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SOME WEIGHTED HARDY-TYPE INEQUALITIES AND APPLICATIONS

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Abstract. We study the two-weighted estimate

$$\Big\| \sum_{k=0}^{n} a_k(x) \int_0^x t^k f(t)dt | L_{q,v}(0,\infty) \Big\| \le c\|f|L_{p,u}(0,\infty)\|, \qquad (*)$$

where the functions $a_k(x)$ are assumed to be non-positive. It is shown that for $1 < p \le q \le \infty$, provided that the weight u satisfies the certain conditions, the estimate $(*)$ holds if and only if the estimate

$$\sum_{k=0}^{n} \Big\| a_k(x) \int_0^x t^k f(t)dt | L_{q,v}(0,\infty) \Big\| \le c\|f|L_{p,u}(0,\infty)\|. \qquad (**)$$

is fulfilled. The necessary and sufficient conditions for $(**)$ to be valid are well-known. The obtained result can be applied to the estimates of differential operators with variable coefficients in some weighted Sobolev spaces.

1. Introduction

The problem of finding the necessary and sufficient conditions imposed on the functions $u(x)$, $v(x)$, for which the estimate of the kind

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$$\left(\int_0^\infty \left|v(x)\int_0^x A(x,t)f(t)dt\right|^q dx\right)^{1/q} \le c\left(\int_0^\infty |u(x)f(x)|^p dx\right)^{1/p} \tag{1.1}$$

are valid, attracted great attention for the last decade (see, e.g., a wide bibliography in the work of V. D. Stepanov [1]). It is well known that the cases $p > q$ and $p \le q$ are in principle distinct. Here and everywhere below we assume that $p \le q$.

For $1 \le p \le q \le \infty$ and $A(x,t) \equiv 1$ the necessary and sufficient condition for validity of the estimate (1.1) has been obtained by J. S. Bradley [2] and V. M. Kokilashvili [3]. The first substantial progress for $A(x,t) \not\equiv 1$ has been reached in their papers by V. D. Stepanov [4], [5] and F. J. Martin–Reyes and E. Sawyer [6] where they investigated the case $A(x,t) = (x-t)^\alpha$, $\alpha > 0$.

For the present the most general classes of kernels $A(x,t)$ are apparently considered by R. Oinarov [7]. The kernels of these classes are positive and satisfy some additional different types restrictions of which the most known is "the Oinarov's condition":

$$c_1 A(x,t) \le A(x,y) + A(y,t) \le c_2 A(x,t), \quad t < y < x, \quad c_1, c_2 > 0.$$

We should like to emphasize the following fact. As far as our knowledge goes, until recently no work has been available in which the criteria for the validity of (1.1) with the kernels $A(x,t)$ of alternating signs would have been considered.

In the recent work [8] the author has, however, shown some class of kernels with alternating signs for which we managed to characterize the admissible weights in (1.1) for $p = q = 2$ provided that the weight u undergoes some restrictions. In the present work we extend this result to all $1 < p \le q \le \infty$ and weaken the restriction to u. As an application we studied in [8] the problem of description of pointwise multipliers in some weighted Sobolev spaces. As an application we solve here more general problem of finding the criteria of restrictive action for differential operators with variable coefficients from the same weighted Sobolev spaces to the weighted spaces L_q.

The paper is presented as follows: the basic results are formulated in §2. The proofs are given in §3 and §4. All the functions are assumed to be measurable and finite almost everywhere.

2. Results

Let $\mathbb{R}^+ = (0, \infty)$ be a half-line, $1 \le p \le \infty$, and u be a non-negative function on $\mathbb{R}$ (weight). Denote by $L_{p,u}$ a weight space of functions $f : \mathbb{R}^+ \to \mathbb{C}$ with a norm

$$\|f|L_{p,u}\| = \left(\int\limits_0^\infty |u(x)f(x)|^p dx \right)^{1/p}, \quad 1 \le p < \infty,$$
$$\|f|L_{\infty,u}\| = \operatorname*{ess\,sup}_{x>0} |u(x)f(x)|.$$

In what follows the last modification is meant everywhere. We denote by p' a conjugate exponent: $p' = p/(p-1)$. The main statement of the present paper is the following

Theorem 2.1. *Given $n \in \mathbb{N}$ and the functions $a_k : \mathbb{R}^+ \to \mathbb{C}$, $k = 0, \dots, n$, let $1 < p \le q \le \infty$, u, v be two non-negative functions on $\mathbb{R}^+$. Moreover there exists a constant $D(u)$ such that*

$$\int\limits_0^r u^{-p'}(x)dx < \infty \quad \forall r > 0, \tag{2.1}$$

$$\int\limits_0^{2r} x^{(n-1)p'} u^{-p'}(x)dx \le$$
$$\le D(u) \int\limits_0^r x^{(n-1)p'} u^{-p'}(x)dx \quad \forall r > 0. \tag{2.2}$$

Then the following three assertions are equivalent:

(i) *The estimate*

$$\left\| \sum_{k=0}^n a_k(x) \int\limits_0^x t^k f(t)dt |L_{q,v} \right\| \le c\|f|L_{p,u}\|$$

holds;

(ii) *the estimates*

$$\left\| a_k(x) \int\limits_0^x t^k f(t)dt |L_{q,v} \right\| \le c_k\|f|L_{p,u}\|, \quad k = 0, \dots, n,$$

hold;

(iii) *the values*

$$S_k = \sup_{r>0} \left(\int_r^\infty |a_k(x)v(x)|^q dx \right)^{1/q} \cdot \left(\int_0^r x^{-kp'} u^{-p'}(x) dx \right)^{1/p'} < \infty,$$
$$k = 0, \dots, n,$$

are finite.

If $\widetilde{C}$ is the best constant in (i), *then*

$$K_1(p,q)\widetilde{c} \le \sum_{k=0}^n S_k \le K_2(D(u), n, p, q)\widetilde{c}.$$

Remark 1. For $p = q = 2$ and under more restrictive condition to u of the kind

$$\int_\Delta u^{-2}(x)dx \le D(u) \int_{\frac{1}{2}\Delta} u^{-2}(x)dx,$$

where Δ runs through all segments in $\mathbb{R}^+$ ($\frac{1}{2}\Delta$ is twice as less segment with the same center), this theorem has been proved by the author in [8].

Remark 2. There exist weights u not satisfying (2.2) for which the assertion of Theorem 2.1 does not hold. For example, for $u(x) = e^{-x}$ the estimate

$$\left\| x \int_0^x f(t)dt - \int_0^x tf(t)dt | L_{2,e^{-x}} \right\| \le c \|f|L_{2,e^{-x}}\|$$

is valid, at the time as the corresponding estimates for either summands in the left-hand side does not hold separately. For details refer to the author's work [8, Prop. 1.2] and also to A. Kufner [9, Example 1].

As an application to Theorem 2.1 we consider the estimates for differential operators with variable coefficients in the weighted Sobolev spaces. For $l \in \mathbb{N}$ we distinguish two types of such spaces:

$$W^l_{p,u} = \left\{ f : \mathbb{R}^+ \to \mathbb{C} \;\Big|\; \|f|W^l_{p,u}\| = \sum_{k=0}^{l-1} |f^{(k)}(0)| + \|f^{(l)}|L_{p,u}\| < \infty \right\},$$

and

$$\overset{\circ}{W}{}^l_{p,u} = \Big\{ f : \mathbb{R}^+ \to \mathbb{C} | f^{(k)}(0) = 0, \quad k = 0, \dots, l-1,$$
$$\|f|\overset{\circ}{W}{}^l_{p,u}\| = \|f^{(l)}|L_{p,u}\| < \infty \Big\}.$$

Under the notation $f^{(l)}$ we mean that the function f has an absolutely continuous derivative $f^{(l-1)}$. Then $f^{(l)} = (f^{(l-1)})'$ exists almost everywhere. In the sequel, the condition (2.1) is assumed to be fulfilled. In this case $\|f^{(l)}|L_{p,u}\| < \infty$ implies the existence of $f^{(k)}(0)$, $k = 0, \dots, l-1$, which may be understood to be the limits $f^{(k)}(0+0)$, and the definitions of the spaces $W^l_{p,u}$ and $\overset{\circ}{W}{}^l_{p,u}$ become correct. Note that all polynomials of degree $\le l-1$ belong to $W^l_{p,u}$. The spaces $W^l_{p,u}$ have been introduced and studied by L. D. Kudryavtsev [10].

Let us consider a differential operator of the l-th order with variable coefficients

$$P(x, D) = \sum_{m=0}^{l} b_m(x) D^m,$$

which acts by the rule

$$P(x, D) f(x) = \sum_{m=0}^{l} b_m(x) f^{(m)}(x).$$

Introduce the notation

$$d_k(x) = \sum_{m=0}^{l-1-k} b_m(x) \frac{x^{l-1-k-m}}{(l-1-k-m)!}, \quad k = 0, \dots, l-1.$$

The following theorem is valid.

Theorem 2.2. *Let $l \in \mathbb{N}$, $1 < p \le q \le \infty$ and u, v be non-negative on $\mathbb{R}^+$ functions and let the condition* (2.1) *be fulfilled. If $l \ge 2$, then let the condition* (2.2) *with $n = l-1$ be also fulfilled.*

(i) *In order for*

$$P(x, D) : \overset{\circ}{W}{}^l_{p,u} \to L_{q,v},$$

it is necessary and sufficient that the conditions

$$\sup_{r>0} \left(\int_r^\infty |d_k(x) v(x)|^q dx \right)^{1/q} \left(\int_0^r x^{kp'} u^{-p'}(x) dx \right)^{1/p'} < \infty, \qquad (2.3)$$
$$k = 0, \dots, l-1,$$

and

$$b_l(x) \equiv 0 \quad \text{for } p < q, \quad \|b_l v u^{-1} | L_\infty(\mathbb{R}^+)\| < \infty \quad \text{for } p = q \qquad (2.4)$$

be fulfilled.

(ii) *In order that*

$$P(x, D) : W^l_{p,u} \to L_{q,v},$$

it is necessary and sufficient that the conditions (2.3) *and* (2.4) *be fulfilled and also that*

$$\|P(x, D)x^k | L_{q,v}\| < \infty, \quad k = 0, \dots, l-1. \tag{2.5}$$

Remark. The theorem given above generalizes Theorem 2.4 on pointwise multipliers in $W^l_{p,u}$ presented in the author's work [8] in which was considered the case $p = q = 2$, and $P(x, D)f = (\varphi f)^{(m)}$, $0 \le m \le l$.

In conclusion let us formulate the following unsolved problem: Find for $l \ge 2$ the necessary and sufficient conditions imposed on the function φ under which

$$\|\varphi f | W^l_{2,e^{-x}}\| \le c\|f | W^l_{2,e^{-x}}\| \quad \forall f \in W^l_{2,e^{-x}}.$$

In other words, it is required to describe pointwise multipliers in the space $W^l_{2,e^{-x}}$. Theorem 2.4 in [8] does not answer this question. Note that for $l = 1$, $1 \le p \le \infty$ and for arbitrary weights u, v we can show that the estimate

$$\|\varphi f | W^1_{p,v}\| \le c\|f | W^1_{p,u}\| \quad \forall f \in W^1_{p,u}$$

holds if and only if the function φ satisfies the conditions

$$\|\varphi v u^{-1} | L_\infty(\mathbb{R}^+)\| < \infty,$$
$$\sup_{r>0} \left(\int_r^\infty |\varphi'(x)v(x)|^p dx \right)^{1/p} \left(\int_0^r u^{-p'}(x) dx \right)^{1/p'} < \infty.$$

3. Proof of Theorem 2.1

Lemma 3.1. *Let X be a Banach space, X^* be its conjugate and let $Y \subset X$ be a closed subspace. Moreover, let $e \in X$, $e \notin Y$ be a fixed vector. Then*

(i) $\sup\{\langle y^*, e\rangle : y^*|_Y = 0,\ \|y^* | X^*\| = 1\} = \operatorname{dist}(e, Y)$.

(ii) *If there exists a vector $y_0 \in Y$ such that* $\operatorname{dist}(e, Y) = \|e - y_0\|$, *then there also exists a functional $y_0^* \in X^*$, $y_0^*|_Y = 0$, $\|y_0^* | X^*\| = 1$ such that*

$$\langle y_0^*, e\rangle = \|e - y_0\|.$$

Remark. The symbol $\langle y^*, e\rangle$ denotes the value of the functional $y^* \in X^*$ on the vector $e \in X$. The writing $y^*|_Y = 0$ means that $\langle y^*, y\rangle = 0$ $\forall y \in Y$. Item (i) of the above lemma is formulated as Exercise 19 in Chapter 4 of W. Rudin's book [11].

Proof. Let $y \in Y$, $y^*|_Y = 0$, $\|y^*|X^*\| = 1$. We see that

$$\langle y^*, e\rangle = \langle y^*, e - y\rangle \le \|e - y\| \le \operatorname{dist}(e, Y),$$

whence

$$\sup\{\langle y^*, e\rangle : y^*|_Y = 0, \quad \|y^*\| = 1\} \le \operatorname{dist}(e, Y). \tag{3.1}$$

Next, since $e \notin Y$, by the Hahn–Banach theorem (see W. Rudin [11, Th.3.3]), for every $y \in Y$ there exists $y^* \in X^*$ such that $\|y^*\| = 1$, $y^*|_Y = 0$ and we achieve the norm of the functional y^* on the vector $e - y$, i.e.,

$$\|e - y\| = \langle y^*, e - y\rangle = \langle y^*, e\rangle.$$

This implies that the inverse inequality in (3.1) is valid and hence item (i) is fulfilled. Item (ii) is proved simultaneously. ■

Lemma 3.2. *Let w be a positive function on $\mathbb{R}^+$; moreover,*

$$\int_0^r w(x)dx < \infty \qquad \forall r > 0.$$

Let $1 \le s < \infty$, $P_{n,r}(x)$ be a polynomial of the n-th degree with a highest degree term x^n, satisfying the conditions

$$\int_0^r |P_{n,r}(x)|^{s-1} \operatorname{sign} P_{n,r}(x) x^k w(x)dx = 0, \quad k = 1, \dots, n.$$

Then

(i) *all roots of the polynomial $P_{n,r}(x)$ are simple, real and lying on the interval $(0, r)$.*

(ii) *Let, in addition, the condition*

$$\int_0^{2r} w(x)dx \le c \int_0^r w(x)dx \qquad \forall r > 0 \tag{3.2}$$

be fulfilled with some constant $c = c(w)$. Let $x_1(r)$ be the least root of the polynomial $P_{n,r}(x)$. Then

$$\int_0^r w(x)dx \le K \int_0^{x_1(r)} w(x)dx \quad \forall r > 0, \tag{3.3}$$

for some constant $K = K(c, n, s)$.

Remark 3. The polynomial $P_{n,r}(x)$ does exist for every $n \in \mathbb{N}$ and $r > 0$, and is the solution of the extremal problem

$$\int_0^r |P(x)|^s x w(x)dx \to \min_P,$$

where $P(x)$ runs through all polynomials with the highest degree term x^n. The interested reader is referred to [12], §2.1.1. Therein is proved item (i) (for $w \equiv 1$, but the proof works also in the general case).

Proof. It remains to prove (ii). Let $x_1, \dots, x_n \in (0, r)$ be the roots of $P_{n,r}(x)$ enumerated in the increasing order. Suppose also that $x_{n+1} = r$. We shall prove that

$$\int_0^{x_{m+1}} wdx \le K_1 \int_0^{x_m} wdx \quad \text{for all} \quad m = 1, \dots, n. \tag{3.4}$$

This will imply the required inequality (3.3) with the constant $K = K_1^n$.

For $x_{m+1} \le 4x_m$ the inequality (3.4) follows from (3.2) with $K_1 = c^2$. Let now $x_{m+1} > 4x_m$. Taking the polynomial

$$R(x) = \prod (x - x_k),$$

where the product is taken over all $k \in \{1, \dots, n\} \backslash \{m\}$, by the definition of $P_{n,r}$ we have

$$\int_0^r |P_{n,r}|^{s-1} \operatorname{sign}(P_{n,r}) Rxwdx = 0,$$

which can be written as

$$\int_{x_m}^r \prod_{k=1}^{m-1} |x - x_k|^s |x - x_m|^{s-1} \prod_{k=m+1}^{n} |x - x_k|^s xwdx = \int_0^{x_m} \text{from the same} \tag{3.5}$$

(with obvious modifications for $m = 1$ and $m = n$).

By virtue of simple estimates

$$\left.\begin{array}{l} |x-x_k| \ge x_m, \quad k=1,\dots,m \\ |x-x_k| \ge x_k/2, \quad k=m+1,\dots,n \\ x \ge x_m \end{array}\right\} \quad x\in[2x_m, x_{m+1/2}]$$

$$\left.\begin{array}{l} |x-x_k| \le x_m, \quad k=1,\dots,m \\ |x-x_k| \le x_k, \quad k=m+1,\dots,n \\ x \le x_m \end{array}\right\} \quad x\in[0, x_m].$$

From (3.5) we have

$$\int\limits_{2x_m}^{x_{m+1/2}} w\,dx \le 2^{s(n-m-1)} \int\limits_0^{x_m} w\,dx.$$

Therefore

$$\int\limits_0^{x_{m+1}} w \le c \int\limits_0^{x_{m+1/2}} w = c\int\limits_0^{2x_m} w + c\int\limits_{2x_m}^{x_{m+1/2}} w \le (c^2 + c2^{s(n-m-1)}) \int\limits_0^{x_m} w. \qquad \blacksquare$$

Lemma 3.3. *Let n, $a_k(x)$, $k=0,\dots,n$, p, q, $v(x)$ be the same as in Theorem* 2.1. *Let the positive on $\mathbb{R}^+$ function $u(x)$ satisfy the condition* (2.1) *and*

$$\int\limits_0^{2r} u^{-p'}(x)dx \le D\int\limits_0^{r} u^{-p'}(x)dx \quad \forall r>0$$

(*which is weaker than* (2.2)). *Let the estimate* (i) *with the constant $c<\infty$ from Theorem* 2.1 *holds. Then*

$$S_0 = \sup_{r>0}\left(\int\limits_r^\infty |a_0(x)v(x)|^q dx\right)^{1/q}\left(\int\limits_0^r u^{-p'}(x)dx\right)^{1/p'} \le$$

$$\le C(D,n,p)c < \infty. \tag{3.6}$$

Proof. Consider the extremal problem

$$\begin{cases} \dfrac{\int\limits_0^r f(t)dt}{\left(\int\limits_0^r |u(t)f(t)|^p dt\right)^{1/p}} \to \max\limits_f, \\ \int\limits_0^r t^k f(t)dt = 0, \quad k=1,\dots,n. \end{cases} \tag{3.7}$$

By substitution $f(t) = u^{-p'}(t)g(t)$, this problem reduces to more convenient one:

$$\begin{cases} \dfrac{\int\limits_0^r g(t)u^{-p'}(t)dt}{\Big(\int\limits_0^r |g(t)|^p u^{-p'}(t)dt\Big)^{1/p}} \to \max\limits_g, \\ \int\limits_0^r g(t)t^k u^{-p'}(t)dt = 0, \quad k = 1\dots,n. \end{cases} \tag{3.8}$$

Let $L_p(A, d\mu)$ denote a space of functions $f : A \to \mathbb{C}$ with the norm $(\int_A |f|^p d\mu)^{1/p}$. Let $X = L_{p'}([0,r], u^{-p'}(t)dt)$, Y be a finite-dimensional subspace X with a basis form by the functions $t, t^2, \dots, t^n$. Note that $X^* = L_p([0,r], u^{-p'}(t)dt)$, the pairing between $f \in X$ and $g \in X^*$ being defined by the formula

$$\langle g, f\rangle = \int\limits_0^r g(t)f(t)u^{-p'}(t)dt.$$

If we denote by e a vector of the space X representing the function $e(x) \equiv 1$, then the extremal problem (3.8) can be written in an abstract manner:

$$\begin{cases} \dfrac{\langle g, e\rangle}{\|g|X^*\|} \to \max\limits_{g\in X^*}, \\ g|_Y = 0. \end{cases}$$

By Lemma 3.1 (i) we have

$$\sup\left\{\frac{\langle g, e\rangle}{\|g|X^*\|} : g|_Y = 0\right\} = \operatorname{dist}(e, Y). \tag{3.9}$$

Moreover, since the subspace Y is finite-dimensional, $\operatorname{dist}(e, Y)$ is by all means realized at some vector $y_0 \in Y$. Hence by Lemma 3.1 (ii), sup is attained in the right-hand side of (3.9), that is there exists $g_0 \in X^*$, $g_0|_Y = 0$ such that

$$\frac{\langle g_0, e\rangle}{\|g_0|X^*\|} = \min_{c_1,\dots,c_n}\left(\int\limits_0^r |1 + c_1 t + \cdots c_n t^n|^{p'} u^{-p'} dt\right)^{1/p'}. \tag{3.10}$$

Assume $P(t) = 1 + c_1 t + \cdots + c_n t^n$, $w(t) = u^{-p'}(t)$. The extremum conditions in the right-hand side of (3.10) have the form

$$\int\limits_0^r |P(t)|^{p'-1} \operatorname{sign} P(t) t^k w(t) dt = 0, \quad k = 1, \dots, n.$$

Therefore it is clear that the extremal polynomial $\widetilde{P}(t)$ differs from the polynomial $P_{n,r}(t)$ from Lemma 3.2 (with $s = p'$) by the factor only. According to this lemma (noting that the condition (3.2) is fulfilled), all the roots of $\widetilde{P}(t)$ are located on $(0, r)$ and the following inequality holds:

$$\int\limits_0^r u^{-p'}(t) dt \le K \int\limits_0^{x_1} u^{-p'}(t) dt \quad \forall r > 0,$$

where x_1 is the least of the roots and the constant K does not depend on r. Using this fact, we derive from (3.10) the following estimate:

$$\frac{\langle g_0, e\rangle}{\|g_0|X^*\|} = \left(\int\limits_0^r |\widetilde{P}(t)|^{p'} u^{-p'}(t) dt \right)^{1/p'} =$$

$$= \left(\int\limits_0^r \prod_{k=1}^n \left| 1 - \frac{x}{x_k} \right|^{p'} u^{-p'}(t) dt \right)^{1/p'} \ge$$

$$\ge \left(\int\limits_0^{x_1} \left(1 - \frac{x}{x_k} \right)^{np'} u^{-p'}(t) dt \right)^{1/p'} \ge$$

$$\ge \left(2^{-np'} \int\limits_0^{x_1/2} u^{-p'}(t) dt \right)^{1/p'} \ge \frac{2^{-n}}{(DK)^{1/p'}} \left(\int\limits_0^r u^{-p'}(t) dt \right)^{1/p'}.$$

Getting back to the original extremal problem (3.7), we see that for every $r > 0$ there exists on $(0, r)$ a function $f_0(t) = u^{-p'}(t) g_0(t)$ such that

$$\int\limits_0^r f_0(t) dt \ge \widetilde{K}(D, n, p) \left(\int\limits_0^r u^{-p'}(t) dt \right)^{1/p'} \left(\int\limits_0^r |u(t) f_0(t)|^p dt \right)^{1/p},$$

$$\int\limits_0^r t^k f_0(t)\, dt = 0, \quad k = 1, \dots, n.$$

Extending f_0 on $[r, \infty)$ by zero and substituting the obtained function into the estimate (i) of Theorem 2.1, we get (3.6) with $C = 1/\widetilde{K}$. ■

Lemma 3.4. *Let $1 \le p \le q \le \infty$, u, v be non-negative on $\mathbb{R}^+$ functions. The estimate*

$$\left\| \int_0^x f(t)\,dt | L_{q,v} \right\| \le c \| f | L_{p,u} \| \tag{3.11}$$

holds if and only if

$$S = \sup_{r>0} \left(\int_r^\infty v^q(x)\,dx \right)^{1/q} \left(\int_0^r u^{-p'}(x)\,dx \right)^{1/p'} < \infty.$$

Moreover, if $\widetilde{c}$ is the best constant in (3.11)*, then $S \le \widetilde{c} \le (q')^{1/p'} q^{1/q} S$. If $p = 1$ or $q = \infty$, then $\widetilde{c} = S$.*

This is the well known criterion for the validity of the weighted Hardy's inequality belonging to J. S. Bradley [2] and V. M. Kokilashvili [3]. The proof can be found in [13], §1.3.1.

Proof of Theorem 2.1. Implication (ii)⇒(i) is obvious. Equivalence (ii)⇔(iii) follows from Lemma 3.4. To complete the proof we have to show that (i)⇒(iii).

We act by induction with respect to n. For $n = 0$ the assertion of the theorem follows from Lemma 3.4. Assume that the theorem is already proved for $n = n_0$ and that the following estimate holds:

$$\left\| \sum_{k=0}^{n_0+1} a_k(x) \int_0^x t^k f(t)\,dt | L_{q,v} \right\| \le c \| f | L_{p,u} \|, \tag{3.12}$$

where the function u satisfies the condition (2.2) with $n = n_0+1$. Lemma 3.3 implies that $S_0 < \infty$. By Lemma 3.4 this is equivalent to the fact that the inequalities

$$\left\| a_0(x) \int_0^x f(t)\,dt | L_{q,v} \right\| \le c' \| f | L_{p,u} \| \tag{3.13}$$

are fulfilled. Inequalities (3.12) and (3.13) yield

$$\left\| \sum_{k=1}^{n_0+1} a_k(x) \int_0^x t^k f(t)\,dt | L_{q,v} \right\| \le c'' \| f | L_{p,u} \|,$$

which by substitution $\widetilde{f}(t) = tf(t)$ reduces to the form

$$\left\| \sum_{k=0}^{n_0} a_{k+1}(x) \int_0^x t^k \widetilde{f}(t)\, dt | L_{q,v} \right\| \leq c'' \|\widetilde{f}|L_{p,\widetilde{u}}\|,$$

where $\widetilde{u}(x) = x^{-1}u(x)$. Since $\widetilde{u}$ satisfies the condition (2.2) with $n = n_0$, by assumption of the induction we obtain the finiteness of the rest constants S_k, $k = 1, \ldots, n_0 + 1$. ■

4. Proof of Theorem 2.2

Lemma 4.1. *Let $[a,b]$ be a segment in $\mathbb{R}$. For any set of functions $h_1, \ldots, h_l \in L_1([a,b])$ there exists a function σ with $|\sigma(x)| = 1$ on $[a,b]$ such that*

$$\int_a^b h_k(x)\sigma(x)\, dx, \quad k = 1, \ldots, l.$$

The proof of this lemma can be found in [14], p. 267.

Lemma 4.2 ([15]). *Let $l \in \mathbb{N}$, $w(x)$ be a non-negative on the segment $[a,b]$ function with $\int_a^b w dx < \infty$. There exists a function $g(x)$, $x \in \mathbb{R}$, such that*
(a) $g(x) = 0$ *for* $x \notin [a,b]$,
(b) $g, g', \ldots, g^{(l-1)}$ *are absolutely continuous on* $\mathbb{R}$,
(c) $|g^{(l)}(x)| = w(x)$, $x \in [a,b]$.

Proof. By Lemma 4.1 there exists a function σ with $|\sigma(x)| = 1$ on $[a,b]$ such that

$$\int_a^b x^k w(x)\sigma(x)\, dx = 0, \quad k = 0, \ldots, l-1.$$

Then

$$g(x) = \begin{cases} \dfrac{1}{(l-1)!} \displaystyle\int_a^x (x-t)^{l-1} w(t)\sigma(t)\, dt, & x \in [a,b], \\ 0, \quad x \notin [a,b], \end{cases}$$

is the required function. ■

Proof of Theorem 2.2.

Step 1. Let $T_f(x) = \sum\limits_{k=0}^{l-1} f^{(k)}(0)x^k/k!$ be the Teylor's polynomial of the function f. Then

$$P(x,D)f(x) = P(x,D)\left(T_f(x) + \int\limits_0^x \frac{(x-t)^{l-1}}{(l-1)!}\, f^{(l)}(t)\,dt\right) =$$

$$= P(x,D)T_f(x) + \sum_{m=0}^{l} b_m(x)D^m \int\limits_0^x \frac{(x-t)^{l-1}}{(l-1)!}\, f^{(l)}(t)\,dt =$$

$$= P(x,D)T_f(x) + \sum_{m=0}^{l-1} b_m(x) \int\limits_0^x \frac{(x-t)^{l-m-1}}{(l-m-1)!}\, f^{(l)}(t)\,dt + b_l(x)f^{(l)}(x) =$$

$$= P(x,D)T_f(x) + \sum_{m=0}^{l-1}\sum_{k=0}^{l-m-1} b_m(x)\,\frac{x^{l-m-1-k}(-1)^k}{(l-m-1-k)!k!} \int\limits_0^x t^k f^{(l)}(t)\,dt +$$

$$+ b_l(x)f^{(l)}(x).$$

Changing in the last expression the order of summation with respect to m and k, we arrive at the formula

$$P(x,D)f(x) = P(x,D)T_f(x) +$$

$$+ \sum_{k=0}^{l-1} \frac{(-1)^k}{k!}\, d_k(x) \int\limits_0^x t^k f^{(l)}(t)dt + b_l(x)f^{(l)}(x). \tag{4.1}$$

Step 2 (*Sufficiency*). From the identity (4.1) there follows the estimate

$$\|P(x,D)f(x)|L_{q,v}\| \le c\sum_{k=0}^{l-1} |f^{(k)}(0)| \cdot \|P(x,D)x^k|L_{q,v}\| +$$

$$+ c\sum_{k=0}^{l-1} \left\| d_k(x) \int\limits_0^x t^k f^{(l)}(t)\,dt \Big| L_{q,v} \right\| + \|b_l f^{(l)}|L_{q,v}\|. \tag{4.2}$$

If conditions (2.3) are fulfilled, then by Lemma 3.4 the second term in the right-hand side (4.2) does not exceed $c'\|f^{(l)}|L_{p,u}\|$. The third term for $p = q$ can be estimated in an obvious manner:

$$\|b_l f^{(l)}|L_{p,v}\| \le \|b_l v u^{-1}|L_\infty(\mathbb{R}^+)\| \cdot \|f^{(l)}|L_{p,u}\|.$$

All the arguments cited at this step imply the sufficiency of conditions in both parts of Theorem 2.2.

Step 3 (*Necessity*). The necessity of the condition (2.5) in item (ii) is obvious. The theorem will be proved if we show that from

$$P(x,D):\overset{\circ}{W}{}^{(l)}_{p,u}\to L_{q,v} \tag{4.3}$$

there follows the fulfilment of (2.3) and (2.4).

We proceed from (2.4). Consider the set $A_\alpha=\{x\in\mathbb{R}^+ : |b_l v u^{-1}(x)|\ge\alpha\}$. Suppose $\operatorname{mes} A_\alpha>0$. Let B be a bounded subset of A of positive measure on which the functions $v(x)d_k(x)$ are bounded, say $B\subset[0,M]$, $|v(x)d_k(x)|\le N$, $x\in B$, $k=0,\dots,l-1$. For every $\varepsilon>0$ there exists a segment $\Delta_\varepsilon\subset\mathbb{R}^+$ of the length ε such that $\operatorname{mes}\Delta_\varepsilon\cap B>0$. By Lemma 4.2, there exists l times differentiable function g with a support on Δ_ε such that

$$|g^{(l)}(x)|=\begin{cases}u^{-p'}(x) & \text{if } x\in\Delta_\varepsilon\cap B,\\ 0 & \text{otherwise.}\end{cases}$$

The inequality

$$\|P(x,D)g|L_{q,v}\|\le K\|g^{(l)}|L_{p,u}\|$$

and the identity (4.1) yield the estimate

$$\|b_l f^{(l)}|L_{q,v}\|\le K\|g^{(l)}|L_{p,u}\|+c\sum_{k=0}^{l-1}\left\|d_k(x)\int\limits_0^x t^k g^{(l)}(t)\,dt|L_{q,v}\right\|. \tag{4.4}$$

Using Hölder's inequality and taking into account the inclusion $\Delta_\varepsilon\cap B\subset A_\alpha$, we can see that

$$\|b_l f^{(l)}|L_{q,v}\|=\Bigg(\int\limits_{\Delta_\varepsilon\cap B}|b_l u^{-p'}v|^q dx\Bigg)^{1/q}\ge$$
$$\ge(\operatorname{mes}\Delta_\varepsilon\cap B)^{1/q-1/p}\Bigg(\int\limits_{\Delta_\varepsilon\cap B}|b_l u^{-p'}v|^p dx\Bigg)^{1/p}\ge$$
$$\ge(\operatorname{mes}\Delta_\varepsilon\cap B)^{1/q-1/p}\alpha\Bigg(\int\limits_{\Delta_\varepsilon\cap B}u^{-p'}(x)\,dx\Bigg)^{1/p}. \tag{4.5}$$

Further,

$$\|g^{(l)}|L_{p,u}\| = \Bigg(\int\limits_{\Delta_\varepsilon \cap B} u^{-p'}(x)\,dx \Bigg)^{1/p}, \tag{4.6}$$

$$\Bigg\| d_k(x) \int\limits_0^x t^k g^{(l)}(t)\,dt | L_{q,v} \Bigg\| \le NM^k \Bigg(\int\limits_{\Delta_\varepsilon \cap B} u^{-p'}(t) dt \Bigg) \times (\operatorname{mes} \Delta_\varepsilon \cap B)^{1/q}. \tag{4.7}$$

Substituting (4.5)–(4.7) into (4.4) we obtain

$$\alpha \le (\operatorname{mes} \Delta_\varepsilon \cap B)^{1/p-1/q} \times$$
$$\times \Bigg(K + cNM^{l-1} \Bigg(\int\limits_{\Delta_\varepsilon \cap B} u^{-p'}(t)\,dt \Bigg)^{1/p'} (\operatorname{mes} \Delta_\varepsilon \cap B)^{1/q} \Bigg). \tag{4.8}$$

We pass to the limit as $\varepsilon \to 0$. Then for $p < q$ from (4.8) there follows that $\alpha = 0$, while for $p = q$ we shall have $\alpha \le K$. Thus the proof of (2.4) is complete.

By virtue of the identity (4.1) and also from (4.3) and (2.4) it follows that the estimate

$$\Bigg\| \sum_{k=0}^{l-1} \frac{(-1)^k}{k!}\, d_k(x) \int\limits_0^x t^k f^{(l)}(t)\,dt | L_{q,v} \Bigg\| \le \|f^{(l)}|L_{p,u}\|$$

holds.

Theorem 2.1 implies (2.3). ■

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