = (1 + 3\sqrt{2l_1l_2}\lambda/8)3/2, \ \ \lambda = \max(|\lambda_1|, |\lambda_2|)$, *hold.*

Proof. Taking into account (3), the positive definitness of the operator L_h follows from the estimate (7), which in its turn follows from easy relations

$$A = A^* \geq 2/3\Lambda, \ B = B^* \geq 0, \ C = -C^*$$

Further, applying

$$\|y_{\overset{\circ}{x}_1\overset{\circ}{x}_2}\| \leq \|y_{\bar{x}_1\bar{x}_2}\|_{\omega^+} = (\Lambda_1 y, \Lambda_2 y),$$

we obtain

$$(By, -\Lambda_\alpha y) \geq \frac{h^2}{12}(\lambda_\alpha^2\|\Lambda_\alpha y\|^2 + \lambda_{3-\alpha}^2\|y_{\bar{x}_1\bar{x}_2}\|_{\omega^+}^2 - 2|\lambda_1\lambda_2| \, \|\Lambda_\alpha y\| \, \|y_{\bar{x}_1\bar{x}_2}\|_{\omega^+}) \geq 0.$$

And as

$$(Ay, \Lambda y) \geq \frac{2}{3}\|\Lambda y\|^2,$$

then

$$((A+B)y, \Lambda y) \geq \frac{2}{3}\|\Lambda y\|^2$$

or

$$\|\Lambda y\| \leq \frac{3}{2}\|(A+B)y\|. \tag{10}$$

It is clear, that

$$\|Cy\| \leq \lambda \sum_{\alpha=1}^{2} \|(E + \frac{h^2}{6}\Lambda_{3-\alpha}\partial_\alpha)y\| \leq \sqrt{2}\lambda\|y\|_{W_2^1(\omega)}. \tag{11}$$

Besides, (7) and the discrete analogue of the Friedrichs inequality yields

$$\|y\|_{W_2^1(\omega)} \leq \frac{3\sqrt{l_1l_2}}{8}\|L_h y\| \tag{12}$$

and therefore the relation

$$\|Cy\| \leq \frac{3\sqrt{2l_1l_2}}{8}\lambda\|L_h y\| \tag{13}$$

follows from (11).

Substituting

$$A + B = L_h - C$$

into (10) and using (13), we obtain (8).

The estimate

$$\|\Lambda y\| \leq c\|L_h^* y\|, \ \forall y \in H \tag{14}$$

can be obtained similarly.

Further, $L_h^* L_h$ is self-adjoint and positive definite operator in H. Therefore

$$\|L_h^{-1} y\| = \|(L_h^* L_h)^{-1} L_h^* y\| = \sup_{v\neq 0} \frac{|(L_h^* y, v)|}{\|L_h^* L_h v\|} = \sup_{v\neq 0} \frac{|(\Lambda^{-1} y, \Lambda L_h v)|}{\|L_h^* L_h v\|} \leq$$

$$\leq \|\Lambda^{-1} y\| \sup_{v\neq 0} \frac{\|\Lambda L_h v\|}{\|L_h^* L_h v\|}$$

from here and the inequality (14) follows (9). This completes the proof of lemma 1. ■

For the error $z = y - u$ the folowing relation is true

$$L_h z = \psi, \ z \in H \tag{15}$$

The approximation error $\psi = \varphi - L_h u$ can be written in the form

$$\psi = \sum_{\alpha=1}^{2} (\Lambda_\alpha \eta_{\alpha 1} + \lambda_\alpha \Lambda_{3-\alpha} \eta_{\alpha 2} + \lambda_\alpha \eta_{\alpha 3 x_\alpha} + \lambda_\alpha^2 \Lambda_\alpha \eta_{\alpha 4}) + \lambda_1 \lambda_2 \eta_{x_1 x_2}, \tag{16}$$

where

$$\eta_{\alpha 1} = u + \frac{h^2}{12} \Lambda_{3-\alpha} u - T_{3-\alpha} u, \ \ \eta_{\alpha 2} = \frac{h^2}{6} \left(\frac{1}{2} (u + u^{(-1_\alpha)}) - S_\alpha^- u \right)_{x_\alpha},$$

$$\eta_{\alpha 3} = \frac{1}{2} (u + u^{(-1_\alpha)}) + \frac{h^2}{12} S_\alpha^- \Lambda_{3-\alpha} u - S_\alpha^- T_{3-\alpha} u - \frac{h^2}{12} T_{3-\alpha} \left(\frac{\partial u}{\partial x_\alpha} \right)_{\bar{x}_\alpha},$$

$$\eta_{\alpha 4} = \frac{h^2}{12} (u - T_{3-\alpha} u), \ \ \eta = \frac{h^2}{6} \left(\frac{1}{4} (u + u^{(-1_1)} + u^{(-1_2)} + u^{(-1_1, -1_2)}) - S_1^- S_2^- u \right)$$

From the positive definitness of operator L_h follows the unique solvability of the difference scheme (5), (6) and problem (15).

Lemma 2. *For the solution of problem* (15) *the following estimates are true*

$$\|z\| \leq c \left(\sum_{\alpha=1}^{2} (\|\eta_{\alpha 1}\| + \frac{\lambda l_\alpha}{\sqrt{8}} \|\eta_{\alpha 3}\|_{(\alpha)} + \lambda^2 \|\eta_{\alpha 4}\| + \lambda \|\eta_{\alpha 2}\|) + \lambda^2 \|\eta\|_{\omega^+} \right), \tag{17}$$

$$\|z\|_{W_2^1(\omega)} \leq \frac{3}{2} \Big(\sum_{\alpha=1}^{2} (\|\eta_{\alpha 1 \bar{x}_\alpha}\|_{(\alpha)} + \lambda \|\eta_{\alpha 3}\|_{(\alpha)} + \lambda^2 \|\eta_{\alpha 4 \bar{x}_\alpha}\|_{(\alpha)} +$$

$$+ \lambda \|\eta_{\alpha 2 \bar{x}_{3-\alpha}}\|_{(3-\alpha)}) + \lambda^2 \|\eta_{x_1}\|_{(2)} \Big), \tag{18}$$

$$\|\Lambda z\| \leq$$

$$\leq c \left(\sum_{\alpha=1}^{2} (\|\Lambda_\alpha \eta_{\alpha 1}\| + \lambda \|\eta_{\alpha 3 x_\alpha}\| + \lambda^2 \|\Lambda_\alpha \eta_{\alpha 4}\| + \lambda \|\Lambda_{3-\alpha} \eta_{\alpha 2}\|) + \lambda^2 \|\eta_{x_1 x_2}\| \right). \tag{19}$$

Proof. The inequality (17) can be obtained using the estimate (9). For establishing the a priori estimation (18) we multiply both parts of (15) scalarly by z and use the formulas of summing by parts and inequality (7). The estimate (19) follows immediately from (15) using (8). Thus, lemma 2 is proved. ■

The inequalities (16)-(18) with the generalization of Bramble-Hilbert lemma enable us to establish the validity of the following statement on the accuracy of difference scheme (5), (6).

Theorem 1. *Let the solution of the boundary value problem* (1) *belong to* $u \in W_2^m(\Omega),\ m \geq 2$. *Then for the solution* $y(x)$ *of the difference scheme* (5), (6) *the estimate of convergence* (2) *with* $m \in [2; 4+s],\ s = 0, 1, 2$ *holds.*

Let now the right-hand side in (5) be defined by the equality

$$\varphi = f + \frac{h^2}{12}(\Lambda_1 + \Lambda_2 + \lambda_1\partial_1 + \lambda_2\partial_2)f. \tag{20}$$

The difference scheme (5), (20) represents the scheme from [1,2].
The following statement is true.

Theorem 2. *Let the solution of the problem* (1) *belong to the space* $W_2^m(\Omega),\ m > 3$. *Then the convergence rate of difference scheme* (5), (20) *is given by estimate* (2) *with* $s = 2, m \in (3; 6]$.

As it was marked in [1], the equation of the type (1) often arise in problems of hydrodinamics at linearization of the equation of movement and it is desirable for the corresponding difference scheme to have enough accuracy for big values of λ_α. Therefore it is a matter of interest to represent the estimate of convergence taking into acount the value λ. Such estimate, for instance in the norm $W_2^2(\omega)$ for the scheme (5), (6) has the form

$$\|y - u\|_{W_2^2(\omega)} \leq Mh^{m-2}(1 + \lambda h^{\varkappa_1} + \lambda^2 h^{\varkappa_2})\|u\|_{W_2^m(\Omega)},\ m \in (3; 6],$$

where $\varkappa_1 = \min(1; 6-m),\ \varkappa_2 = \min(2; 6-m)$.

Remark 1. Using the imbedding $W_2^1(\omega)$ into $C(\omega)$ for the mesh functions of two variables with coefficient $|\ln h|^{0.5}$ (*see* [6]), we obtain the estimate

$$\|y - u\|_{C(\omega)} \leq M|\ln h|^{0.5}h^{m-1}\|u\|_{W_2^m(\Omega)}$$

for the schemes (5), (6) and (5), (20) where respectively $m \in [2; 5)$ and $m \in (3; 5]$. Using the imbedding of $W_2^2(\omega)$ into $C(\omega)$ (*see* [5]), for the both schemes we obtain

$$\|y - u\|_{C(\omega)} \leq Mh^{m-2}\|u\|_{W_2^m(\Omega)}$$

with $m \in [2; 6]$ and $m \in (3; 6]$ respectively.

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