Proceedings of A.Razmadze
Mathematical Institute,
Vol. **112**, 1997

TWO-WEIGHT INEQUALITIES FOR POTENTIALS DEFINED ON HOMOGENEOUS GROUPS

A. MESKHI

Abstract. In this paper we prove two-weight inequalities for the operator of potential type defined on homogeneous groups. In some special case we have found the necessary and sufficient conditions for pairs of weights ensuring the validity of strong and weak type inequalities for the operator of potential type.

Despite the fact that two-weight problems for potential are already solved (see [1–4]), it is important to find new, easily verifiable, criteria which will be more convenient in applications.

The necessary and sufficient conditions for pairs of weights ensuring the validity of two-weight inequalities of a strong type for the operator of potential type T_α defined on homogeneous groups are found in some particular case. The sufficient conditions for which inequalities of a weak type are valid are also found. Analogous problems are studied in [5–6].

A homogeneous group is a nilpotent connected simply connected Lie group G on whose Lie algebra g is given one-parameter group of extensions $\delta_t = \exp(A \ln t)$, where A is a diagonalized linear operator on g whose eigenvalues are positive (see, for e.g., [7], p. 5).

If G is a homogeneous group, the mappings $\exp \circ \delta_t \circ \exp^{-1}$, $t > 0$, are the group automorphisms; denote them by δ_t. The number $Q = trA$ is said to be homogeneous dimension of the group G.

1991 *Mathematics Subject Classification.* 42B25, 43A80, 47G10.
Key words and phrases. Homogeneous group, potential, two-weight inequality.

As examples of homogeneous groups may serve an n-dimensional Euclidean space with a fixed diagonal matrix $A=(a_{ij})$ $i,j=1,2,\dots,n$, Heisenberg groups H^n and so on.

Homogeneous norm is defined on homogeneous groups. This is a continuous function $r : G \to [0,\infty)$ of the class C^∞ on $G\backslash\{e\}$ possessing the following properties:

(1) $r(x)=r(x^{-1})$ for arbitrary $x\in G$;

(2) $r(x)=0$ if and only if $x=e$;

(3) $r(\delta_t x)=tr(x)$ for arbitrary $x\in G$ and $t>0$;

(4) there exists a constant $c_0>0$ such that

$$r(xy^{-1})\le c_0(r(x)+r(y))$$

for all $x,y\in G$.

For $x\in G$, $\rho>0$ we assume that

$$B(x,\rho)=\{y\in G : r(xy^{-1})<\rho\}.$$

Next, let

$$S(x,\rho)=\{y\in G : r(xy^{-1})=\rho\}.$$

Fix on G a normed Haar measure so that the measure of the unit ball $B(e,1)$ is equal to 1. The Haar measure of any measurable set $E\subset G$ we denote by $|E|$ and the integral on E with respect to the given measure is denoted by $\int_E f(x)dx$.

We can easily verify that

$$|\delta_t(E)|=t^Q|E|,\quad d(\delta_t x)=t^Q dx.$$

In particular, $|B(x,\rho)|=\rho^Q$.

Definition 1. Almost everywhere positive, locally summable function $w : G\to\mathbb{R}$ is called the weight.

Definition 2. The function f is said to belong to $f\in L^p_w(G)$ for $1\le p<\infty$ if f is measurable and

$$\|f\|_{L^p_w(G)}=\left(\int\limits_G |f(x)|^p w(x)dx\right)^{\frac{1}{p}}<\infty.$$

Definition 3. The weight w is said to belong to $A_p(G)$, $1<p<\infty$ if

$$\sup\left(\frac{1}{|B|}\int\limits_B w(x)dx\right)\left(\frac{1}{|B|}\int\limits_B w^{1-p'}(x)dx\right)^{p-1}<\infty\quad\left(p'=\frac{p}{p-1}\right).$$

where the exact upper bound is taken over all balls $B \subset G$.

The classes $A_p(G)$ are the analoges of the well-known Muckenhoupt weight classes.

For the function $f : G \to \mathbb{R}^1$ let us consider the potential

$$T_\alpha f(x) = \int_G \frac{f(y)}{r(xy^{-1})^{Q-\alpha}} dy \quad (0 < \alpha < Q).$$

Definition 4. The function $\rho : G \to \mathbb{R}^1_+$ is said to be radial if there exists a function $\beta : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ such that $\rho(x) = \beta(r(x))$. In what follows instead of the function β we shall use the symbol ρ.

The following theorems are known.

Theorem A (see [7], p. 14). *Let G be a homogeneous group, $S_G = \{x \in G : r(x) = 1\}$. There exists the measure σ concentrated on S_G such that for all $u \in L^1(G)$ the equality*

$$\int_G u(x)dx = \int_0^\infty \bigg(\int_{S_G} u(\delta_\rho y)\rho^{Q-1} d\sigma(y) \bigg) d\rho$$

is valid.

Theorem B (see [7], p. 188). *Let $0 < \alpha < Q$, $\frac{1}{q} = 1 - \frac{\alpha}{Q}$. Then there exists a constant $c > 0$ such that for an arbitrary $f \in L^1(G)$ the inequality*

$$\Big| \Big\{ x : |T_\alpha f(x)| > \lambda \Big\} \Big| \le c \Big(\frac{1}{\lambda} \int_G |f(x)| dx \Big)^q$$

is valid.

Theorem C (see [8]). *Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$. If the weight $w \in A_{1+\frac{q}{p'}}(G)$, then there exists a positive constant c such that for arbitrary $f \in L^p_w(G)$ the inequality*

$$\bigg(\int_G \Big| T_\alpha (f w^{\frac{\alpha}{Q}})(x) \Big|^q w(x) dx \bigg)^{\frac{1}{q}} \le c \bigg(\int_G |f(x)|^p w(x) dx \bigg)^{\frac{1}{p}}$$

holds.

Theorem D (see [9–10]). *Let $1 < p \le q < \infty$ and let $\alpha(t)$ and $\beta(t)$ be positive functions on $(0, \infty)$.*

1) *For the validity of the inequality*

$$\left(\int_0^\infty \alpha(t)\left|\int_0^t F(\tau)d\tau\right|^q dt\right)^{\frac{1}{q}} \leq c_1\left(\int_0^\infty \beta(t)|F(t)|^p dt\right)^{\frac{1}{p}}$$

with the constant c_1 not depending on F, it is necessary and sufficient to fulfil the condition

$$\sup_{t>0}\left(\int_t^\infty \alpha(\tau)d\tau\right)^{\frac{p}{q}}\left(\int_0^t \beta^{1-p'}(\tau)d\tau\right)^{p-1} < \infty.$$

2) *For the validity of the inequality*

$$\left(\int_0^\infty \alpha(t)\left|\int_t^\infty F(\tau)d\tau\right|^q dt\right)^{\frac{1}{q}} \leq c_2\left(\int_0^\infty \beta(t)|F(t)|^p dt\right)^{\frac{1}{p}}$$

with the constant c_2 not depending on F, it is necessary and sufficient to fulfil the condition

$$\sup_{t>0}\left(\int_0^t \alpha(\tau)d\tau\right)^{\frac{p}{q}}\left(\int_t^\infty \beta^{1-p'}(\tau)d\tau\right)^{p-1} < \infty.$$

For the locally integrable function $f : G \to \mathbb{R}$ let us define the maximal function

$$Mf(x) = \sup \frac{1}{|B|}\int_B |f(y)|dy,$$

where the supremum is taken over all balls $B \ni x$.

The following theorem is valid.

Theorem E (see [11]). *Let $1 < p < \infty$. If the weight $w \in A_p(G)$, then there exists a constant $c > 0$ such that for an arbitrary $f \in L^p_w(G)$ the inequality*

$$\|Mf\|_{L^p_w(G)} \leq c\|f\|_{L^p_w(G)}$$

is fulfilled.

Let us mention and prove several lemmas used in proving the basic theorems.

Lemma 1. *Let $1 < p < \infty$ and the radial function $\rho \in A_p(G)$, $0 \le c_1 < c_2 \le c_3 < c_4 < \infty$. Then there exists a constant $c > 0$ such that for an arbitrary $t > 0$ the inequality*

$$\int_{c_3t}^{c_4t} \rho(\tau)\tau^{Q-1}d\tau \le c\int_{c_1t}^{c_2t} \rho(\tau)\tau^{Q-1}d\tau \tag{1}$$

is valid.

Proof. Denote

$$\begin{aligned}\Gamma &= \{x \in G : c_1t < r(x) < c_2t\},\\ \Gamma_1 &= \{x \in G : c_3t < r(x) < c_4t\},\\ B &= \{x \in G : r(x) < c_4t\}.\end{aligned}$$

It can be easily verified that

$$M\varphi(x) > \frac{1}{|B|}\int_B |\varphi(y)|dy \cdot \chi_{\Gamma_1}(x) \ge \frac{c(Q)}{|\Gamma|}\int_\Gamma |\varphi(y)|dy \cdot \chi_{\Gamma_1}(x),$$

where the constant $c(Q)$ depends only on Q. Taking next into account Theorem E, we obtain

$$\int_{\Gamma_1}\left(\frac{1}{|\Gamma|}\int_\Gamma \varphi(y)dy\right)^p \rho(x)dx \le c\int_G |\varphi(y)|^p\rho(y)dy.$$

But if we take $\varphi(y) = \chi_\Gamma(y)$ then we arrive at

$$\int_{c_3t<r(x)<c_4t} \rho(x)dx \le c \int_{c_1t<r(x)<c_2t} \rho(x)dx. \quad \blacksquare$$

By Theorem A we have (1).

Lemma 2. *Let $0 < \alpha < Q$, $0 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $\beta \ge 1$, the radial function $\rho \in A_{1+\frac{q}{p'}}(G)$, and let σ and u be positive, increasing on $(0,\infty)$ functions. Assume that $v = \sigma\rho$ and $w = u\rho$. If the condition*

$$\sup_{t>0}\left(\int_t^\infty v(\tau)^{-1-\frac{Qq}{p'}}d\tau\right)^{\frac{p}{q}}\left(\int_0^{\frac{t}{2}}\left(\rho^{-\frac{\alpha p}{Q}}(\tau)w(\tau)\right)^{1-p'}\tau^{Q-1}d\tau\right)^{p-1} < \infty \tag{2}$$

is fulfilled, then there exists a constant c such that for an arbitrary $t > 0$ we have the inequality

$$\sigma^{\frac{p}{q}}(\beta t) \le cu(t).$$

Proof. Since σ and u are the increasing functions, then the following inequalities hold:

$$\bigg(\int\limits_t^\infty v(\tau)\tau^{-1-\frac{qQ}{p'}}d\tau\bigg)^{\frac{p}{q}} \ge \bigg(\int\limits_{\beta t}^{2\beta t} \sigma(\tau)\rho(\tau)\tau^{-1-\frac{qQ}{p'}}d\tau\bigg)^{\frac{p}{q}} \ge$$

$$\ge \sigma^{\frac{p}{q}}(\beta t)\bigg(\int\limits_{\beta t}^{2\beta t}\rho(\tau)\tau^{-1-\frac{Qq}{p'}}d\tau\bigg)^{\frac{p}{q}},$$

$$\bigg(\int\limits_0^{\frac{t}{2}} w^{1-p'}(\tau)\rho^{-\frac{\alpha p(1-p')}{Q}}(\tau)\tau^{Q-1}d\tau\bigg)^{p-1} \ge \frac{1}{u(\frac{t}{2})}\bigg(\int\limits_0^{\frac{t}{2}}\rho^{-\frac{p'}{q}}(\tau)\tau^{Q-1}d\tau\bigg)^{p-1} \ge$$

$$\ge \frac{1}{u(t)}\bigg(\int\limits_0^{\frac{t}{2}}\rho^{-\frac{p'}{q}}(\tau)\tau^{Q-1}d\tau\bigg)^{p-1}.$$

It is not difficult to verify that if $\rho \in A_{1+\frac{q}{p'}}(G)$, then $\rho^{-\frac{p'}{q}} \in A_{1+\frac{p'}{q}}(G)$. Using Lemma 1, we get

$$\frac{\sigma^{\frac{p}{q}}(\beta t)}{u(t)} \le c_1\frac{\sigma^{\frac{p}{q}}(\beta t)}{u(t)}\bigg(\int\limits_{\beta t}^{2\beta t}\rho(\tau)\tau^{-1-\frac{Qq}{p'}}d\tau\bigg)^{\frac{p}{q}}\bigg(\int\limits_{\beta t}^{2\beta t}\rho^{-\frac{p'}{q}}(\tau)\tau^{Q-1}d\tau\bigg)^{\frac{p}{p'}}$$

$$\le c_2\frac{\sigma^{\frac{p}{q}}(\beta t)}{u(t)}\bigg(\int\limits_{\beta t}^{2\beta t}\rho(\tau)\tau^{-1-\frac{Qq}{p'}}d\tau\bigg)^{\frac{p}{q}}\bigg(\int\limits_0^{\frac{t}{2}}\rho^{-\frac{p'}{q}}(\tau)\tau^{Q-1}d\tau\bigg)^{p-1} \le c. \blacksquare$$

The lemma below is proved analogously.

Lemma 3. *Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $\beta \ge 1$ and the radial function $\rho \in A_{1+\frac{q}{p'}}(G)$. Assume next that u and σ are the positive, decreasing on $(0,\infty)$ functions, and $v = \sigma\rho$, $w = u\rho$. If the condition*

$$\sup_{t>0}\bigg(\int\limits_0^{\frac{t}{2}} v(\tau)\tau^{Q-1}d\tau\bigg)^{\frac{p}{q}}\bigg(\int\limits_t^\infty\Big(\rho^{-\frac{\alpha p}{Q}}(\tau)w(\tau)\Big)^{1-p'}\tau^{-1-\frac{Qp'}{q}}d\tau\bigg)^{p-1} < \infty, \qquad (3)$$

is fulfilled, then the following inequality holds:

$$\sigma^{\frac{p}{q}}(\frac{t}{\beta}) \le cu(t),$$

where the constant $c > 0$ does not depend on t.

Lemma 4. *Let $q > 1$, and let α and β be positive functions on $(0, \infty)$.*
1) *If for α and β the inequality*

$$\sup_{t>0} \left(\int_t^\infty \alpha(\tau) d\tau \right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau \in (0,2t)} \frac{1}{\beta(\tau)} < \infty$$

is valid, then we have the inequality

$$\left(\int_0^\infty \alpha(t) \left| \int_0^t F(\tau) d\tau \right|^q dt \right)^{\frac{1}{q}} \le c_1 \int_0^\infty \beta(t) |F(t)| dt$$

with the constant c_1 not depending on F;
2) *If the condition*

$$\sup_{t>0} \left(\int_0^t \alpha(\tau) d\tau \right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau \in (\frac{t}{2},\infty)} \frac{1}{\beta(\tau)},$$

is satisfied, then there exists $c_2 > 0$ such that for an arbitrary F the following inequality is fulfilled:

$$\left(\int_0^\infty \alpha(t) \left| \int_t^\infty F(\tau) d\tau \right|^q dt \right)^{\frac{1}{q}} \le c_2 \int_0^\infty \beta(t) |F(t)| dt.$$

Proof. Let us prove (1). Owing to the generalized Minkowsky's inequality, we have

$$\left(\int_0^\infty \alpha(t) \left| \int_0^t F(\tau) d\tau \right|^q dt \right)^{\frac{1}{q}} \le \int_0^\infty |F(\tau)| \left(\int_\tau^\infty \alpha(t) dt \right)^{\frac{1}{q}} d\tau \le$$

$$\le \int_0^\infty \beta(\tau) |F(\tau)| \left(\sup_{\tau>0} \left(\int_\tau^\infty \alpha(t) dt \right)^{\frac{1}{p}} \operatorname*{ess\,sup}_{t \in (0,2\tau)} \frac{1}{\beta(t)} \right) d\tau \le$$

$$\le c_1 \int_0^\infty |F(\tau)| \beta(\tau) d\tau.$$

(2) is proved similarly. ■

Lemma 5. *Let $q > 1$, $\beta \geq 1$ and let v and w be the increasing, positive functions on* $(0, \infty)$.

If the condition

$$\sup_{t>0} \left(\int_t^\infty \frac{v(\tau)}{\tau} d\tau \right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau \in (0,2t)} \frac{1}{w(\tau)} < \infty \tag{4}$$

is fulfilled, then there exists a constant $c_1 > 0$ such that for an arbitrary $t > 0$ the inequality

$$v^{\frac{1}{q}}(\beta t) \leq c_1 w(t),$$

holds, while if v and w are the decreasing, positive functions on $(0, \infty)$ *satisfying the condition*

$$\sup_{t>0} \left(\int_0^t v(\tau) \tau^{Q-1} d\tau \right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau \in (\frac{t}{2}, \infty)} \frac{1}{w(\tau) \tau^{\frac{Q}{q}}} < \infty, \tag{5}$$

then we have the inequality

$$v^{\frac{1}{q}}\Big(\frac{t}{\beta}\Big) \leq c_2 w(t),$$

where the constant c_2 does not depend on t.

Proof. Let us prove the first part of the lemma. Since v is the increasing function, we have

$$\left(\int_t^\infty \frac{v(\tau)}{\tau} d\tau \right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau \in (0,2t)} \frac{1}{w(\tau)} \geq$$
$$\geq \left(\int_{\beta t}^{2\beta t} \frac{v(\tau)}{\tau} d\tau \right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{\tau \in (0,2t)} \frac{1}{w(\tau)} \geq c_1 \frac{v^{\frac{1}{q}}(\beta t)}{w(t)} .$$

The second part of the lemma is proved analogously. ■

Theorem 1. *Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ and let σ and u be the positive, increasing functions on* $(0, \infty)$, *the radial function $\rho \in A_{1+\frac{q}{p'}}(G)$, $v = \sigma\rho$ and $w = u\rho$. If for (v, w) the condition* (2) *is* *fulfilled, then there exists a constant $c > 0$ such that for an arbitrary $f \in L^p_w(G)$ the inequality*

$$\Big(\int_G |T_\alpha(f\cdot\rho^{\frac{\alpha}{Q}})(x)|^q v(r(x))\Big)^{\frac{1}{q}} \le c\Big(\int_G |f(x)|^p w(r(x))dx\Big)^{\frac{1}{p}} \tag{6}$$

is valid.

Proof. Without restriction of generality we may assume that the function σ has the form

$$\sigma(t) = \sigma(0) + \int_0^t \varphi(u)du,$$

where $\sigma(0) = \lim_{t\to 0}\sigma(t)$, and $\varphi(t) \ge 0$ on the interval $(0,\infty)$. We may also assume that the inequality $c_0 \ge 1$ holds for the homogeneous norm r.

Clearly,

$$\Big(\int_G |T_\alpha(f\cdot\rho^{\frac{\alpha}{Q}})(x)|^q v(r(x))dx\Big)^{\frac{1}{q}} \le$$
$$\le \sigma^{\frac{1}{q}}(0)\Big(\int_G |T_\alpha(f\cdot\rho^{\frac{\alpha}{Q}})(x)|^q \rho(x)dx\Big)^{\frac{1}{q}} +$$
$$+\Big(\int_G |T_\alpha(f\cdot\rho^{\frac{\alpha}{Q}})(x)|^q\Big(\int_0^{r(x)}\varphi(t)dt\Big)\rho(x)dx\Big)^{\frac{1}{q}} = I_1 + I_2.$$

If $\sigma(0) = 0$, then $I_1 = 0$, while if $\sigma(0) \ne 0$, then by Theorem C we have

$$I_1 \le c_1\sigma^{\frac{1}{q}}(0)\Big(\int_G |f(x)|^p\rho(x)dx\Big)^{\frac{1}{p}} \le c_1\Big(\int_G |f(x)|^p\rho(x)\sigma^{\frac{p}{q}}(r(x))dx\Big)^{\frac{1}{p}}.$$

Applying Lemma 2, we obtain

$$I_1 \le c_2\Big(\int_G |f(x)|^p w(r(x))dx\Big)^{\frac{1}{p}}.$$

Let us now estimate I_2.

$$I_2=\bigg(\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>t}|T_\alpha(f\cdot\rho^{\frac{\alpha}{Q}})(x)|^q\rho(x)dx\bigg)dt\bigg)^{\frac1q}\le$$

$$\le\bigg(\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>t}\rho(x)\bigg|\int\limits_{r(y)>\frac{t}{2c_0}}\frac{f(y)\rho^{\frac{\alpha}{Q}}(y)}{r(xy^{-1})^{Q-\alpha}}dy\bigg|^q dx\bigg)dt\bigg)^{\frac1q}+$$

$$+\bigg(\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>t}\rho(x)\bigg|\int\limits_{r(y)\le\frac{t}{2c_0}}\frac{f(y)\rho^{\frac{\alpha}{Q}}(y)}{r(xy^{-1})^{Q-\alpha}}dy\bigg|^q dx\bigg)dt\bigg)^{\frac1q}=I_{21}+I_{22}.$$

By virtue of Theorem C and owing to the Minkowsky's inequality with the exponent $\frac{q}{p}\ge1$, we have

$$I_{21}\le c_3\bigg(\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>\frac{t}{2c_0}}|f(x)|^p\rho(x)dx\bigg)^{\frac{q}{p}}dt\bigg)^{\frac1q}\le$$

$$\le c_3\bigg(\int\limits_G|f(x)|^p\rho(x)\bigg(\int\limits_0^{2c_0r(x)}\varphi(t)dt\bigg)^{\frac{p}{q}}dx\bigg)^{\frac1p}.$$

Using Lemma 2, we get

$$I_{21}\le c_4\bigg(\int\limits_G|f(x)|^pw(r(x))dx\bigg)^{\frac1p}.$$

Estimate now I_{22}. For $r(x)>t$, $r(y)<\frac{t}{2c_0}$ we have the inequality $r(xy^{-1})\ge\frac{1}{2c_0}r(x)$. Passing on the group G to the spherical coordinates, by formulas $x=\delta_\tau\overline{x}$, $y=\delta_\varepsilon\overline{y}$, $\tau=r(x)$, $\varepsilon=r(y)$, $\overline{x}\in S_G$, $\overline{y}\in S_G$ (see Theorem A), we obtain

$$I_{22}\le c_5\bigg[\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>t}\frac{\rho(x)}{r(x)^{q(Q-\alpha)}}dx\bigg)\bigg(\int\limits_{r(y)\le\frac{t}{2}}|f(y)|\rho^{\frac{\alpha}{Q}}(y)dy\bigg)^q dt\bigg]^{\frac1q}=$$

$$=c_6\bigg(\int\limits_0^\infty\varphi(2s)\bigg(\int\limits_{2s}^\infty\rho(\tau)\tau^{-1-\frac{Qq}{p'}}d\tau\bigg)\times$$

$$\times\bigg(\int\limits_0^s \rho^{\frac{\alpha}{Q}}(\varepsilon)\varepsilon^{Q-1}\bigg(\int\limits_{S_G}|f(\delta_\varepsilon\overline{y})|d\sigma(\overline{y})\bigg)d\varepsilon\bigg)^q ds\bigg)^{\frac{1}{q}}.$$

If we take into consideration that

$$\int\limits_t^\infty \varphi(2s)\bigg(\int\limits_{2s}^\infty \tau^{-1-\frac{Qq}{p'}}\rho(\tau)d\tau\bigg)ds =$$

$$= \int\limits_{2t}^\infty \rho(\tau)\tau^{-1-\frac{qQ}{p'}}\bigg(\int\limits_{2t}^\tau \varphi(s)ds\bigg)d\tau \le \int\limits_{2t}^\infty v(\tau)\tau^{-1-\frac{Qq}{p'}}d\tau,$$

then using Theorem D, we arrive at

$$I_{22} \le c_7\bigg(\int\limits_0^\infty w(t)t^{Q-1}\bigg(\int\limits_{S_G}|f(\delta_t\xi)|d\sigma(\xi)\bigg)^p dt\bigg)^{\frac{1}{p}},$$

and due to Theorem A, we finally get

$$I_{22} \le c_8\bigg(\int\limits_G |f(x)|^p w(r(x))dx\bigg)^{\frac{1}{p}}. \quad \blacksquare$$

Theorem 2. *Let* $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, *let* σ *and* u *be the positive, decreasing functions on* $(0,\infty)$ *and the radial function* $\rho \in A_{1+\frac{q}{p'}}(G)$. *Assume further that* $v = \sigma\rho$ *and* $w = u\rho$. *If the condition* (3) *is fulfilled, then the inequality* (6) *holds.*

Proof. Without restriction of generality assume the function σ to have the form

$$\sigma(t) = \sigma(\infty) + \int\limits_t^\infty \psi(u)du,$$

where $\sigma(\infty) = \lim\limits_{t\to+\infty}\sigma(t)$ and $\psi(t) \ge 0$ on $(0,\infty)$.

We have

$$\bigg(\int\limits_G |T_\alpha(f\cdot\rho^{\frac{\alpha}{Q}})(x)|^q v(r(x))dx\bigg)^{\frac{1}{q}} \le$$

$$\le \bigg(\int\limits_G |T_\alpha(f\cdot\rho^{\frac{\alpha}{Q}})(x)|^q \sigma(\infty)\rho(x)dx\bigg)^{\frac{1}{q}} +$$

$$+\Bigg(\int\limits_{G}|T_\alpha(f\cdot\rho^{\frac{\alpha}{Q}})(x)|^q\rho(x)\Bigg(\int\limits_{r(x)}^{\infty}\psi(t)dt\Bigg)dx\Bigg)^{\frac{1}{q}}=I_1+I_2.$$

If $\sigma(\infty)=0$, then $I_1=0$. However if $\sigma(\infty)\neq 0$, then by Theorem C we obtain

$$I_1\le c_1\sigma^{\frac{1}{q}}(\infty)\Bigg(\int\limits_{G}|f(x)|^p\rho(x)dx\Bigg)^{\frac{1}{p}}\le c_1\Bigg(\int\limits_{G}|f(x)|^p\rho(x)\sigma^{\frac{p}{q}}(r(x))dx\Bigg)^{\frac{1}{p}}.$$

Using Lemma 3, we get

$$I_1\le c_2\Bigg(\int\limits_{G}|f(x)|^p w(x)dx\Bigg)^{\frac{1}{p}}.$$

Estimate now I_2.

$$I_2=\Bigg(\int\limits_{0}^{\infty}\psi(t)\Bigg(\int\limits_{r(x)<t}|T_\alpha(f\cdot\rho^{\frac{\alpha}{Q}})(x)|^q\rho(x)dx\Bigg)dt\Bigg)^{\frac{1}{q}}\le$$

$$\le\Bigg(\int\limits_{0}^{\infty}\psi(t)\Bigg(\int\limits_{r(x)<t}\rho(x)\Bigg|\int\limits_{r(y)<2c_0}\frac{f(y)\rho^{\frac{\alpha}{Q}}(y)}{r(xy^{-1})^{Q-\alpha}}dy\Bigg|^q dx\Bigg)dt\Bigg)^{\frac{1}{q}}+$$

$$+\Bigg(\int\limits_{0}^{\infty}\psi(t)\Bigg(\int\limits_{r(x)<t}\rho(x)\Bigg|\int\limits_{r(y)\ge 2c_0}\frac{f(y)\rho^{\frac{\alpha}{Q}(y)}}{r(xy^{-1})^{Q-\alpha}}dy\Bigg|^q dx\Bigg)dt\Bigg)^{\frac{1}{q}}=I_{21}+I_{22}.$$

By Theorem C and the generalized Minkowsky's inequality, we have

$$I_{21}\le c_3\Bigg(\int\limits_{0}^{\infty}\varphi(t)\Bigg(\int\limits_{r(x)<2c_0}|f(x)|^p\rho(x)dx\Bigg)^{\frac{q}{p}}dt\Bigg)^{\frac{1}{q}}\le$$

$$\le c_3\Bigg(\int\limits_{G}|f(x)|^p\rho(x)\Bigg(\int\limits_{\frac{r(x)}{2c_0}}^{\infty}\varphi(t)dt\Bigg)^{\frac{p}{q}}dx\Bigg)^{\frac{1}{p}}.$$

Using Lemma 3, we obtain

$$I_{21}\le c_4\Bigg(\int\limits_{G}|f(x)|^p w(r(x))dx\Bigg)^{\frac{1}{p}}.$$

Estimate I_{22}. For $r(x) < t$, $r(y) > 2c_0t$ the inequality $\frac{1}{2c_0}r(y) < r(xy^{-1})$ is fulfilled. Passing to the spherical coordinates, we obtain

$$I_{22} \le c_5\bigg[\int_0^\infty \psi(t)\bigg(\int_{r(x)<t} \rho(x)dx\bigg)\bigg(\int_{r(y)>2t} \frac{|f(y)|\rho^{\frac{\alpha}{Q}}(y)}{r(y)^{Q-\alpha}}dy\bigg)^q dt\bigg]^{\frac{1}{q}} =$$

$$= c_6\bigg(\int_0^\infty \psi\Big(\frac{s}{2}\Big)\bigg(\int_0^{\frac{s}{2}} \rho(\tau)\tau^{Q-1}d\tau\bigg) \times$$

$$\times\bigg(\int_0^s \rho^{\frac{\alpha}{Q}}(\varepsilon)\varepsilon^{\alpha-1}\bigg(\int_{S_G} |f(\delta_\varepsilon \overline{y})|d\sigma(\overline{y})\bigg)d\varepsilon\bigg)^q ds\bigg)^{\frac{1}{q}}.$$

Besides we have

$$\int_0^t \psi(\frac{s}{2})\bigg(\int_0^{\frac{s}{2}} \rho(\tau)\tau^{Q-1}d\tau\bigg)ds \le \int_0^{\frac{t}{2}} v(\tau)\tau^{Q-1}d\tau.$$

Using the second part of Theorem D and Theorem A, we have

$$I_{22} \le c_7\bigg(\int_0^\infty t^{Q-1}w(t)\bigg(\int_{S_G} |f(\delta_t\overline{x})d\sigma(\overline{x})\bigg)^p dt\bigg)^{\frac{1}{p}} \le$$

$$\le c_8\bigg(\int_G |f(x)|^p w(r(x))dx\bigg)^{\frac{1}{p}}. \quad \blacksquare$$

The following inverse assertion is also valid.

Theorem 3. *Let* $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$. *If the weight radial functions* ρ, v, w *satisfy the inequality* (6), *then they satisfy the conditions* (2) *and* (3) *as well.*

Proof. Choose the function f as follows:

$$f(y) = \begin{cases} \rho^{-\frac{\alpha}{Q(1-p)}}(r(y))w^{1-p'}(r(y)) & \text{if } r(y) < \frac{t}{2}, \\ 0 & \text{if } r(y) \ge \frac{t}{2}. \end{cases}$$

Passing on the group G to the spherical coordinates, we obtain

$$\bigg(\int_G |f(y)|^p w(r(y))dy\bigg)^{\frac{1}{p}} =$$

$$= \Bigg(\int\limits_{r(y)<\frac{t}{2}} \rho^{-\frac{\alpha p}{Q(1-p)}}(r(y)) w^{p(1-p')}(r(y)) w(r(y)) dy \Bigg)^{\frac{1}{p}} =$$

$$= c_1 \Bigg(\int\limits_0^{\frac{t}{2}} \Big(\rho^{-\frac{\alpha p}{Q}}(\tau) w(\tau) \Big)^{1-p'} \tau^{Q-1} d\tau \Bigg)^{\frac{1}{p}}. \tag{7}$$

On the other hand, we have

$$\Bigg(\int\limits_G |T_\alpha(f \cdot \rho^{\frac{\alpha}{Q}})(x)|^q v(r(x)) dx \Bigg)^{\frac{1}{q}} \geq$$

$$\geq \Bigg(\int\limits_{r(x)>t} v(r(x)) \Bigg(\int\limits_{r(y)<\frac{t}{2}} r(xy^{-1})^{\alpha-Q} \rho^{-\frac{\alpha p}{Q(1-p)}}(r(y)) w^{1-p'}(r(y)) dy \Bigg)^q dx \Bigg)^{\frac{1}{q}}.$$

For $r(x) > t$ and $r(y) < \frac{t}{2}$ the inequality $r(xy^{-1}) \leq \frac{3c_0}{2} r(x)$ holds. Then

$$\Bigg(\int\limits_G |T_\alpha(f \cdot \rho^{\frac{\alpha}{Q}})(x)|^q v(r(x)) dx \Bigg)^{\frac{1}{q}} \geq$$

$$\geq c_2 \Bigg(\int\limits_{r(x)>t} \frac{v(r(x))}{r(x)^{(Q-\alpha)q}} dx \Bigg)^{\frac{1}{q}} \Bigg(\int\limits_{r(y)\leq\frac{t}{2}} \rho^{-\frac{\alpha p}{Q(1-p)}}(r(y)) w^{1-p'}(r(y)) dy \Bigg) =$$

$$= c_3 \Bigg(\int\limits_t^\infty v(\tau) \tau^{-1-\frac{Qq}{p'}} d\tau \Bigg)^{\frac{1}{q}} \Bigg(\int\limits_0^{\frac{t}{2}} \Big(\rho^{-\frac{\alpha p}{Q}}(\tau) w(\tau) \Big)^{1-p'} \tau^{Q-1} d\tau \Bigg).$$

Taking into account the latter estimate and equality (7), we get

$$\Bigg(\int\limits_t^\infty v(\tau) \tau^{-1-\frac{Qq}{p'}} d\tau \Bigg)^{\frac{1}{q}} \Bigg(\int\limits_0^{\frac{t}{2}} \Big(\rho^{-\frac{\alpha p}{Q}}(\tau) w(\tau) \Big)^{1-p'} \tau^{Q-1} d\tau \Bigg)^{\frac{1}{p'}} \leq c$$

and

$$\sup_{t>0} \Bigg(\int\limits_t^\infty v(\tau) \tau^{-1-\frac{Qq}{p'}} d\tau \Bigg)^{\frac{p}{q}} \Bigg(\int\limits_0^{\frac{t}{2}} \Big(\rho^{-\frac{\alpha p}{Q}}(\tau) w(\tau) \Big)^{1-p'} \tau^{Q-1} d\tau \Bigg)^{p-1} < \infty.$$

Defining the function f as

$$f(y)=\begin{cases} w^{1-p'}(r(y))\rho^{-\frac{\alpha p}{Q(1-p)}}(r(y))r(y)^{-\frac{Q(p'+q)}{pq}} & \text{if} \quad r(y)>t, \\ 0 & \text{if} \quad r(y)<t, \end{cases}$$

we obtain analogously the condition (3). ■

Theorem 4. *Let* $0<\alpha<Q$, $\frac{1}{q}=1-\frac{\alpha}{Q}$, *and let* v *and* w *be increasing, positive functions on* $(0,\infty)$ *satisfying the condition* (4). *Then the following inequality is valid:*

$$\int\limits_{\{x:|T_\alpha f(x)|>\lambda\}} v(r(x))dx \le c\bigg(\frac{1}{\lambda}\int\limits_G |f(x)|dx\bigg)^q, \tag{8}$$

where the constant $c>0$ *does not depend on* $f\in L^1(G)$ *and* $\lambda>0$.

Proof. Represent the function as follows:

$$v(t)=v(0)+\int\limits_0^t \varphi(u)du,$$

where $v(0)=\lim\limits_{t\to 0} v(t)$ and $\varphi(t)\ge 0$ on $(0,\infty)$.

We have

$$\int\limits_{\{x:|T_\alpha f(x)|>\lambda\}} v(r(x))dx =$$

$$=\int\limits_{\{x:|T_\alpha f(x)|>\lambda\}} v(0)dx+\int\limits_{\{x:|T_\alpha f(x)|>\lambda\}}\bigg(\int\limits_0^{r(x)}\varphi(t)dt\bigg)dx=I_1+I_2.$$

If $v(0)=0$, then $I_1=0$, while if $v(0)\ne 0$, then by Theorem B we have

$$I_1\le c_1\frac{v(0)}{\lambda^q}\bigg(\int\limits_G |f(x)|dx\bigg)^q \le \frac{c_1}{\lambda^q}\bigg(\int\limits_G |f(x)|v^{\frac{1}{q}}(r(x))dx\bigg)^q.$$

Using Lemma 5, we get

$$I_1\le c_2\bigg(\frac{1}{\lambda}\int\limits_G |f(x)|w(r(x))dx\bigg)^q.$$

Estimate now I_2.

$$I_2=\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>t}\chi_{\{z:T_\alpha f(z)>\lambda\}}(x)dx\bigg)dt\le$$

$$\le\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>t}\chi\bigg\{x:\bigg|\int\limits_{r(y)>\frac{t}{2c_0}}\frac{f(y)}{r(xy^{-1})^{Q-\alpha}}dy\bigg|>\frac{\lambda}{2}\bigg\}dx\bigg)dt+$$

$$+\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>t}\chi\bigg\{x:\bigg|\int\limits_{r(y)\le\frac{t}{2c_0}}\frac{f(y)}{r(xy^{-1})^{Q-\alpha}}dy\bigg|>\frac{\lambda}{2}\bigg\}dx\bigg)dt=I_{21}+I_{22}.$$

By Theorem B and Minkowsky's inequality with the exponent $q>1$, we have

$$I_{21}\le\frac{c_3}{\lambda^q}\bigg(\int\limits_0^\infty\varphi(t)\bigg(\int\limits_G|f(x)|\chi_{\{r(z)>\frac{t}{2c_0}\}}(x)dx\bigg)^q dt\le$$

$$\le\frac{c_3}{\lambda^q}\bigg(\int\limits_G|f(x)|\bigg(\int\limits_0^{2c_0r(x)}\varphi(t)dt\bigg)^{\frac{1}{q}}dx\bigg)^q.$$

By virtue of Lemma 5, we get

$$I_{22}\le c_4\bigg(\frac{1}{\lambda}\int\limits_G|f(x)|w(r(x))dx\bigg)^q.$$

Estimate now I_{22}. For $r(x)>t$, $r(y)<\frac{t}{2c_0}$ the inequality $r(xy^{-1})\ge\frac{1}{2c_0}r(x)$ is valid. Passing on G to the spherical coordinates, we have

$$I_{22}\le\frac{c_4}{\lambda^q}\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>t}\bigg|\int\limits_{r(y)\le\frac{t}{2c_0}}\frac{f(y)}{r(xy^{-1})^{Q-\alpha}}dy\bigg|^q dx\bigg)dt\le$$

$$\le\frac{c_5}{\lambda^q}\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{r(x)>t}r(x)^{(\alpha-Q)q}dx\bigg)\bigg(\int\limits_{r(y)\le\frac{t}{2}}|f(y)|dy\bigg)^q dt=$$

$$=\frac{c_6}{\lambda^q}\bigg(\int\limits_0^\infty\varphi(2s)\bigg(\int\limits_{2s}^\infty\frac{d\gamma}{\gamma}\bigg)\bigg(\int\limits_0^s t^{Q-1}\bigg(\int\limits_{S_G}|f(\delta_t(\overline{y}))|d\sigma(\overline{y})\bigg)dt\bigg)^q ds.$$

It can be easily verified that

$$\int_t^\infty \varphi(2s)\bigg(\int_{2s}^\infty \frac{d\gamma}{\gamma}d\gamma\bigg)ds \le \int_{2t}^\infty \frac{v(\gamma)}{\gamma}d\gamma$$

and owing to Lemma 4, we obtain

$$I_{22} \le \frac{c_7}{\lambda^q}\bigg(\int_0^\infty t^{Q-1}\bigg(\int_{S_G} |f(\delta_t\xi)|d\sigma(\xi)\bigg)w(t)dt\bigg)^q =$$

$$= \frac{c_8}{\lambda^q}\bigg(\int_G |f(x)|w(r(x))dx\bigg)^q. \ \blacksquare$$

The theorem below is proved analogously.

Theorem 5. *Let* $0 < \alpha < Q$, $\frac{1}{q} = 1 - \frac{\alpha}{Q}$ *and let* v *and* w *be the positive, decreasing on* $(0,\infty)$ *functions. If the pair* (v,w) *satisfies the condition* (5), *then the inequality* (8) *is fulfilled.*

The sufficient conditions for general radial weights ensuring the validity of the two-weight strong or weak type inequalities for the operator T_α are given in the following theorems.

Theorem 6. *Let* $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, *and let* v *and* w *be the positive functions on* $(0,\infty)$ *satisfying the following conditions:*

1) *there exists a constant* $b > 0$ *such that for an arbitrary* $t > 0$ *the inequality*

$$\bigg(\sup_{\frac{t}{c_0} < \tau \le 8c_0 t} v(\tau)\bigg)^{\frac{p}{q}} \le b \inf_{\frac{t}{c_0} \le \tau \le 8c_0 t} w(\tau)$$

holds;

2) $\displaystyle \sup_{t>0}\bigg(\int_t^\infty v(\tau)\tau^{-1-\frac{Qq}{p'}}d\tau\bigg)^{\frac{p}{q}}\bigg(\int_0^t w^{1-p'}(\tau)\tau^{Q-1}d\tau\bigg)^{p-1} < \infty,$

3) $\displaystyle \sup_{t>0}\bigg(\int_0^t v(\tau)\tau^{Q-1}d\tau\bigg)^{\frac{p}{q}}\bigg(\int_t^\infty w^{1-p'}(\tau)\tau^{-1-\frac{Qp'}{q}}d\tau\bigg)^{p-1} < \infty.$

Then the inequality

$$\bigg(\int_G |T_\alpha f(x)|^q v(r(x))dx\bigg)^{\frac{1}{q}} \le c\bigg(\int_G |f(x)|^p w(r(x))dx\bigg)^{\frac{1}{p}} \tag{9}$$

is valid, where the constant $c > 0$ *does not depend on* f.

Proof. Represent the left-hand side of inequality (9) as follows:

$$\bigg(\int_G |T_\alpha f(x)|^q v(r(x))dx\bigg)^{\frac{1}{q}} = \bigg(\sum_{k\in\mathbb{Z}}\int_{2^k<r(x)\le 2^{k+1}} |T_\alpha f(x)|^q v(r(x))dx\bigg)^{\frac{1}{q}} \le$$

$$\le c_1\bigg(\sum_{k\in\mathbb{Z}}\int_{2^k<r(x)\le 2^{k+1}} \Big|T_\alpha\Big(f\cdot \chi_{\{r(y)\le\frac{2^{k-1}}{c_0}\}}\Big)(x)\Big|^q v(r(x))dx\bigg)^{\frac{1}{q}} +$$

$$+c_1\bigg(\sum_{k\in\mathbb{Z}}\int_{2^k<r(x)\le 2^{k+1}} \Big|T_\alpha\Big(f\cdot \chi_{\{\frac{2^{k-1}}{c_0}<r(y)\le c_0 2^{k+2}\}}\Big)(x)\Big|^q v(r(x))dx\bigg)^{\frac{1}{q}} +$$

$$+c_1\bigg(\sum_{k\in\mathbb{Z}}\int_{2^k<r(x)\le 2^{k+1}} \Big|T_\alpha\Big(f\cdot \chi_{\{r(y)>c_0 2^{k+2}\}}\Big)(x)\Big|^q v(r(x))dx\bigg)^{\frac{1}{q}} =$$

$$= c_1\Big(\sum_k p_{1k} + \sum_k p_{2k} + \sum_k p_{3k}\Big).$$

Estimate $\sum_k P_{1k}$. For $2^k < r(x) \le 2^{k+1}$, $r(y) \le \frac{2^{k-1}}{c_0}$ the inequalities $r(y) \le r(x)$ and $r(xy^{-1}) \ge \frac{r(x)}{2c_0}$ are valid. Using Theorem D, we get

$$\sum_k p_{1k} \le c_2\bigg(\sum_k \int_{2^k<r(x)\le 2^{k+1}} v(r(x))r(x)^{(\alpha-Q)q}\bigg(\int_{r(y)\le r(x)} |f(y)|dy\bigg)^q dx\bigg)^{\frac{1}{q}} =$$

$$= c_2\bigg(\int_G v(r(x))r(x)^{(\alpha-Q)q}\bigg(\int_0^{r(x)} \tau^{Q-1}\bigg(\int_{S_G} |f(\delta_\tau\overline{y})|d\sigma(\overline{y})\bigg)d\tau\bigg)^q dx\bigg)^{\frac{1}{q}} =$$

$$= c_3\bigg(\int_0^\infty v(t)t^{-1-\frac{Qq}{p'}}\bigg(\int_0^t \tau^{Q-1}\bigg(\int_{S_G} |f(\delta_\tau\overline{y})|d\sigma(\overline{y})\bigg)d\tau\bigg)^q dt\bigg)^{\frac{1}{q}} \le$$

$$\le c_4\bigg(\int_0^\infty w(t)t^{Q-1}\bigg(\int_{S_G} |f(\delta_t\overline{y})|d\sigma(\overline{y})\bigg)^p dt\bigg)^{\frac{1}{p}} \le$$

$$\le c_5\bigg(\int_0^\infty w(t)t^{Q-1}\bigg(\int_{S_G} |f(\delta_t\overline{y})|^p d\sigma(\overline{y})\bigg)dt\bigg)^{\frac{1}{p}} =$$

$$= c_5\Bigg(\int\limits_G |f(y)|^p w(r(y))dy\Bigg)^{\frac{1}{p}}.$$

Let us now estimate $\sum_k p_{3k}$. For $2^k < r(x) \le 2^{k+1}$, $r(y) \le \frac{2^{k-1}}{c_0}$ the inequalities $r(x) \le r(y)$ and $r(xy^{-1}) \ge \frac{r(y)}{2c_0}$ are valid. Using again Theorem D, we obtain

$$\sum_k p_{3k} \le c_6\Bigg(\sum_k \int\limits_{2^k<r(x)\le 2^{k+1}} v(r(x))\Bigg(\int\limits_{r(y)\ge r(x)} \frac{|f(y)|}{r(y)^{Q-\alpha}}dy\Bigg)^q dx\Bigg)^{\frac{1}{q}} =$$

$$= c_6\Bigg(\int\limits_G v(r(x))\Bigg(\int\limits_{r(x)}^{\infty} \tau^{\alpha-1}\Bigg(\int\limits_{S_G} |f(\delta_\tau\overline{y})|d\sigma(\overline{y})\Bigg)^q d\tau\Bigg)dx\Bigg)^{\frac{1}{q}} =$$

$$= c_7\Bigg(\int\limits_0^{\infty} v(t)t^{Q-1}\Bigg(\int\limits_t^{\infty} \tau^{\alpha-1}\Bigg(\int\limits_{S_G} |f(\delta_\tau\overline{y})|d\sigma(\overline{y})\Bigg)^q d\tau\Bigg)dt\Bigg)^{\frac{1}{q}} \le$$

$$\le c_8\Bigg(\int\limits_0^{\infty} w(t)t^{Q-1}\Bigg(\int\limits_{S_G} |f(\delta_t\overline{y})|d\sigma(\overline{y})\Bigg)^p dt\Bigg)^{\frac{1}{p}} \le$$

$$\le c_9\Bigg(\int\limits_G |f(y)|^p w(r(y))dy\Bigg)^{\frac{1}{p}}.$$

Due to the strong type inequality (p,q) for the operator T_α, we have

$$\sum_k p_{2k} \le \Bigg(\sum_k \Bigg(\sup_{\frac{2^{k-1}}{c_0}<r(x)\le c_0 2^{k+2}} v(r(x))\Bigg) \times$$

$$\times \int\limits_G \Big|T_\alpha\Big(f\cdot\chi_{\{\frac{2^{k-1}}{c_0}<r(x)\le c_0 2^{k+2}\}}\Big)(x)\Big|^q dx\Bigg)^{\frac{1}{q}} \le$$

$$\le c_{10}\Bigg(\sum_k \Bigg(\sup_{\frac{2^{k-1}}{c_0}<r(x)\le c_0 2^{k+2}} v(r(x))\Bigg) \times$$

$$\times \Bigg(\int\limits_G \Big|f\cdot\chi_{\{\frac{2^{k-1}}{c_0}<r(x)\le c_0 2^{k+2}\}}(x)\Big|^p dx\Bigg)^{\frac{q}{p}}\Bigg)^{\frac{1}{q}} \le$$

$$\leq c_{12}\bigg(\sum_k\bigg(\int\limits_{\frac{2^{k-1}}{c_0}<r(x)\leq c_0 2^{k+2}}|f(x)|^p w(r(x))dx\bigg)^{\frac{q}{p}}\bigg)^{\frac{1}{q}}\leq$$

$$\leq c_{11}\bigg(\int\limits_G |f(x)|^p w(r(x))dx\bigg)^{\frac{1}{p}}. \ \blacksquare$$

The theorem below is proved analogously.

Theorem 7. *Let* $0<\alpha<Q$, $\frac{1}{q}=1-\frac{\alpha}{Q}$ *and let* v *and* w *be positive functions on* $(0,\infty)$ *satisfying the conditions:*

1) *there exists a positive constant* b *such that for an arbitrary* $t>0$ *we have the inequality*

$$\bigg(\sup_{\frac{t}{c_0}<\tau\leq 8c_0t} v(\tau)\bigg)^{\frac{1}{q}}\leq b\inf_{\frac{t}{c_0}<\tau\leq 8c_0t} w(\tau),$$

2) $\sup\limits_{t>0}\bigg(\int\limits_t^\infty \frac{v(\tau)}{\tau}d\tau\bigg)^{\frac{1}{q}}\operatorname*{ess\,sup}\limits_{\tau\in(0,2t)}\frac{1}{w(\tau)}<\infty,$

3) $\sup\limits_{t>0}\bigg(\int\limits_0^t v(\tau)\tau^{Q-1}d\tau\bigg)^{\frac{1}{q}}\operatorname*{ess\,sup}\limits_{\tau\in(\frac{t}{2},\infty)}\frac{1}{w(\tau)\tau^{Q/q}}<\infty.$

Then the inequality (8) *is valid.*

REFERENCES

1. V. Kokilashvili and M. Gabidzashvili, Two-weight weak type inequalities for fractional type integrals. *Preprint. Math. Inst. Czech. Acad. Sci. Prague* **45**(1989).

2. E. T. Sawyer, R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Preprint*, 1989.

3. A. Gogatishvili, V. Kokilashvili, Criteria of weight inequalities for integral transforms defined on homogeneous type spaces. In: Topological Vector Spaces, Algebras and Related Areas. *Longman. Pitman Research Notes in Mathematics* **316**(1994), 251–362.

4. I. Genebashvili, A. Gogatishvili, and V. Kokilashvili, Solution of two-weight problems for integral transforms with positive kernels. *Georgian Math. J.* **3**(1996), No. 1, 319–342.

5. S. K. Abdullaev, On some classes of integral operators in spaces of summable functions (Russian). *Dokl. Acad. Nauk SSSR*, **283**(1985), No. 4, 777–780.

6. V. S. Guliev, On a certain class of anisotropic integral operators and weight embedding theorems (Russian). *Dokl. Acad. Nauk SSSR* **304**(1983), No. 6, 1289–1293.

7. G. B. Folland and E. M. Stein, Hardy spaces on homogeneous groups. *Princeton University Press and University of Tokyo Press, Princeton, New Jersey*, 1982.

8. B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for fractional integrals. *Trans. Amer. Math. Soc.* **192**(1974), 261–274.

9. V. M. Kokilashvili, On Hardy's inequalities in weighted spaces. (Russian) *Bull. Acad. Sci. Georgian SSR* **96**(1979), 37–40.

10. B. Muckenhoupt, Hardy's inequalities with weight. *Studia Math.* **44**(1972), No. 1, 31–38.

11. B. Muckenhoupt, Weighted norm inequalities for the Hardy-Littlewood maximal function. *Trans. Amer. Math. Soc.* **165**(1972), 207–226.

(Received 08.02.1996)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia