Proceedings of A.Razmadze
Mathematical Institute,
Vol. **112**, 1997

TWO-WEIGHT INEQUALITIES FOR SINGULAR INTEGRALS DEFINED ON HOMOGENEOUS GROUPS

V. KOKILASHVILI AND A. MESKHI

Abstract. The sufficient conditions for pairs of weights ensuring the validity of two-weight strong (p,p) and weak $(1,1)$ type inequalities for singular integrals defined on homogeneous groups, are found. In some cases these conditions are as well necessary for the corresponding inequalities to be fulfilled. The sufficient conditions for which are valid two-weight inequalities of a strong type (p,q) for $1<q<p<\infty$ and of a weak type $(p,1)$ for $p>1$ for singular integrals, are also found. In most cases the examples for pairs of weights are given.

(p,p)

$(1,1)$

(p,q)

$1<q<p<\infty$

Introduction

The sufficient conditions for pairs of weights ensuring the validity of two-weight strong and weak type inequalities for singular integrals defined on homogeneous groups, are found.

In the first section the notion of homogeneous groups is given and some well-known results about singular integrals, maximal functions and Hardy operators are brought.

1991 *Mathematics Subject Classification.* 42B25, 43A80, 46E40, 47G10.

Key words and phrases. Homogeneous group, singular integral, weight function, two-weight inequality.

In the second section we present all the results concerning two-weight inequalities of a strong type (p,p) for $1<p<\infty$ and of a weak type $(1,1)$. Two-weight inequalities of a strong (p,p) type have been established in [1] for the Hilbert transform in the case of monotone weights. Analogous problem for singular integrals defined on Euclidean spaces has been considered in [2] and this result was generalized in [3] for singular integrals defined on the Heisenberg group. Optimal conditions for the pair of weights ensuring the validity of two-weight inequalities of a strong (p,p) type for Calderon-Zygmund singular integrals can be found in [4].

The sufficient conditions for pairs of weights guaranteeing the validity of two-weight inequalities of a strong (p,q), $1<q<p<\infty$, type and of a weak $(p,1)$, $p>1$, type for singular integrals defined on homogeneous groups are established in the third section. Similar results for the Hilbert transform are obtained in [5]. Particular interest to the results of this section is due to the fact that the operators invariant with respect to a shift are bounded from L^p to L^q if and only if $1<p\le q<\infty$.

In the fourth section the two-weight inequalities for Multiplicator of Fourier transforms on the line are proved.

§ 1. Homogeneous Groups, Some Well-known Results

Now let us present the notion of homogeneous groups.

Homogeneous group is referred to as a connected, simply connected, nilpotent Lie group G on whose Lie algebra g is given one-parameter group of extensions $\delta_t=\exp(A\ln t)$, $t>0$, where A is a diagonalized on g operator whose eigenvalues are positive (see [6], p. 5).

For the homogeneous group G the mappings $\exp\circ\delta_t\circ\exp^{-1}$, $t>0$, are automorphisms on G. We denote them by the symbol δ_t.

The number $Q=trA$ is called a homogeneous dimension of the group G.

As examples of homogeneous groups may serve an n-dimensional Euclidean space, Heisenberg groups, etc.

On the homogeneous group G we introduce a homogeneous norm, a continuous function $r:G\to[0,\infty)$ which is smooth on $G\backslash\{e\}$ and satisfies the following conditions:

1. $r(x)=r(x^{-1})$ for any $x\in G$;
2. $r(\delta_t x)=tr(x)$ for arbitrary $x\in G$ and $t>0$;
3. $r(x)=0\Leftrightarrow x=e$;

4. there exists a constant $c_0 > 0$ such that for arbitrary x, y from G the inequality $r(xy) \le c_0(r(x) + r(y))$ holds.

For $x \in G$ and $\rho > 0$ assume $B(x, \rho) = \{y \in G : r(xy^{-1}) < \rho\}$. Let, further, $S(x, \rho) = \{y \in G : r(xy^{-1}) = \rho\}$. Note that $\delta_\rho B(e, 1) = B(e, \rho)$.

On the group G we fix a normed Haar measure so that measure of the unit ball $B(e, 1)$ equals 1. The Haar measure of any measurable set $E \subset G$ we denote by $|E|$ and the integral on E with respect to the given measure by $\int_E f(x)dx$.

As is easily seen

$$|\delta_t E| = t^Q |E|, \quad d(\delta_t x) = t^Q dx.$$

In particular, $|B(x, \rho)| = \rho^Q$ for arbitrary $x \in G$ and $\rho > 0$.

Definition 1. Almost everywhere positive, locally integrable function $w : G \to \mathbb{R}^1$ will be called a weight.

In the sequel, by $L^p_w(G)$ $(1 \le p < \infty)$ we denote a set of summable functions $f : G \to \mathbb{R}^1$ for which the norm

$$\|f\|_{L^p_w(G)} = \left(\int_G |f(x)|^p w(x) dx \right)^{1/p}$$

is finite.

Definition 2. By the definition, the weight $w \in A_p(G)$ $(1 < p < \infty)$, if

$$\sup \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w^{1-p'}(x) dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset G$. Further, $w \in A_1(G)$ if there exists a positive number c_1 such that for an arbitrary ball $B \subset G$ the inequality

$$\frac{1}{|B|} \int_B w(x) dx \le c_1 \operatorname*{ess\,inf}_{x \in B} w(x)$$

holds.

The classes $A_p(G)$ are the analogues of the well known B. Muckenhoupt weight classes.

Let $k : G \to \mathbb{R}^1$ be a measurable function such that

1) $|k(x)| \le \frac{c}{r(x)^Q}$ for an arbitrary element $x \ne e$;

2) there exists a positive constant c_1 such that for arbitrary $x, y \in G$ with the condition $r(xy^{-1}) < \frac{1}{2}r(x)$ the following inequality holds:

$$|k(x) - k(y)| \leq c_1 w\Big(\frac{r(xy^{-1})}{r(x)}\Big)\frac{1}{r(x)^Q},$$

where $w : [0,1] \to \mathbb{R}^1$ is a non-decreasing function such that $w(0) = 0$, $w(2t) \leq c_2 w(t)$ for any $t > 0$ and $\int_0^1 \frac{w(t)}{t}dt < \infty$.

The kernel k together with the foregoing conditions will always be assumed to satisfy the condition: a singular integral

$$Tf(x) = p \cdot v \cdot \int\limits_G k(xy^{-1})f(y)dy \stackrel{def}{=} \lim_{\varepsilon \to 0} \int\limits_{G \setminus B(x,\varepsilon)} k(xy^{-1})f(y)dy$$

defines the operator bounded in $L^2(G)$.

The following theorems are known.

Theorem A ([7]). *Let $1 < p < \infty$ and the weight function $w \in A_p(G)$. Then the operator T is bounded in $L^p_w(G)$.*

Theorem B ([7]). *Let a weight function $w \in A_1(G)$. Then there exists a constant c such that for any $f \in L^1_w(G)$ and $\lambda > 0$ the inequality*

$$\int\limits_{\{x \in G : |Tf(x)| > \lambda\}} w(x)dx \leq \frac{c}{\lambda}\int\limits_G |f(x)|w(x)dx$$

holds.

Let $f : G \to \mathbb{R}$ be a locally integrable function. Define for it a maximal function:

$$Mf(x) = \sup \frac{1}{|B|}\int\limits_B |f(y)|dy,$$

where the supremum is taken over all balls containing the point x.

Theorem C ([8]). *In order that the operator M be bounded in $L^p_w(G)$, $1 < p < \infty$, it is necessary and sufficient that $w \in A_p(G)$.*

Definition 3. The function $\rho : G \to \mathbb{R}^1_+$ is said to be radial if there exists a function $\beta : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ such that $\rho(x) = \beta(r(x))$ for any element $x \in G$. In what follows instead of β the use will be made of the symbol ρ.

We will also need the following theorems.

Theorem D ([9–11]). *Let $1 < p \leq q < \infty$ and let $\alpha(t)$ and $\beta(t)$ be positive functions on the interval $(0, \infty)$.*

1) *For the fulfillment of the inequality where $c_1 > 0$ is independent of F, it is necessary and sufficient that*

$$\Big(\int_0^\infty \alpha(t)\Big|\int_0^t F(\tau)d\tau\Big|^q dt\Big)^{\frac{1}{q}} \le c_1\Big(\int_0^\infty |F(t)|^p\beta(t)dt\Big)^{\frac{1}{p}}$$

to be fulfilled, where $c_1 > 0$ is independent of F, it is necessary and sufficient that

$$\sup_{t>0}\Big(\int_t^\infty \alpha(\tau)d\tau\Big)^{\frac{p}{q}}\Big(\int_0^t \beta^{1-p'}(\tau)d\tau\Big)^{p-1} < \infty.$$

2) *The inequality*

$$\Big(\int_0^\infty \alpha(t)\Big|\int_t^\infty F(\tau)d\tau\Big|^q dt\Big)^{\frac{1}{q}} \le c_2\Big(\int_0^\infty |F(t)|^p\beta(t)dt\Big)^{\frac{1}{p}}$$

is fulfilled, where the constant $c_2 > 0$ is independent of F, if and only if

$$\sup_{t>0}\Big(\int_0^t \alpha(\tau)d\tau\Big)^{\frac{p}{q}}\Big(\int_t^\infty \beta^{1-p'}(\tau)d\tau\Big)^{p-1} < \infty.$$

Theorem E ([12]). *Let $1 < q < p < \infty$ and let $\alpha(t)$ and $\beta(t)$ be positive functions on $(0,\infty)$.*

1) *For the fulfillment of the inequality where $c_1 > 0$ is independent of F, it is necessary and sufficient that*

$$\Big(\int_0^\infty \alpha(t)\Big|\int_0^t F(\tau)d\tau\Big|^q dt\Big)^{\frac{1}{q}} \le c_1\Big(\int_0^\infty |F(t)|^p\beta(t)dt\Big)^{\frac{1}{p}}$$

to be fulfilled, where the constant $c_1 > 0$ is independent of F, it is necessary and sufficient that

$$\int_0^\infty \Big[\Big(\int_\tau^\infty \alpha(t)dt\Big)\Big(\int_0^\tau \beta^{1-p'}(t)dt\Big)^{q-1}\Big]^{\frac{p}{p-q}}\beta^{1-p'}(\tau)d\tau < \infty.$$

2) *The inequality*

$$\Big(\int_0^\infty \alpha(t)\Big|\int_t^\infty F(\tau)d\tau\Big|^q dt\Big)^{\frac{1}{q}} \le c_2\Big(\int_0^\infty |F(t)|^p\beta(t)dt\Big)^{\frac{1}{p}}$$

is fulfilled, where the constant $c_2 > 0$ *is independent of* F, *if and only if*

$$\int_0^\infty \left[\left(\int_0^\tau \alpha(t)dt\right)\left(\int_\tau^\infty \beta^{1-p'}(t)dt\right)^{q-1}\right]^{\frac{p}{p-q}} \beta^{1-p'}(\tau)d\tau < \infty.$$

Let us prove the following

Lemma A. *Let* $\alpha(t)$ *and* $\beta(t)$ *be positive on* $(0,\infty)$ *functions satisfying the condition*

$$\sup_{t>0}\left(\int_t^\infty \alpha(\tau)d\tau\right) \operatorname*{ess\,sup}_{\tau\in(0,2t)} \frac{1}{\beta(\tau)} < \infty.$$

Then the inequality

$$\int_0^\infty \alpha(t)\left|\int_0^t F(\tau)d\tau\right|dt \le c\int_0^\infty |F(t)|\beta(t)dt$$

holds, where the constant $c > 0$ *does not depend on* F.

Proof. We have:

$$\int_0^\infty \alpha(t)\left|\int_0^t F(\tau)d\tau\right|dt \le \int_0^\infty \alpha(t)\left(\int_0^t |F(\tau)|d\tau\right)dt =$$

$$= \int_0^\infty |F(\tau)|\left(\int_\tau^\infty \alpha(t)dt\right)d\tau \le \int_0^\infty |F(\tau)|\beta(\tau)\left(\int_\tau^\infty \alpha(t)dt\right) \operatorname*{ess\,sup}_{t\in(0,2\tau)} \frac{1}{\beta(t)}d\tau \le$$

$$\le c\int_0^\infty |F(\tau)|\beta(\tau)d\tau. \ \blacksquare$$

The lemma below is proved analogously.

Lemma B. *Let* $\alpha(t)$ *and* $\beta(t)$ *be positive on* $(0,\infty)$ *functions satisfying the condition*

$$\sup_{t>0}\left(\int_0^t \alpha(\tau)d\tau\right) \operatorname*{ess\,sup}_{\tau\in(\frac{t}{2},\infty)} \frac{1}{\beta(\tau)} < \infty.$$

Then the inequality

$$\int_0^\infty \alpha(t)\left|\int_t^\infty F(\tau)d\tau\right|dt \le c\int_0^\infty |F(t)|\beta(t)dt$$

holds, where the constant $c > 0$ does not depend on F.

The following lemma is valid.

Lemma C. *Let $p > 1$, and let $\alpha(t)$ and $\beta(t)$ be positive on $(0, \infty)$ functions. If a pair (α, β) satisfies the condition*

$$\int_0^\infty \left(\int_\tau^\infty \alpha(t)dt \right)^{p'} \beta^{1-p'}(\tau)d\tau < \infty,$$

then there exists positive constant c such that for an arbitrary F the inequality

$$\int_0^\infty \alpha(t) \left| \int_0^t F(\tau)d\tau \right| dt \le c \left(\int_0^\infty |F(t)|^p \beta(t)dt \right)^{\frac{1}{p}}$$

holds.

Proof. If we change the order of integration and use the Hölder's inequality, then we obtain:

$$\int_0^\infty \alpha(t) \left| \int_0^t F(\tau)d\tau \right| dt \le \int_0^\infty \alpha(t) \left(\int_0^t |F(\tau)|d\tau \right) dt =$$

$$= \int_0^\infty |F(\tau)| \left(\int_\tau^\infty \alpha(t)dt \right) \frac{\beta^{\frac{1}{p}}(\tau)}{\beta^{\frac{1}{p}}(\tau)} d\tau \le$$

$$\le \left(\int_0^\infty |F(\tau)|^p \beta(\tau)d\tau \right)^{\frac{1}{p}} \left(\int_0^\infty \left(\int_\tau^\infty \alpha(t)dt \right)^{p'} \beta^{-\frac{p'}{p}}(\tau)d\tau \right)^{\frac{1}{p'}} \le$$

$$\le c \left(\int_0^\infty |F(\tau)|^p \beta(\tau)d\tau \right)^{\frac{1}{p}}. \quad \blacksquare$$

The lemma below is proved similarly.

Lemma D. *Let $p > 1$, and let $\alpha(t)$ and $\beta(t)$ be positive on $(0, \infty)$ functions satisfying the condition*

$$\int_0^\infty \left(\int_0^\tau \alpha(t)dt \right)^{p'} \beta^{1-p'}(\tau)d\tau < \infty.$$

Then the following inequality holds:

$$\int_0^\infty \alpha(t)\bigg|\int_t^\infty F(\tau)d\tau\bigg|dt \le c\bigg(\int_0^\infty |F(t)|^p\beta(t)dt\bigg)^{\frac{1}{p}},$$

where the constant $c > 0$ is independent of F.

We will also need the following theorems:

Theorem F ([12]). *Let $1 \le q < p < \infty$, and let v and w be weight functions on G. In order thatr the inequality*

$$\bigg(\int_G |f(x)|^q v(x)dx\bigg)^{\frac{1}{q}} \le c\bigg(\int_G |f(x)|^p w(x)dx\bigg)^{\frac{1}{p}}$$

be valid, where the constant $c > 0$ is independent of f, it is necessary and sufficient that the condition

$$\int_G v^{\frac{p}{p-q}}(x)w^{-\frac{q}{p-q}}(x)dx < \infty$$

be fulfilled.

Theorem G (see [6], p. 14). *Let G be a homogeneous group, $S_G = \{x \in G : r(x) = 1\}$. There exists a unique countably additive measure σ defined on S_G such that for any $f \in L^1(G)$ the equality*

$$\int_G f(x)dx = \int_0^\infty t^{Q-1}\bigg(\int_{S_G} f(\delta_t\xi)d\sigma(\xi)\bigg)dt$$

holds.

§ 2. Two-weight Inequalities of a Strong (p, p) and a Weak $(1, 1)$ Types for Singular Integrals Defined on Homogeneous Groups

In this section the sufficient conditions are found for the pairs of weights which guarantee the validity of two-weight inequalities of a strong (p, p), $1 < p < \infty$, type and a weak $(1, 1)$ type for singular integrals defined on homogeneous groups. Note that the found conditions are necessary for the Hilbert transform. Appropriate examples for pairs of weights are given in most of the cases.

We start with the proofs of some lemmas.

Lemma 2.1. *Let $1 < p < \infty$, the radial function $\rho \in A_p(G)$, $u(t)$ be a positive, increasing on $(0, \infty)$, function. It is also assumed that $w = u\rho$ and for an arbitrary $t > 0$ the condition*

$$\int_0^t w^{1-p'}(\tau)\tau^{Q-1}d\tau < \infty$$

holds. Then for every $\varphi \in L^p_w(G)$ the singular integral $T\varphi(x)$ exists almost everywhere on G.

Proof. Let us fix the number $\alpha > 0$ and let

$$S_\alpha = \Big\{x \in G : r(x) > \frac{\alpha}{2}\Big\}.$$

Take the function $\varphi \in L^p_w(G)$ and write it as follows:

$$\varphi(x) = \varphi_1(x) + \varphi_2(x),$$

where $\varphi_1(x) = \varphi(x) \cdot \chi_{S_\alpha}(x)$, $\varphi_2(x) = \varphi(x) - \varphi_1(x)$. Since the function u is increasing, we get

$$\int_G |\varphi_1(x)|^p \rho(x)dx = \int_{S_\alpha} |\varphi(x)|^p \rho(x)dx \le \frac{1}{u(\frac{\alpha}{2})} \int_{S_\alpha} |\varphi(x)|^p \rho(x)u(r(x))dx \le$$
$$\le c_1 \int_G |\varphi(x)|^p \rho(x)u(r(x))dx.$$

The last integral is finite, and the condition $\rho \in A_p(G)$ results in $T\varphi_1 \in L^p_\rho(G)$. This implies that $T\varphi_1(x)$ exists almost everywhere on G.

Let us now show that $T\varphi_2(x)$ converges absolutely almost everywhere on G.

For a homogeneous norm r the condition $c_0 \ge 1$ is assumed to be fulfilled.

Take the element $x \in G$, $r(x) > \alpha c_0$. If $r(x) > \alpha c_0$ and $r(y) \le \frac{\alpha}{2}$, then $r(xy^{-1}) \ge \frac{1}{c_0} r(x) - r(y) \ge \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}$. We obtain:

$$|T\varphi_2(x)| \le c_2 \int_{r(y)\le\frac{\alpha}{2}} \frac{|\varphi(y)|}{r(xy^{-1})^Q} dy \le$$
$$\le c_3 \alpha^{-Q} \int_{r(y)\le\frac{\alpha}{2}} |\varphi(y)| w^{\frac{1}{p}}(r(y)) w^{-\frac{1}{p}}(r(y))dy \le$$

$$\leq c_3\alpha^{-Q}\bigg(\int\limits_G |\varphi(y)|^p w(r(y))dy\bigg)^{\frac{1}{p}}\bigg(\int\limits_{r(y)\leq\frac{\alpha}{2}} w^{1-p'}(r(y))dy\bigg)^{\frac{1}{p'}}.$$

Since the number $\alpha > 0$ is taken to be arbitrary, we have that $T\varphi_2(x)$ converges absolutely almost everywhere on G. ■

The lemma below is proved analogously.

Lemma 2.2. *Let $1 < p < \infty$, the radial function $\rho \in A_p(G)$, and let $u(t)$ be a positive, decreasing on $(0,\infty)$ function. It is also assumed that $w = u\rho$ and for an arbitrary $t > 0$ the condition*

$$\int\limits_t^\infty w^{1-p'}(\tau)\tau^{-1-\frac{Q}{p-1}}d\tau < \infty$$

is fulfilled. Then for every $\varphi \in L^p_w(G)$ the singular integral $T\varphi(x)$ exists almost everywhere on G.

We will need the following

Lemma 2.3. *Let $1 < p < \infty$ and the radial function $\rho \in A_p(G)$, $0 \leq c_1 < c_2 \leq c_3 < c_4$. Then there exists a positive constant c such that for an arbitrary $t > 0$ we have the inequality*

$$\int\limits_{c_3t}^{c_4t} \rho(\tau)\tau^{Q-1}d\tau \leq c\int\limits_{c_1t}^{c_2t} \rho(\tau)\tau^{Q-1}d\tau. \tag{2.1}$$

Proof. Let

$$\Gamma = \Big\{x \in G : c_1t < r(x) < c_2t\Big\},$$
$$\Gamma_1 = \Big\{x \in G : c_3t < r(x) < c_4t\Big\},$$
$$B = \Big\{x \in G : r(x) < c_4t\Big\}.$$

From the definition of a maximal function we obtain

$$M\varphi(x) > \bigg(\frac{1}{|B|}\int\limits_B |\varphi(y)|dy\bigg)\cdot\chi_{\Gamma_1}(x) \geq \bigg(\frac{c}{|\Gamma|}\int\limits_\Gamma |\varphi(y)|dy\bigg)\chi_{\Gamma_1}(x),$$

where $\varphi \in L^p_\rho(G)$ and the constant $c > 0$ is independent of φ. By Theorem C we get

$$\int\limits_{\Gamma_1} \Big(\frac{1}{|\Gamma|} \int\limits_{\Gamma} \varphi(y)dy\Big)^p \rho(x)dx \le c \int\limits_{G} |\varphi(y)|^p \rho(y)dy.$$

If we replace the function φ by χ_Γ, then we obtain

$$\int\limits_{\{x: c_3 t < r(x) < c_4 t\}} \rho(x)dx \le c \int\limits_{\{x: c_1 t < r(x) < c_2 t\}} \rho(x)dx.$$

Finally, by Theorem G we arrive at inequality (2.1). ∎

Lemma 2.4. *Let $1 \le p < \infty$, $\beta \ge 1$, the radial function $\rho \in A_p(G)$, and let $\sigma(t)$ and $u(t)$ be positive increasing on $(0, \infty)$ functions, $v = \sigma\rho$, $w = u\rho$. If v and w satisfy the condition*

$$\sup_{t>0} \Big(\int\limits_t^\infty v(\tau)\tau^{Q-Qp-1}d\tau\Big)\Big(\int\limits_0^{\frac{t}{2}} w^{1-p'}(\tau)\tau^{Q-1}d\tau\Big)^{p-1} < \infty$$

for $1 < p < \infty$ and the condition

$$\sup_{t>0} \Big(\int\limits_t^\infty \frac{v(\tau)}{\tau} d\tau\Big) \operatorname*{ess\,sup}_{\tau \in (0,t)} \frac{1}{w(\tau)} < \infty,$$

for $p = 1$, then there exists a positive constant c (depending only on Q and β) such that for any $t > 0$ the inequality

$$\sigma(\beta t) \le c u(t)$$

holds.

Proof. Let $1 < p < \infty$. By the condition, σ and u are the positive increasing function. Therefore we have:

$$\int\limits_t^\infty \sigma(\tau)\rho(\tau)\tau^{Q-Qp-1}d\tau \ge$$

$$\ge \int\limits_{\beta t}^{2\beta t} \sigma(\tau)\rho(\tau)\tau^{Q-Qp-1}d\tau \ge \sigma(\beta t) \int\limits_{\beta t}^{2\beta t} \rho(\tau)\tau^{Q-Qp-1}d\tau,$$

$$\left(\int_0^{\frac{t}{2}} u^{1-p'}(\tau)\rho^{1-p'}(\tau)\tau^{Q-1}\right)^{p-1} \geq \frac{1}{u(\frac{t}{2})}\left(\int_0^{\frac{t}{2}} \rho^{1-p'}(\tau)\tau^{Q-1}d\tau\right)^{p-1}.$$

According to Hölder's inequality and also by Lemma 2.3, we obtain

$$\frac{\sigma(\beta t)}{u(\frac{t}{2})} \leq c_1\frac{\sigma(\beta t)}{u(\frac{t}{2})}\left(\int_{\beta t}^{2\beta t} \rho(\tau)\tau^{Q-Qp-1}d\tau\right)\left(\int_{\beta t}^{2\beta t} \rho^{1-p'}(\tau)\tau^{Q-1}d\tau\right)^{p-1} \leq$$

$$\leq c_2\frac{\sigma(\beta t)}{u(\frac{t}{2})}\left(\int_{\beta t}^{2\beta t} \rho(\tau)\tau^{Q-Qp-1}d\tau\right)\left(\int_0^{\frac{t}{2}} \rho^{1-p'}(\tau)\tau^{Q-1}d\tau\right)^{p-1} \leq c.$$

Hence we have

$$\sigma(\beta t) \leq cu\left(\frac{t}{2}\right) \leq cu(t),$$

where the constant $c > 0$ is independent of t.

Suppose now that $p = 1$. We have

$$\int_t^{\infty} \sigma(\tau)\rho(\tau)\tau^{-1}d\tau \geq \int_{\beta t}^{2\beta t} \sigma(\tau)\rho(\tau)\tau^{-1}d\tau \geq \sigma(\beta t)\int_{\beta t}^{2\beta t} \rho(\tau)\tau^{-1}d\tau.$$

It can be easily verified that if $\rho \in A_1(G)$, then $\rho \in A_p(G)$, $p > 1$. By Hölder's inequality and also by Lemma 2.3 we get

$$\frac{\sigma(\beta t)}{u(t)} \leq c_1\frac{\sigma(\beta t)}{u(t)}t^{-pQ}\left(\int_{\beta t}^{2\beta t} \rho(\tau)\tau^{Q-1}d\tau\right)\left(\int_{\beta t}^{2\beta t} \rho^{1-p'}(\tau)\tau^{Q-1}d\tau\right)^{p-1} \leq$$

$$\leq c_2\frac{\sigma(\beta t)}{u(t)}t^{-pQ}\left(\int_{\beta t}^{2\beta t} \rho(\tau)\tau^{Q-1}d\tau\right)\left(\int_0^{\frac{t}{2}} \rho^{1-p'}(\tau)\tau^{Q-1}d\tau\right)^{p-1} \leq$$

$$\leq c_3\sigma(\beta t)t^{-pQ}\left(\int_{\beta t}^{2\beta t} \rho(\tau)\tau^{Q-1}d\tau\right)\left(\int_0^{t} w^{1-p'}(\tau)\tau^{Q-1}d\tau\right)^{p-1} \leq$$

$$\leq c_3\sigma(\beta t)\left(\int_{\beta t}^{2\beta t} \rho(\tau)\tau^{-1}d\tau\right)\operatorname*{ess\,sup}_{\tau\in(0,t)}\frac{1}{w(\tau)} \leq c. \quad \blacksquare$$

The following lemma is proved analogously.

Lemma 2.5. *Let $1 \le p < \infty$, $\beta \ge 1$, the radial function $\rho \in A_p(G)$, and let σ and u be positive decreasing on $(0,\infty)$ functions, $v = \sigma\rho$, $w = u\rho$. If the pair (v,w) satisfies the condition*

$$\sup_{t>0}\left(\int_0^{\frac{t}{2}} v(\tau)\tau^{Q-1}d\tau\right)\left(\int_t^\infty w^{1-p'}(\tau)\tau^{-1-\frac{Q}{p-1}}d\tau\right)^{p-1} < \infty$$

for $1 < p < \infty$ and the condition

$$\sup_{t>0}\left(\int_0^t v(\tau)\tau^{Q-1}d\tau\right)\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\frac{1}{w(\tau)\tau^Q} < \infty$$

for $p = 1$, then there exists a positive constant c (depending only on Q and β) such that for an arbitrary $t > 0$ the inequality

$$\sigma\left(\frac{t}{\beta}\right) \le cu(t)$$

holds.

Theorem 2.1. *Let $1 < p < \infty$. Let σ and u be positive increasing on $(0,\infty)$ functions, the radial function $\rho \in A_p(G)$, and $v = \sigma\rho$ and $w = u\rho$. If the pair (v,w) satisfies the condition*

$$\sup_{t>0}\left(\int_t^\infty v(\tau)\tau^{Q-Qp-1}d\tau\right)\left(\int_0^t w^{1-p'}(\tau)\tau^{Q-1}d\tau\right)^{p-1} < \infty,$$

then there exists a constant c such that for any f the inequality

$$\int_G |Tf(x)|^p v(r(x))dx \le c\int_G |f(x)|^p w(r(x))dx \tag{2.2}$$

holds.

Proof. Without restriction of generality we may assume that the function σ has the form

$$\sigma(t) = \sigma(0) + \int_0^t \varphi(u)du,$$

where $\sigma(0) = \lim_{t\to 0}\sigma(t)$ and φ is a positive on $(0,\infty)$ function.

Estimate the left-hand side of inequality (2.2):

$$\int\limits_G |Tf(x)|^p v(r(x))dx = \int\limits_G |Tf(x)|^p \rho(x)\Big(\sigma(0) + \int\limits_0^{r(x)} \varphi(t)dt\Big)dx =$$

$$= \int\limits_G |Tf(x)|^p \rho(x)\sigma(0)dx + \int\limits_G |Tf(x)|^p \rho(x)\Big(\int\limits_0^{r(x)} \varphi(t)dt\Big)dx = I_1 + I_2.$$

If $\sigma(0) = 0$, then $I_1 = 0$. However if $\sigma(0) \neq 0$, then by Theorem A and also by Lemma 2.4 we have

$$I_1 = \sigma(0)\int\limits_G |Tf(x)|^p \rho(x)dx \leq c_1\sigma(0)\int\limits_G |f(x)|^p \rho(x)dx \leq$$

$$\leq c_1 \int\limits_G |f(x)|^p \rho(x)\sigma(r(x))dx \leq c_2 \int\limits_G |f(x)|^p \rho(x)u(r(x))dx =$$

$$= c_2 \int\limits_G |f(x)|^p w(r(x))dx.$$

Without loss of generality we may assume that for a homogeneous norm r the condition $c_0 \geq 1$ is fulfilled.

Estimate I_2:

$$I_2 = \int\limits_G |Tf(x)|^p \rho(x)\Big(\int\limits_0^{r(x)} \varphi(t)dt\Big)dx =$$

$$= \int\limits_0^{\infty} \varphi(t)\Big(\int\limits_{r(x)>t} |Tf(x)|^p \rho(x)dx\Big)dt \leq$$

$$\leq c_3 \int\limits_0^{\infty} \varphi(t)\Big(\int\limits_{r(x)>t} \Big|\int\limits_{r(y)>\frac{t}{2c_0}} k(xy^{-1})f(y)dy\Big|^p \rho(x)dx\Big)dt +$$

$$+ c_3 \int\limits_0^{\infty} \varphi(t)\Big(\int\limits_{r(x)>t} \Big|\int\limits_{r(y)\leq\frac{t}{2c_0}} k(xy^{-1})f(y)dy\Big|^p \rho(x)dx\Big)dt = I_{21} + I_{22}.$$

Using again Theorem A and Lemma 2.4, for the integral I_{21} we obtain

$$I_{21} \leq c_3 \int_0^\infty \varphi(t)\bigg(\int_G \bigg|\int_G k(xy^{-1})f(y)\chi_{\{z:r(z)>\frac{t}{2c_0}\}}(y)dy\bigg|^p \rho(x)dx\bigg)dt \leq$$

$$\leq c_4 \int_0^\infty \varphi(t)\bigg(\int_{r(x)>\frac{t}{2c_0}} |f(x)|^p\rho(x)dx\bigg)dt =$$

$$= c_4 \int_G |f(x)|^p\rho(x)\bigg(\int_0^{2c_0 r(x)} \varphi(t)dt\bigg)dx \leq$$

$$\leq c_5 \int_G |f(x)|^p\rho(x)u(r(x))dx = c_5 \int_G |f(x)|^p w(r(x))dx.$$

Estimate now I_{22}. If $r(x) > t$ and $r(y) \leq \frac{t}{2c_0}$, then $r(xy^{-1}) \geq \frac{1}{c_0}r(x) - r(y) \geq \frac{1}{c_0}r(x) - \frac{t}{2c_0} \geq \frac{1}{c_0}r(x) - \frac{1}{2c_0}r(x) = \frac{r(x)}{2c_0}$. This yields

$$I_{22} \leq c_6 \int_0^\infty \varphi(t)\bigg(\int_{r(x)>t} \rho(x)\bigg(\int_{r(y)\leq\frac{t}{2c_0}} \frac{|f(y)|}{r(xy^{-1})^Q}dy\bigg)^p dx\bigg)dt \leq$$

$$\leq c_7 \int_0^\infty \varphi(t)\bigg(\int_{r(x)>t} \frac{\rho(x)}{r(x)^{Qp}}dx\bigg)\bigg(\int_{r(y)\leq\frac{t}{2c_0}} |f(y)|dy\bigg)^p dt \leq$$

$$\leq c_7 \int_0^\infty \varphi(2s)\bigg(\int_{r(x)>2s} \frac{\rho(x)}{r(x)^{Qp}}dx\bigg)\bigg(\int_{r(y)\leq s} |f(y)|dy\bigg)^p ds =$$

$$= c_8 \int_0^\infty \varphi(2s)\bigg(\int_{2s}^\infty \rho(\tau)\tau^{Q-Qp-1}d\tau\bigg)\bigg(\int_0^s t^{Q-1}\bigg(\int_{S_G} |f(\delta_t\overline{y})|d\sigma(\overline{y})\bigg)dt\bigg)^p ds.$$

Besides we have the following estimates:

$$\int_t^\infty \varphi(2s)\bigg(\int_{2s}^\infty \rho(\tau)\tau^{Q-Qp-1}d\tau\bigg)ds = \int_{2t}^\infty \varphi(s)\bigg(\int_s^\infty \rho(\tau)\tau^{Q-Qp-1}d\tau\bigg)ds =$$

$$= \int_{2t}^\infty \rho(\tau)\tau^{Q-Qp-1}\bigg(\int_{2t}^\tau \varphi(s)ds\bigg)d\tau \leq \int_{2t}^\infty \rho(\tau)\tau^{Q-Qp-1}\sigma(\tau)d\tau.$$

Owing to the last inequalities, by Theorem D (part one) and Theorem G we arrive at

$$\int_0^\infty \varphi(2s)\bigg(\int_{2s}^\infty \rho(\tau)\tau^{Q-Qp-1}d\tau\bigg)\bigg(\int_0^s t^{Q-1}\bigg(\int_{S_G}|f(\delta_t\overline{y})|d\sigma(\overline{y})\bigg)dt\bigg)^p ds \leq$$

$$\leq c_9\int_0^\infty w(t)t^{Q-1}\bigg(\int_{S_G}|f(\delta_t\overline{y})|d\sigma(\overline{y})\bigg)^p dt \leq$$

$$\leq c_{10}\int_0^\infty w(t)t^{Q-1}\bigg(\int_{S_G}|f(\delta_t\overline{y})|^p d\sigma(\overline{y})\bigg)dt =$$

$$= c_{11}\int_G |f(x)|^p w(r(x))dx. \ \blacksquare$$

Writing the function $\sigma(t)$ in terms of $\sigma(t) = \sigma(+\infty) + \int_t^\infty \psi(u)du$, where $\sigma(+\infty) = \lim\limits_{t\to\infty}\sigma(t)$ and $\psi(t)$ is a positive on $(0,\infty)$ function, using Theorem A, Theorem D (part two), Lemma 2.5 and arguing as in proving Theorem 2.1, we can easily get the following

Theorem 2.2. *Let $1 < p < \infty$. Let σ and u be positive decreasing on $(0,\infty)$ functions, the radial function $\rho \in A_p(G)$, and $v = \sigma\rho$ and $w = u\rho$. If v and w satisfy the condition*

$$\sup_{t>0}\bigg(\int_0^{\frac{t}{2}} v(\tau)\tau^{Q-1}d\tau\bigg)\bigg(\int_t^\infty w^{1-p'}(\tau)\tau^{-1-\frac{Q}{p-1}}d\tau\bigg)^{p-1} < \infty,$$

then the inequality (2.2) *holds.*

Similar theorems for classical singular integrals have been proved in [4].

Example 2.1. *Let*

$$w(t) = \begin{cases} t^{Q(p-1)}\ln^p\dfrac{1}{t} & \text{for} \quad t\in(0,\frac{1}{2}) \\ \Big(2^{\beta-p+1}\ln^p 2\Big)t^\beta & \text{for} \quad t\in[\frac{1}{2},\infty) \end{cases},$$

$$v(t) = \begin{cases} t^{Q(p-1)} & \text{for} \quad t\in(0,\frac{1}{2}) \\ 2^{\alpha-p+1}t^\alpha & \text{for} \quad t\in[\frac{1}{2},\infty) \end{cases},$$

where $0 < \alpha \le \beta < Q(p-1)$. *Then the pair* (v, w) *satisfies the condition of Theorem* 2.1.

Theorem 2.3. *Let* σ *and* u *be positive increasing on* $(0,\infty)$ *functions, the radial function* $\rho \in A_1(G)$, $v = \sigma\rho$ *and* $w = u\rho$. *If the pair* (v, w) *satisfies the condition*

$$\sup_{t>0}\left(\int\limits_t^\infty \frac{v(\tau)}{\tau}d\tau\right)\operatorname*{ess\,sup}_{\tau\in(0,t)}\frac{1}{w(\tau)} < \infty,$$

then there exists a positive constant c *such that for arbitrary* λ *and* f *the inequality*

$$\int\limits_{\{x:|Tf(x)|>\lambda\}} v(r(x))dx \le \frac{c}{\lambda}\int\limits_G |f(x)|w(r(x))dx \tag{2.3}$$

holds.

Proof. Without restriction of generality we represent the function σ as follows:

$$\sigma(t) = \sigma(0) + \int\limits_0^t \varphi(u)du,$$

where $\sigma(0) = \lim\limits_{t\to 0}\sigma(t)$ and φ is a positive on $(0,\infty)$ function.

Estimate the left-hand side of inequality (2.3).

$$\int\limits_{\{x:|Tf(x)|>\lambda\}} v(r(x))dx = \int\limits_{\{x:|Tf(x)|>\lambda\}} \rho(x)\left(\sigma(0) + \int\limits_0^{r(x)}\varphi(t)dt\right)dx =$$

$$= \int\limits_{\{x:|Tf(x)|>\lambda\}} \rho(r(x))\sigma(0)dx + \int\limits_{\{x:|Tf(x)|>\lambda\}} \rho(x)\left(\int\limits_0^{r(x)}\varphi(t)dt\right)dx = I_1 + I_2.$$

If $\sigma(0) = 0$, then $I_1 = 0$. However if $\sigma(0) \neq 0$, then by Theorem B and also by Lemma 2.4 we have

$$I_1 = \sigma(0)\int\limits_{\{x:|Tf(x)|>\lambda\}} \rho(x)dx \le \frac{c_1\sigma(0)}{\lambda}\int\limits_G |f(x)|\rho(x)dx \le$$

$$\le \frac{c_1}{\lambda}\int\limits_G |f(x)|\rho(x)\sigma(r(x))dx \le \frac{c_2}{\lambda}\int\limits_G |f(x)|w(r(x))dx.$$

Now estimate the integral I_2.

$$I_2=\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>t}\chi\{x:|Tf(x)|>\lambda\}\rho(x)dx\bigg)dt\le$$

$$\le\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>t}\chi\bigg\{x:\bigg|\int\limits_{r(y)>\frac{t}{2c_0}}k(xy^{-1})f(y)dy\bigg|>\frac{\lambda}{2}\bigg\}\rho(x)dx\bigg)dt+$$

$$+\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>t}\chi\bigg\{x:\bigg|\int\limits_{r(y)\le\frac{t}{2c_0}}k(xy^{-1})f(y)dy\bigg|>\frac{\lambda}{2}\bigg\}\rho(x)dx\bigg)dt=$$

$$=I_{21}+I_{22}.$$

Using again Theorem B and Lemma 2.4, we obtain

$$I_{21}\le\int\limits_0^\infty \varphi(t)\bigg(\int\limits_G\chi\bigg\{x:\bigg|\int\limits_{r(y)>\frac{t}{2c_0}}k(xy^{-1})f(y)dy\bigg|>\frac{\lambda}{2}\bigg\}\rho(x)dx\bigg)dt\le$$

$$\le\frac{c_3}{\lambda}\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>\frac{t}{2c_0}}|f(x)|\rho(x)dx\bigg)dt=$$

$$=\frac{c_3}{\lambda}\int\limits_G|f(x)|\rho(x)\bigg(\int\limits_0^{2c_0r(x)}\varphi(t)dt\bigg)dx\le$$

$$\le\frac{c_4}{\lambda}\int\limits_G|f(x)|w(r(x))dx.$$

Estimate the integral I_{22}. If $r(x)>t$ and $r(y)\le\frac{t}{2c_0}$, then $r(xy^{-1})\ge\frac{r(x)}{2c_0}$. We have:

$$I_{22}=\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>t}\rho(x)\chi\bigg\{x:\bigg|\int\limits_{r(y)\le\frac{t}{2c_0}}k(xy^{-1})f(y)dy\bigg|>\frac{\lambda}{2}\bigg\}dx\bigg)dt\le$$

$$\le\frac{2}{\lambda}\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>t}\rho(x)\bigg|\int\limits_{r(y)\le\frac{t}{2c_0}}k(xy^{-1})f(y)dy\bigg|dx\bigg)dt\le$$

$$\leq \frac{c_5}{\lambda}\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>t}\rho(x)r(x)^{-Q}dx\bigg)\bigg(\int\limits_{r(y)\leq\frac{t}{2}}|f(y)|dy\bigg)dt=$$

$$=\frac{c_6}{\lambda}\int\limits_0^\infty \varphi(2s)\bigg(\int\limits_{2s}^\infty \frac{\rho(\gamma)}{\gamma}d\gamma\bigg)\bigg(\int\limits_0^s t^{Q-1}\bigg(\int\limits_{S_G}|f(\delta_t\overline{y})|d\sigma(\overline{y})\bigg)dt\bigg)ds.$$

Besides we have the following estimates:

$$\int\limits_t^\infty \varphi(2s)\bigg(\int\limits_{2s}^\infty \frac{\rho(\gamma)}{\gamma}d\gamma\bigg)ds=\int\limits_{2t}^\infty \varphi(s)\bigg(\int\limits_s^\infty \frac{\rho(\gamma)}{\gamma}d\gamma\bigg)ds=$$

$$=\int\limits_{2t}^\infty \frac{\rho(\gamma)}{\gamma}\bigg(\int\limits_{2t}^\gamma \varphi(s)ds\bigg)d\gamma\leq\int\limits_{2t}^\infty \frac{\rho(\gamma)\sigma(\gamma)}{\gamma}d\gamma,$$

By Lemma A we have:

$$\frac{c_6}{\lambda}\int\limits_0^\infty \varphi(2s)\bigg(\int\limits_{2s}^\infty \frac{\rho(\gamma)}{\gamma}d\gamma\bigg)\bigg(\int\limits_0^s t^{Q-1}\bigg(\int\limits_{S_G}|f(\delta_t\overline{y})|d\sigma(\overline{y})\bigg)dt\bigg)ds\leq$$

$$\leq\frac{c_7}{\lambda}\int\limits_0^\infty t^{Q-1}w(t)\bigg(\int\limits_{S_G}|f(\delta_t\overline{y})|d\sigma(\overline{y})\bigg)dt=\frac{c_8}{\lambda}\int\limits_G |f(y)|w(r(y))dy. \ \blacksquare$$

If we represent the function σ in terms of $\sigma(t)=\sigma(+\infty)+\int_t^\infty \psi(u)du$, where $\sigma(+\infty)=\lim\limits_{t\to+\infty}\sigma(t)$, and the function ψ is positive on $(0,\infty)$, then using Theorem B, Lemma B and Lemma 2.5, and arguing as in proving Theorem 2.3, we obtain the following

Theorem 2.4. *Let σ and u be positive decreasing on $(0,\infty)$ functions, the radial function $\rho\in A_1(G)$, and $v=\sigma\rho$, $w=u\rho$. If the pair (v,w) satisfies the condition*

$$\sup_{t>0}\bigg(\int\limits_0^t v(\tau)\tau^{Q-1}d\tau\bigg)\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\frac{1}{w(\tau)\tau^Q}<\infty,$$

then inequality (2.3) *holds.*

Example 2.2. *Let*

$$w(t)=\begin{cases}\dfrac{1}{t^Q}\ln^\gamma\dfrac{1}{t} & \textit{for}\ \ t<c\\ \Big(c^{-Q-\alpha}\ln^\gamma\dfrac{1}{c}\Big)t^\alpha & \textit{for}\ \ t\ge c\end{cases},$$

$$v(t)=\begin{cases}\dfrac{1}{t^Q}\ln^\beta\dfrac{1}{t} & \textit{for}\ \ t<c\\ \Big(c^{-Q-\lambda}\ln^\beta\dfrac{1}{c}\Big)t^\lambda & \textit{for}\ \ t\ge c\end{cases},$$

where $\beta+1<\gamma\le 0$, $-Q<\lambda<\alpha<0$, $c=e^{\frac{\beta}{Q}}$. *Then the pair* (v,w) *satisfies the condition of Theorem* 2.4.

The sufficient conditions for the pairs of general radial weights guaranteeing the validity of two-weight inequalities of a strong type for singular integrals are found in the following

Theorem 2.5. *Let* $1<p<\infty$. *Let* $v(t)$ *and* $w(t)$ *be positive on* $(0,\infty)$ *functions satisfying the following conditions:*

1) *there exists a positive constant* b *such that for any* $t>0$ *the inequality*

$$\sup_{\frac{t}{c_0}<\tau\le 8c_0t} v(\tau)\le b\inf_{\frac{t}{c_0}<\tau\le 8c_0t} w(\tau)$$

holds;

2) $\displaystyle\sup_{t>0}\bigg(\int_t^\infty v(\tau)\tau^{Q-Qp-1}d\tau\bigg)\bigg(\int_0^t w^{1-p'}(\tau)\tau^{Q-1}d\tau\bigg)^{p-1}<\infty;$

3) $\displaystyle\sup_{t>0}\bigg(\int_t^t v(\tau)\tau^{Q-1}d\tau\bigg)\bigg(\int_0^\infty w^{1-p'}(\tau)\tau^{-1-\frac{Q}{p-1}}d\tau\bigg)^{p-1}<\infty.$

Then inequality (2.2) *holds.*

Proof. Estimate the left-hand side of inequality (2.2) as follows:

$$\int_G |Tf(x)|^p v(r(x))dx=\sum_{k\in\mathbb{Z}}\int_{2^k<r(x)\le 2^{k+1}}|Tf(x)|^p v(r(x))dx\le$$

$$\le c_1\bigg(\sum_{k\in\mathbb{Z}}\int_{2^k<r(x)\le 2^{k+1}}\Big|T\Big(f\cdot\chi_{\{r(y)\le\frac{2^{k-1}}{c_0}\}}\Big)(x)\Big|^p v(r(x))dx\bigg)+$$

$$+c_1\Bigg(\sum_{k\in\mathbb{R}}\int\limits_{2^k<r(x)\le 2^{k+1}}\Big|T\Big(f\cdot\chi_{\{\frac{2^{k-1}}{c_0}<r(y)\le c_0 2^{k+2}\}}\Big)(x)\Big|^p v(r(x))dx\Bigg)+$$

$$+c_1\Bigg(\sum_{k\in\mathbb{R}}\int\limits_{2^k<r(x)\le 2^{k+1}}\Big|T\Big(f\cdot\chi_{\{r(y)>c_0 2^{k+2}\}}\Big)(x)\Big|^p v(r(x))dx\Bigg)=$$

$$=c_1\Bigg(\sum_k P_{1k}+\sum_k p_{2k}+\sum_k p_{3k}\Bigg).$$

Estimate now $\sum\limits_k p_{1k}$. If $2^k<r(x)\le 2^{k+1}$ and $r(y)\le\frac{2^{k-1}}{c_0}$, then obviously, $r(y)\le\frac{2^{k-1}}{c_0}=\frac{2^k}{2c_0}\le\frac{r(x)}{2c_0}\le r(x)$, $r(xy^{-1})\ge\frac{r(x)}{2c_0}$. We have:

$$\sum_k p_{1k}\le c_2\sum_k\int\limits_{2^k<r(x)\le 2^{k+1}}v(r(x))\Bigg(\int\limits_{r(y)\le\frac{2^{k-1}}{c_0}}\frac{|f(y)|}{r(xy^{-1})^Q}dy\Bigg)^p dx\le$$

$$\le c_3\sum_k\int\limits_{2^k<r(x)\le 2^{k+1}}v(r(x))r(x)^{-Qp}\Bigg(\int\limits_{r(y)\le r(x)}|f(y)|dy\Bigg)^p dx=$$

$$=c_4\int\limits_G v(r(x))r(x)^{-Qp}\Bigg(\int\limits_0^{r(x)}\tau^{Q-1}\Bigg(\int\limits_{S_G}|f(\delta_\tau\overline{y})|d\sigma(\overline{y})\Bigg)d\tau\Bigg)^p dx=$$

$$=c_5\int\limits_0^\infty v(t)t^{Q-Qp-1}\Bigg(\int\limits_0^t\tau^{Q-1}\Bigg(\int\limits_{S_G}|f(\delta_\tau\overline{y})|d\sigma(\overline{y})\Bigg)d\tau\Bigg)^p dt\le$$

$$\le c_6\int\limits_0^\infty w(t)t^{Q-1}\Bigg(\int\limits_{S_G}|f(\delta_\tau\overline{y})|d\sigma(\overline{y})\Bigg)^p dt\le$$

$$\le c_7\int\limits_G|f(y)|^p w(r(y))dy.$$

Here we have used Theorem D (part one).

Estimate now $\sum\limits_k p_{3k}$. Note that if $2^k<r(x)\le 2^{k+1}$ and $r(y)>c_0 2^{k+2}$, then $r(x)\le r(y)$ and $r(xy^{-1})\ge\frac{r(y)}{2c_0}$. We have:

$$\sum_k p_{3k}\le c_8\sum_k\int\limits_{2^k<r(x)\le 2^{k+1}}v(r(x))\Bigg(\int\limits_{r(x)\le r(y)}\frac{|f(y)|}{r(y)^Q}dy\Bigg)^p dx=$$

$$= c_9 \int\limits_0^\infty v(t)t^{Q-1}\bigg(\int\limits_t^\infty \tau^{-1}\Big(\int\limits_{S_G} |f(\delta_\tau \overline{y})|d\sigma(\overline{y})\Big)d\tau\bigg)^p dt \le$$

$$\le c_{10} \int\limits_0^\infty w(t)t^{Q-1}\bigg(\int\limits_{S_G} |f(\delta_t \overline{y})|d\sigma(\overline{y})\bigg)^p dt \le c_{11} \int\limits_G |f(y)|^p w(r(y))dy.$$

Here the use has been made of Theorem D (part two).

Using the boundedness of the operator T in the space $L^p(G)$, we get for $\sum_k p_{2k}$

$$\sum_k p_{2k} \le$$

$$\le \sum_k \bigg(\sup_{\frac{2^{k-1}}{c_0}<r(x)\le 2^{k+2}c_0} v(r(x))\bigg) \int\limits_G |T(f\cdot \chi_{\{\frac{2^{k-1}}{c_0}<r(y)\le 2^{k+2}c_0\}})(x)|^p dx \le$$

$$\le c_{12} \sum_k \bigg(\sup_{\frac{2^{k-1}}{c_0}<r(x)\le 2^{k+2}c_0} v(r(x))\bigg) \int\limits_{\frac{2^{k-1}}{c_0}<r(x)\le 2^{k+2}c_0} |f(x)|^p dx \le$$

$$\le c_{13} \sum_k \int\limits_{\frac{2^{k-1}}{c_0}<r(x)\le 2^{k+2}c_0} |f(x)|^p w(r(x))dx \le c_{14} \int\limits_G |f(x)|^p w(r(x))dx. \quad \blacksquare$$

In [4] it has been proved that conditions (2) and (3) of Theorem 2.5 are necessary for inequality (2.2) to be valid in the case of the Hilbert transform.

Corollary 2.1. *Let* $1 < p < \infty$*. Let* $v(t)$ *and* $w(t)$ *be positive increasing on* $(0, \infty)$ *functions satisfying condition* (2) *of Theorem* 2.5*. Then inequality* (2.2) *holds.*

Corollary 2.2. *Let* $1 < p < \infty$*. Let* $v(t)$ *and* $w(t)$ *be positive decreasing on* $(0, \infty)$ *functions satisfying condition* (3) *of Theorem* 2.5*. Then inequality* (2.2) *holds.*

These corollaries can be easily proved due to the monotonicity of weights. If the functions $v(t)$ and $w(t)$ are increasing (decreasing) and satisfy condition (2) (condition (3)) of Theorem 2.5, then the rest of the conditions can be easily fulfilled.

Using Lemmas A and B and taking into account the proof of Theorem 2.5, we obtain

Theorem 2.6. *Let $v(t)$ and $w(t)$ be positive on $(0,\infty)$ functions satisfying the following conditions:*

1) *there exists a constant b such that for any $t>0$ the inequality*

$$\sup_{\frac{t}{c_0}<\tau\le 8c_0t} v(\tau)\le b \inf_{\frac{t}{c_0}<\tau\le 8c_0t} w(\tau)$$

holds;

2) $\sup\limits_{t>0}\Bigg(\int\limits_t^\infty \dfrac{v(\tau)}{\tau}d\tau\Bigg)\operatorname*{ess\,sup}\limits_{\tau\in(0,2t)}\dfrac{1}{w(\tau)}<\infty;$

3) $\sup\limits_{t>0}\Bigg(\int\limits_0^t v(\tau)\tau^{Q-1}d\tau\Bigg)\operatorname*{ess\,sup}\limits_{\tau\in(\frac{t}{2},\infty)}\dfrac{1}{w(\tau)\tau^Q}<\infty.$

Then inequality (2.3) *holds.*

Because of monotonicity of the weights v and w we can easily get the following

Corollary 2.3. *Let $v(t)$ and $w(t)$ be positive increasing on $(0,\infty)$ functions satisfying condition* 2) *of Theorem* 2.6. *Then inequality* (2.3) *holds.*

Corollary 2.4. *Let $1<p<\infty$. Let $v(t)$ and $w(t)$ be positive decreasing on $(0,\infty)$ functions satisfying condition* 3) *of Theorem* 2.6. *Then inequality* (2.3) *holds.*

§ 3. Two-weight Inequalities of a Strong (p,q), $1<q<p<\infty$, and of a Weak $(p,1)$, $p>1$, Types for Singular Integrals Defined on Homogeneous Groups

In this section the sufficient conditions for the pairs of weights ensuring the validity of two-weight inequalities of a strong (p,q), $1<q<p<\infty$, and of a weak $(p,1)$, $p>1$, type are found for singular integrals defined on homogeneous groups.

Corresponding examples for the pairs of weights are given.

Lemma 3.1. *Let $1<q<p<\infty$, $\beta\ge 1$, and let $v(t)$ and $w(t)$ be positive increasing on $(0,\infty)$ functions satisfying the condition*

$$I=\int\limits_0^\infty\Bigg[\Bigg(\int\limits_t^\infty v(\tau)\tau^{Q-Qq-1}d\tau\Bigg)\Bigg(\int\limits_0^{\frac{t}{2}} w^{1-p'}(\tau)\tau^{Q-1}d\tau\Bigg)^{q-1}\Bigg]^{\frac{p}{p-q}}\times$$
$$\times w^{1-p'}\Big(\frac{t}{2}\Big)t^{Q-1}dt<\infty.$$

Then the inequality

$$\int_0^\infty v^{\frac{p}{p-q}}(\beta t)w^{-\frac{q}{p-q}}(\frac{t}{2})t^{Q-1}dt < \infty$$

holds.

Proof. Since v and w are monotonous, we have

$$I \geq \int_0^\infty \Bigg[\bigg(\int_{t\beta}^{2t\beta} v(\tau)\tau^{Q-Qq-1}d\tau\bigg)\bigg(\int_0^{\frac{t}{2}} w^{1-p'}(\tau)\tau^{Q-1}d\tau\bigg)^{q-1}\Bigg]^{\frac{p}{p-q}} \times$$
$$\times w^{1-p'}\Big(\frac{t}{2}\Big)t^{Q-1}dt \geq c_1\int_0^\infty v^{\frac{p}{p-q}}(\beta t)w^{-\frac{q}{p-q}}\Big(\frac{t}{2}\Big)t^{Q-1}dt. \ \blacksquare$$

Lemmas below are as well proved trivially.

Lemma 3.2. *Let $1 < q < p < \infty$, $\beta \geq 1$, and let $v(t)$ and $w(t)$ be positive decreasing on $(0,\infty)$ functions satisfying the condition*

$$\int_0^\infty \Bigg[\bigg(\int_0^{\frac{t}{2}} v(\tau)\tau^{Q-1}d\tau\bigg)\bigg(\int_t^\infty w^{1-p'}(\tau)\tau^{-1-\frac{Q}{p-1}}d\tau\bigg)^{q-1}\Bigg]^{\frac{p}{p-q}} \times$$
$$\times w^{1-p'}(t)t^{-1-\frac{Q}{p-1}}dt < \infty.$$

Then the inequality

$$\int_0^\infty v^{\frac{p}{p-q}}\Big(\frac{t}{2\beta}\Big)w^{-\frac{q}{p-q}}(t)t^{Q-1}dt < \infty$$

holds.

Lemma 3.3. *Let $1 < p < \infty$ and let v and w be positive on $(0,\infty)$ functions, provided the function v is increasing and the pair (v,w) satisfies the condition*

$$\int_0^\infty \bigg(\int_t^\infty \frac{v(\tau)}{\tau}d\tau\bigg)^{p'} w^{1-p'}\Big(\frac{t}{2}\Big)t^{Q-1}dt < \infty.$$

Then the following inequality holds:

$$\int\limits_0^\infty v^{p'}(\beta t) w^{1-p'}\Big(\frac{t}{2}\Big) t^{Q-1} dt < \infty,$$

where $\beta \geq 1$.

Lemma 3.4. *Let* $1 < p < \infty$, $\beta \geq 1$, *and let* v *and* w *be positive on* $(0,\infty)$ *functions, provided the function* v *is decreasing and the pair* (v,w) *satisfies the condition*

$$\int\limits_0^\infty \Big[\int\limits_0^{\frac{t}{2}} v(\tau)\tau^{Q-1} d\tau\Big]^{p'} w^{1-p'}(t) t^{Q-Qp'-1} dt < \infty.$$

Then the inequality

$$\int\limits_0^\infty v^{p'}\Big(\frac{t}{2\beta}\Big) w^{1-p'}(t) t^{Q-1} dt < \infty$$

holds.

Theorem 3.1. *Let* $1 < q < p < \infty$, *and let* u *and* w *be positive increasing on* $(0,\infty)$ *functions satisfying the condition*

$$\int\limits_0^\infty \Big[\Big(\int\limits_t^\infty v(\tau)\tau^{Q-Qq-1} d\tau\Big)\Big(\int\limits_0^{\frac{t}{2}} w^{1-p'}(\tau)\tau^{Q-1} d\tau\Big)^{q-1}\Big]^{\frac{p}{p-q}} \times$$
$$\times w^{1-p'}\Big(\frac{t}{2}\Big) t^{Q-1} dt < \infty.$$

Then there exists a positive constant $c > 0$ *such that for any* f *the inequality*

$$\Big(\int\limits_G |Tf(x)|^q v(r(x)) dx\Big)^{\frac{1}{q}} \leq c\Big(\int\limits_G |f(x)|^p w(r(x)) dx\Big)^{\frac{1}{p}} \tag{3.1}$$

holds.

Proof. Without restriction of generality we represent the function v as:

$$v(t) = v(0) + \int\limits_0^t \varphi(u) du,$$

where $v(0)=\lim\limits_{t\to 0} v(t)$ and φ is a positive on $(0,\infty)$ function.

Estimate the left-hand side of inequality (3.1):

$$\bigg(\int\limits_G |Tf(x)|^q v(r(x))dx\bigg)^{\frac{1}{q}} \le c_1\bigg(\int\limits_G |Tf(x)|^q v(0)dx\bigg)^{\frac{1}{q}} +$$

$$+c_1\bigg(\int\limits_G |Tf(x)|^q\bigg(\int\limits_0^{r(x)} \varphi(t)dt\bigg)dx\bigg)^{\frac{1}{q}} = I_1+I_2.$$

If $v(0)=0$, then $I_1=0$, but if $v(0)\neq 0$, then due to the fact that T is bounded in the space $L^q(G)$, by Theorem F and Lemma 3.1 we obtain

$$I_1 = c_1 v^{\frac{1}{q}}(0)\bigg(\int\limits_G |Tf(x)|^q dx\bigg)^{\frac{1}{q}} \le c_2 v^{\frac{1}{q}}(0)\bigg(\int\limits_G |f(x)|^q dx\bigg)^{\frac{1}{q}} \le$$

$$\le c_2\bigg(\int\limits_G |f(x)|^q v(2c_0 r(x))dx\bigg)^{\frac{1}{q}} \le c_3\bigg(\int\limits_G |f(x)|^p w(r(x))dx\bigg)^{\frac{1}{p}}.$$

Let us estimate integral I_2.

$$I_2 = c_1\bigg(\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>t} |Tf(x)|^q dx\bigg)dt\bigg)^{\frac{1}{q}} \le$$

$$\le c_4\bigg(\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>t}\bigg|\int\limits_{r(y)>\frac{t}{2c_0}} k(xy^{-1})f(y)dy\bigg|^q dx\bigg)dt\bigg)^{\frac{1}{q}} +$$

$$+c_4\bigg(\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{r(x)>t}\bigg|\int\limits_{r(y)\le\frac{t}{2c_0}} k(xy^{-1})f(y)dy\bigg|^q dx\bigg)dt\bigg)^{\frac{1}{q}} = I_{21}+I_{22}$$

Using again the boundedness of the operator T in $L^q(G)$ and also Theorem F and Lemma 3.1, we get

$$I_{21} \le c_4\bigg(\int\limits_0^\infty \varphi(t)\bigg(\int\limits_G\bigg|\int\limits_G f(y)k(xy^{-1})\cdot\chi_{\{z:r(z)>\frac{t}{2c_0}\}}(y)\bigg|^q dx\bigg)dt\bigg)^{\frac{1}{q}} \le$$

$$\le c_5\bigg(\int\limits_0^\infty \varphi(t)\bigg(\int\limits_G |f(x)|^q\chi_{\{z:r(z)>\frac{t}{2c_0}\}}(x)dx\bigg)dt\bigg)^{\frac{1}{q}} =$$

$$= c_5\bigg(\int_G |f(x)|^q\bigg(\int_0^{2c_0r(x)} \varphi(t)dt\bigg)dx\bigg)^{\frac{1}{q}} \le c_5\bigg(\int_G |f(x)|^q v(2c_0r(x))dx\bigg)^{\frac{1}{q}} \le$$

$$\le c_6\bigg(\int_G |f(x)|^p w(r(x))dx\bigg)^{\frac{1}{p}}.$$

Estimate now I_{22}. Note that if $r(x) > t$ and $r(y) \le \frac{t}{2c_0}$, then $r(xy^{-1}) \ge \frac{r(x)}{2c_0}$. We have

$$I_{22} \le c_7\bigg(\int_0^\infty \varphi(t)\bigg(\int_{r(x)>t}\bigg(\int_{r(y)\le\frac{t}{2c_0}} \frac{|f(y)|}{r(xy^{-1})^Q}dy\bigg)^q dx\bigg)dt\bigg)^{\frac{1}{q}} \le$$

$$\le c_8\bigg(\int_0^\infty \varphi(t)\bigg(\int_{r(x)>t} \frac{dx}{r(x)^{Qq}}\bigg)\bigg(\int_{r(y)\le\frac{t}{2}} |f(y)|dy\bigg)^q dt\bigg)^{\frac{1}{q}} =$$

$$= c_9\bigg(\int_0^\infty \varphi(2s)\bigg(\int_{2s}^\infty \tau^{Q-Qq-1}d\tau\bigg)\bigg(\int_0^s t^{Q-1}\bigg(\int_{S_G} |f(\delta_t\overline{y})|d\sigma(\overline{y})\bigg)dt\bigg)^q ds\bigg)^{\frac{1}{q}}.$$

Taking into account that

$$\int_t^\infty \varphi(2s)\bigg(\int_{2s}^\infty \tau^{Q-Qq-1}d\tau\bigg)ds \le \int_{2t}^\infty v(\tau)\tau^{Q-Qq-1}d\tau,$$

by Theorem E we obtain that the last integral in the above estimate does not exceed

$$c_{10}\bigg(\int_0^\infty w(t)t^{Q-1}\bigg(\int_{S_G} |f(\delta_t\overline{y})|d\sigma(\overline{y})\bigg)^p dt\bigg)^{\frac{1}{p}} \le$$

$$\le c_{11}\bigg(\int_0^\infty w(t)t^{Q-1}\bigg(\int_{S_G} |f(\delta_t\overline{y})|^p d\sigma(\overline{y})\bigg)dt\bigg)^{\frac{1}{p}} =$$

$$= c_{12}\bigg(\int_G |f(y)|^p w(r(y))dy\bigg)^{\frac{1}{p}}. \blacksquare$$

Example 3.1. *Let*

$$w(t)=\begin{cases} t^{p-1}\ln^{\gamma}\dfrac{1}{t} & for \quad t<\frac{1}{c} \\ \Big(c^{\alpha-p+1}\ln^{\gamma}c\Big)t^{\alpha} & for \quad t\ge\frac{1}{c} \end{cases},$$

$$v(t)=\begin{cases} t^{q-1}\ln^{\beta}\dfrac{1}{t} & for \quad t<\frac{1}{c} \\ \Big(c^{\lambda-q+1}\ln^{\beta}c\Big)t^{\lambda} & for \quad t\ge\frac{1}{c} \end{cases},$$

where $c=e^{\frac{p}{p-q}}$, $p-1<\gamma<p$, $\beta<\frac{q}{p}\gamma-q+\frac{q}{p}-1$, $0\le\lambda<\alpha\frac{q}{p}+\frac{q}{p}-1$, $\frac{p}{q}-1<\alpha<p-1$. *Then the pair* (v,w) *satisfies the condition of Theorem* 3.1 *for* $Q=1$ *and the operator* H *is bounded from* L^p_w *to* L^q_v, *where* H *is the Hilbert transform.*

Representing the function v in terms of $v(t)=v(+\infty)+\int_t^\infty\psi(u)du$, where $v(+\infty)=\lim\limits_{t\to\infty}v(t)$, ψ is a positive on $(0,\infty)$ function and using Theorem F and Theorem E, Lemma 3.2 and the fact that the operator T is bounded in $L^q(G)$, we obtain

Theorem 3.2. *Let* $1<q<p<\infty$ *and let* v *and* w *be positive decreasing on* $(0,\infty)$ *functions satisfying the condition*

$$\int\limits_0^\infty\Bigg[\Bigg(\int\limits_0^{\frac{t}{2}}v(\tau)\tau^{Q-1}d\tau\Bigg)\Bigg(\int\limits_t^\infty w^{1-p'}(\tau)\tau^{-1-\frac{Q}{p-1}}d\tau\Bigg)^{q-1}\Bigg]^{\frac{p}{p-q}}w^{1-p'}(t)t^{-1-\frac{Q}{p-1}}dt<\infty.$$

Then inequality (3.1) *holds.*

Theorem 3.3. *Let* $p>1$, *and let* v *and* w *be positive on* $(0,\infty)$ *functions, provided the function* v *is increasing and the pair* (v,w) *satisfies the condition*

$$\int\limits_0^\infty\Bigg(\int\limits_t^\infty\frac{v(\tau)}{\tau}d\tau\Bigg)^{p'}w^{1-p'}\Big(\frac{t}{2}\Big)t^{Q-1}dt<\infty.$$

Then there exists a positive constant c *such that for arbitrary* $\lambda>0$ *and* f *the inequality*

$$\int\limits_{\{x:|Tf(x)|>\lambda\}}v(r(x))dx\le\frac{c}{\lambda}\Bigg(\int\limits_G|f(x)|^p w(r(x))dx\Bigg)^{\frac{1}{p}} \tag{3.2}$$

holds.

Proof. Represent the function v as:

$$v(t) = v(0) + \int_0^t \varphi(u)du,$$

where $v(0) = \lim\limits_{t\to 0} v(t)$ and φ is a positive on $(0,\infty)$ function. Write the left-hand side of inequality (3.2) in terms of

$$\int\limits_{\{x:|Tf(x)|>\lambda\}} v(r(x))dx =$$

$$= \int\limits_{\{x:|Tf(x)|>\lambda\}} v(0)dx + \int\limits_{\{x:|Tf(x)|>\lambda\}} \Bigg(\int_0^{r(x)} \varphi(t)dt\Bigg)dx =$$
$$= I_1 + I_2.$$

If $v(0) = 0$, then $I_1 = 0$, but if $v(0) \neq 0$, then due to the weak $(1,1)$ type inequality for the operator T, Theorem F and Lemma 3.3, we get

$$I_1 = v(0)|\{x : Tf(x) > \lambda\}| \le$$
$$\le \frac{c_1}{\lambda} v(0) \int_G |f(x)|dx \le \frac{c_2}{\lambda}\Bigg(\int_G |f(x)|^p w(r(x))dx\Bigg)^{\frac{1}{p}}.$$

Estimate I_2.

$$I_2 = \int_0^\infty \varphi(t)\Bigg(\int\limits_{r(x)>t} \chi\{x : |Tf(x)| > \lambda\}dx\Bigg)dt \le$$

$$\le \int_0^\infty \varphi(t)\Bigg(\int\limits_{r(x)>t} \chi\Bigg\{x : \Bigg|\int\limits_{r(y)>\frac{t}{2c_0}} k(xy^{-1})f(y)dy\Bigg| > \frac{\lambda}{2}\Bigg\}dx\Bigg)dt +$$

$$+ \int_0^\infty \varphi(t)\Bigg(\int\limits_{r(x)>t} \chi\Bigg\{x : \Bigg|\int\limits_{r(y)\le\frac{t}{2c_0}} k(xy^{-1})f(y)dy\Bigg| > \frac{\lambda}{2}\Bigg\}dx\Bigg)dt = I_{21} + I_{22}.$$

Using again weak $(1,1)$ type inequality for the operator T, Theorem F and Lemma 3.3, we obtain

$$I_{21} \leq \frac{c_3}{\lambda} \int\limits_0^\infty \varphi(t) \bigg(\int\limits_{r(x) > \frac{t}{2c_0}} |f(x)| dx \bigg) dt \leq \frac{c_4}{\lambda} \bigg(\int\limits_G |f(x)|^p w(r(x)) dx \bigg)^{\frac{1}{p}}.$$

Estimate the integral I_{22}. If $r(x) > t$ and $r(y) \leq \frac{t}{2c_0}$, then $r(xy^{-1}) \geq \frac{r(x)}{2c_0}$ and we have

$$I_{22} \leq \frac{2}{\lambda} \int\limits_0^\infty \varphi(t) \bigg(\int\limits_{r(x)>t} \bigg| \int\limits_{r(y) \leq \frac{t}{2c_0}} f(y) k(xy^{-1}) dy \bigg| dx \bigg) dt \leq$$

$$\leq \frac{c_5}{\lambda} \int\limits_0^\infty \varphi(t) \bigg(\int\limits_{r(x)>t} \frac{dx}{r(x)^Q} \bigg) \bigg(\int\limits_{r(y) \leq \frac{t}{2}} |f(y)| dy \bigg) dt =$$

$$= \frac{c_6}{\lambda} \int\limits_0^\infty \varphi(2s) \bigg(\int\limits_{2s}^\infty \frac{d\tau}{\tau} \bigg) \bigg(\int\limits_0^s t^{Q-1} \bigg(\int\limits_{S_G} |f(\delta_t \overline{y})| d\sigma(\overline{y}) \bigg) dt \bigg) ds.$$

Taking into account that

$$\int\limits_t^\infty \varphi(2s) \bigg(\int\limits_{2s}^\infty \frac{d\tau}{\tau} \bigg) ds = \int\limits_{2t}^\infty \frac{1}{\tau} \bigg(\int\limits_{2t}^\tau \varphi(s) ds \bigg) d\tau \leq \int\limits_{2t}^\infty \frac{v(\tau)}{\tau} \, d\tau,$$

by Lemma C we finally have that the last integral I_{22} in the above estimate does not exceed

$$\frac{c_7}{\lambda} \bigg(\int\limits_G |f(x)|^p w(r(x)) dx \bigg)^{\frac{1}{p}}. \ \blacksquare$$

Representing the function v in terms of $v(t) = v(+\infty) + \int_t^\infty \psi(u) du$, where $v(+\infty) = \lim\limits_{t \to 0} v(t)$ and ψ is a positive on $(0,\infty)$ function and using Theorem F, Lemma D, Lemma 3.4 and weak $(1,1)$ type inequality for the operator T, we obtain the following

Theorem 3.4. *Let $p > 1$ and let v and w be positive on $(0,\infty)$ functions, provided the function v is decreasing and the pair (v,w) satisfies* *the condition*

$$\int_0^\infty \Big(\int_0^{\frac{t}{2}} v(\tau)\tau^{Q-1}d\tau \Big)^{p'} w^{1-p'}(t)t^{Q-Qp'-1}dt < \infty.$$

Then inequality (3.2) *holds.*

Example 3.2. *Let*

$$w(t) = \begin{cases} \dfrac{1}{t}\ln^\gamma \dfrac{1}{t} & \textit{for} \quad t < c \\ \Big(c^{-\alpha-1}\ln^\gamma \dfrac{1}{c}\Big)t^\alpha & \textit{for} \quad t \ge c \end{cases},$$

$$v(t) = \begin{cases} \dfrac{1}{t}\ln^\beta \dfrac{1}{t} & \textit{for} \quad t < c \\ \Big(c^{-\lambda-1}\ln^\beta \dfrac{1}{c}\Big)t^\lambda & \textit{for} \quad t \ge c \end{cases},$$

where $\beta < -1$, $\gamma > p(\beta+2)+1$, $-1 < \lambda < 0$, $\alpha > (\lambda+1)p-1$, $c = e^{2\beta}$. *Then the pair* (v, w) *satisfies the condition of Theorem* 3.4 *for* $Q = 1$ *and inequality* (3.2) *is fulfilled in the case of the Hilbert transform.*

§ 4. Two-Weighted Estimates of Strong Type for Multiplicators of Fourier Transforms on The Line

In what follows S will denote a space of basic L. Schwartz functions. S consists of infinitely differentiable functions φ vanishing at infinity more rapidly than any power of $|x|$. For $\varphi \in S$, we denote by $\widetilde{\varphi}$ the Fourier transform of φ:

$$\widetilde{\varphi}(\lambda) = (2\pi)^{-1/2}\int_{\mathbb{R}^1} \varphi(x)\exp(-i\lambda x)\,dx.$$

Conversely,

$$\varphi(x) = (2\pi)^{-1/2}\int_{\mathbb{R}^1} \widetilde{\varphi}(\lambda)\exp(i\lambda x)\,d\lambda.$$

Since weight functions w are locally summable, it can easily be shown that for any w the space S will be dense everywhere in $L^p_w(\mathbb{R}^1)$.

We have

Theorem 4.1. *Let a measurable function* $\Phi(\lambda)$ *be defined by the formula*

$$\Phi(\lambda)=\int\limits_{-\infty}^{\lambda} d\mu(t),$$

where $\mu(t)$ *is the finite measure in* $\mathbb{R}^1$.

Moreover, let $1<p<\infty$, v *and* w *be positive functions on* $(0,\infty)$ *satisfying the conditions of Theorem* 2.5. *Then the transform* $\mathcal{K}$ *given in terms of Fourier images by the formula*

$$\widetilde{\mathcal{K}\varphi}(\lambda)=\Phi(\lambda)\widetilde{\varphi}(\lambda),\quad \varphi\in S,$$

will define the bounded mapping of $L^p_{w(|x|)}(\mathbb{R}^1)$ *into* $L^p_{v(|x|)}(\mathbb{R}^1)$.

Proof. We consider the Heaviside function

$$\widetilde{\chi}(\lambda)=\begin{cases}1 & \text{for } \lambda>0,\\ 0 & \text{for } \lambda<0.\end{cases}$$

The transform $T_{\lambda_0}:\varphi\to\psi$ given on S by the relation

$$\widetilde{g}(\lambda)=\widetilde{\chi}(\lambda-\lambda_0)\widetilde{\varphi}(\lambda) \tag{4.1}$$

generates the bounded transform from $L^p_w(\mathbb{R}^1)$ into $L^p_v(\mathbb{R}^1)$.

Assume first that $\lambda_0=0$. Equality (4.1) corresponds to the convolution in the initial space

$$g=\chi*\varphi,$$

where the generalized function χ has the form

$$\chi(x)=(2\pi)^{1/2}\Big(\delta(x)/2+1/(2\pi i x)\Big).$$

Now by Theorem 2.5 we obtain the boundedness of the operator T_0 from $L^p_w(\mathbb{R}^1)$ into $L^p_v(\mathbb{R}^1)$ for $\lambda_0=0$. Next, the function $\widetilde{\chi}$ has a shift of the argument by λ_0, which corresponds to multiplication of χ by $\exp(i\lambda_0 x)$ and which does not alter the estimate. Obviously, the norm of the operator T_{λ_0} coincides with the norm T_0. By virtue of the definition of the operator $\mathcal{K}$ and the function Φ we have

$$\mathcal{K}\varphi(x)=(2\pi)^{-1/2}\int\limits_{\mathbb{R}^1}\Phi(\lambda)\widetilde{\varphi}(\lambda)\exp(i\lambda x)\,d\lambda=(2\pi)^{-1/2}\int\limits_{\mathbb{R}^1}T_t\varphi(t)\,d\mu(t)$$

for any $\varphi\in S$.

Now we shall estimate the integral of the vector-valued function by using the boundedness of the opertor T_t from $L^p_w(\mathbb{R}^1)$ into $L^p_v(\mathbb{R}^1)$ with the norm not depending on t. We obtain

$$\|\mathcal{K}\varphi\|_{L^p_v(\mathbb{R}^1)} \leq (2\pi)^{-1/2} \int\limits_{\mathbb{R}^1} \|T_t\varphi\|_{L^p_v(\mathbb{R}^1)} |d\mu(t)| \leq c(2\pi)^{-1/2} \int\limits_{\mathbb{R}^1} |d\mu| \cdot \|\varphi\|_{L^p_w(\mathbb{R}^1)}.$$

Finally, using the density of the space S in $L^p_w(\mathbb{R}^1)$, we find that $\mathcal{K}$ has a bounded extention in $L^p_v(\mathbb{R}^1)$. ■

References

1. B. Muckenhoupt and R. L. Wheeden, Two-weight function norm inequalities for the Hardy–Littlewood maximal function and the Hilbert transform. *Studia Math.* **55**(1976), No. 3, 279–294.

2. E. G. Gusseinov, Singular integrals in the spaces of functions summable with monotone weight. (Russian) *Mat. Sb.* **132(174)** (1977), No. 1, 28–44.

3. V. Guliev, Two-weighted L_p inequalities for singular integral operators on Heisenberg groups. *Georgian Math. J.* **1**(1994), No. 4, 367–376.

4. D. E. Edmunds and V. M. Kokilashvili, Two-weighted inequalities for singular integrals. *Canad. Math. Bull.* **38**(1995), No. 3, 295–303.

5. V. Kokilashvili and A. Meskhi, Two-weight inequalities for Hilbert transforms and multipliers. *Journal of Inequalities and Applications. Singapore* (*to appear*).

6. G. B. Folland and E. M. Stein, Hardy spaces on homogeneous groups. *Princeton University Press and University of Tokyo Press, Princeton, New Jersey,* 1982.

7. S. Hoffman, Weighted norm inequalities and vector valued inequalities for certain rough operators. *Indiana Univ. Math. J.* **42**(1993), No. 1, 1–14.

8. B. Muckenhoupt, Weighted norm inequalities for the Hardy-Littlewood maximal function. *Trans. Amer. Math. Soc.* **165**(1972), 207–226.

9. V. M. Kokilashvili, On Hardy's inequalities in weighted spaces. (Russian) *Bull. Acad. Sci. Georgian SSR* **96**(1979), 37–40.

10. B. Muckenhoupt, Hardy's inequalities with weight. *Studia Math.* **44**(1972), No. 1, 31–38.

11. J. S. Bradley, Hardy's inequalities with mixed norms. *Canad. Math. Bull.* **21**(1978), 405–408.

12. V. G. Maz'ya, Sobolev spaces. *Springer, Berlin,* 1985.

13. V. P. Kabaila, On the imbedding $L_p(\nu)$ into $L_p(\mu)$. (Russian) *Litovsky Mat. Sb.* **21**(1981), 143–148.

(Received 02.11.1996)

Authors' address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia