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WEAK TYPE WEIGHTED INEQUALITIES FOR ITERATED MAXIMAL FUNCTIONS AND THE WEIGHTED SPACES $L(1+\log^+ L)^K$

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Abstract. The condition A_1 and the condition of the type A_∞ are shown to be necessary and sufficient respectively for the existence of direct and reverse weak weighted inequalities for the iterated maximal function. Moreover, the characteristic of the spaces $L(1+\log^+ L)^k_w$ is given in terms of iterated maximal functions.

A_1 A_∞

$L(1+\log^+ L)^k_w$

1. Introduction

The object of our investigation is the so-called iterated maximal function.

Let M be a Hardy-Littlewood maximal function defined for $f \in L_{\rm loc}(R^n)$ by the formula

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int\limits_Q |f(y)|dy,$$

where the exact upper bound is taken over all cubes Q containing the point $x \in R^n$. The function

$$M^k f(x) = \underbrace{M \dots M}_{k} f(x), \quad k \geq 1,$$

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is called the iterated maximal function.

N. A. Fava [1] has introduced the spaces $\mathcal{R}_k$ containing those functions f prescribed in the space X with measure μ for which

$$\int\limits_{\{x\in X:|f(x)|>t\}} \frac{|f(y)|}{t}\Big(\log\frac{|f(y)|}{t}\Big)^k d\mu<\infty$$

for any $t>0$. He has shown that $\mathcal{R}_k$ is a linear subspace of Zygmund's class $L(\log^+ L)^k$, and the following inclusions take place:

$$R_0\supset R_1\supset\ldots\supset\bigcup_{p>1}L^p \quad \text{and} \quad L\subset\mathcal{R}_0\subset L+L^\infty.$$

Consider the spaces $L(1+\log^+ L)^k$. As is easily seen $L(1+\log^+ L)^k=L\cap L(\log^+ L)^k$, it is also the subspace of the class $L(\log^+ L)^k$, and the inclusions $L(1+\log^+ L)^k\supset L(1+\log^+ L)^{k+1}$ and $L(1+\log^+ L)^k\subset\mathcal{R}_k\subset L(\log^+ L)^k$ hold.

These three classes coincide if and only if $\mu X<\infty$. Otherwise, $\mathcal{R}_k$ is a maximal linear subspace of $L(\log^+ L)^k$, i.e., it is closed.

As is shown in [1], $\mathcal{R}_k$ is the natural region of definition for iterated maximal function. In particular, if $f\in\mathcal{R}_k$, then $M^kf\in L^1+L^\infty$, i.e., M^kf is integrable on any set $E\subset X$ of finite measure. Hence the existence of the next maximal function is guaranteed. Therefore in the sequel when considering the functions M^kf we shall always assume $f\in\mathcal{R}_{k-1}$.

It is stated in [1] that for any function $f\in\mathcal{R}_k$ and any $t>0$

$$\begin{aligned}&|\{x\in R^n: M^kf(x)>4t\}|\le\\ &\le C_k\int\limits_{\{x\in R^n:|f(x)|>t\}}\frac{|f(x)|}{t}\Big(\log\frac{|f(x)|}{t}\Big)^{k-1}dx,\end{aligned}\tag{1}$$

where $c>0$ does not depend on f and t.

The aim of the present paper is to find the criteria for the weighted analogue of inequality (1), as well as its reverse. First of all we prove the following

Theorem 1. *Let Q_0 be a cube from R^n. Then for any function f from $\mathcal{R}_{k-1}$ such that* $\operatorname{supp} f\subset Q_0$ *and for any $t>\frac{1}{|Q_0|}\int_{Q_0}M^{k-1}f(x)dx$ we* *have*

$$|\{x \in Q_0 : M^k f(x) > t\}| \geq$$
$$\geq c_k \int\limits_{\{x \in Q_0 : |f(x)| > t\}} \frac{|f(x)|}{t} \left(\log \frac{|f(x)|}{t} \right)^{k-1} dx. \tag{2}$$

This theorem generalizes Stein's theorem [2], where the case $k = 1$ has been considered. Note that in inequality (2) $c_k = \frac{c_1^k}{(k-1)!}$, where c_1 is the constant from the Stein's inequality.

The function is a weight if it is non-negative and locally integrable. The weight function w is said to belong to the class A_1 if for any cube Q

$$\frac{wQ}{|Q|} \leq c \operatorname*{ess\,inf}_{x \in Q} w(x),$$

where $wQ = \int_Q w(x)dx$ and $|Q|$ is the Lebesgue measure of Q.

In what follows, for the measurable set E by $|E|$ we denote its Lebesgue measure, and by wE — $\int_E w(x)dx$. Under f_Q we understand an integral mean of f on Q — $\frac{1}{|Q|} \int_Q |f(x)|dx$.

We shall say that $w \in A'_\infty$ if for any cube Q

$$\frac{w(2Q)}{|Q|} \geq c \operatorname*{ess\,sup}_{x \in Q} w(x),$$

where $2Q$ is a cube concentric to Q with twice as much sides. Unlike A_∞, we write here $w(2Q)$ instead of $w(Q)$ which distinguishes essentially A'_∞ from A_∞.

If for some cube Q_0 and any cube $Q \subset Q_0$

$$\frac{w(2Q \cap Q_0)}{|Q|} \geq c \operatorname*{ess\,sup}_{x \in Q} w(x),$$

then we say that $w \in A'_\infty(Q_0)$.

Determine also the weight class $\mathcal{R}_k$. We shall say that $f \in \mathcal{R}_k(w)$ if

$$\int\limits_{\{x \in R^n : |f(x)| > t\}} \frac{|f(y)|}{t} \left(\log \frac{|f(y)|}{t} \right)^k w(y)dy < \infty$$

for any $t > 0$.

Now we can formulate our main results.

Theorem 2. *In order that inequality*

$$w\{x \in R^n : M^k f(x) > 4t\} \leq$$
$$\leq c_k \int\limits_{\{x \in R^n : |f(x)| > t\}} \frac{|f(x)|}{t} \Big(\log \frac{|f(x)|}{t} \Big)^{k-1} w(x)dx \tag{3}$$

be valid for any function $f \in \mathcal{R}_{k-1}(w)$ $(k \geq 1)$ *and any* $t > 0$ *it is necessary and sufficient that* $w \in A_1$.

B. Muckenhoupt [3] has proved this theorem for $k = 1$.

Theorem 3. *Let* I_0 *be an arbitrary interval from* R^1. *Then in order that ineqquality*

$$w\{x \in I_0 : M^k f(x) > t\} \geq$$
$$\geq c_k \int\limits_{\{x \in I_0 : |f(x)| > t\}} \frac{|f(x)|}{t} \Big(\log \frac{|f(x)|}{t} \Big)^{k-1} w(x)dx \tag{4}$$

be valid for any function f *such that support* $f \subset I_0$ *and* $f \in \mathcal{R}_{k-1}(w)$ $(k \geq 1)$ *and for any* $t > \frac{1}{|I_0|} \int_{I_0} M^{k-1} f(x)dx$ *it is necessary and sufficient that*

$$w \in A'_\infty(I_0). \tag{5}$$

In case when $k = 1$ this theorem has been proved by K.F. Andersen and W.-S. Young [4] for (4)⇒(5) and by B. Muckenhoupt [5] for (5)⇒(4).

Unfortunately, we managed to prove this theorem only for one-dimensional case which concerns the implication (4)⇒(5) only. The proof of implication (5)⇒(4) for multi-dimensional case is nothing but that of one-dimensional case.

2. Proofs

Proof of Theorem 1. As we already said, inequality (2) has been obtained by Stein [2] for $k = 1$. We shall prove inequality (2) for $k > 1$ by induction.

Assume that (2) holds for some $k \geq 1$ and let $f \in \mathcal{R}_k$ and $t > (M^k f)_{Q_0}$. Then

$$|\{x \in Q_0 : M^{k+1} f(x) > t\}| \geq |\{x \in Q_0 : M(\chi_{Q_0} M^k f(x)) > t\}| \geq$$
$$\geq c_1 \int\limits_{\{x \in Q_0 : M^k f(x) > t\}} \frac{M^k f(x)}{t} dx = c_1 \int\limits_{Q_0 \cap \{x : M^k \frac{f}{t}(x) > 1\}} M^k \frac{f}{t}(x) dx.$$

Denote $E_t = \{x \in R^n : M^k \frac{f}{t}(x) > 1\}$. Then

$$|\{x \in Q_0 : M^{k+1} f(x) > t\}| \geq c_1 \int\limits_{\{Q_0 \cap E_t\}} M^k \frac{f}{t}(x) dx =$$
$$= c_1 \int\limits_0^\infty |\{x \in Q_0 \cap E_t : M^k \frac{f}{t}(x) > \lambda\}| d\lambda =$$
$$= c_1 \int\limits_0^\infty |\{x \in Q_0 : M^k \frac{f}{t} x > 1, M^k \frac{f}{t}(x) > \lambda\}| d\lambda \geq$$
$$\geq c_1 \int\limits_1^\infty |\{x \in Q_0 : M^k \frac{f}{t}(x) > \lambda\}| d\lambda.$$

But

$$\frac{1}{|Q_0|} \int\limits_{Q_0} M^{k-1} \frac{f}{t}(x) dx = \frac{1}{t} (M^{k-1} f)_{Q_0} < \frac{(M^{k-1} f)_{Q_0}}{(M^k f)_{Q_0}} \leq 1 \leq \lambda.$$

Therefore from our induction hypothesis we obtain

$$|\{x \in Q_0 : M^{k+1} f(x) > t\}| \geq \frac{c_1^{k+1}}{(k-1)!} \int\limits_1^\infty d\lambda \int\limits_{Q_0} \frac{|f(x)|}{t\lambda} \Big(\log^+ \frac{|f(x)|}{t\lambda} \Big)^{k-1} dx.$$

Denoting $\frac{|f(x)|}{t}=\varphi(x)$ and changing the order of integration, we get

$$|\{x\in Q_0 : M^{k+1}f(x)>t\}|\geq$$

$$\geq \frac{c_1^{k+1}}{(k-1)!}\int_1^{\infty}d\lambda\int_{Q_0}\frac{\varphi(x)}{\lambda}\Big(\log^+\frac{\varphi(x)}{\lambda}\Big)^{k-1}dx=$$

$$=\frac{c_1^{k+1}}{(k-1)!}\int_{\{x\in Q_0:\varphi(x)>1\}}dx\int_1^{\infty}\frac{\varphi(x)}{\lambda}\Big(\log^+\frac{\varphi(x)}{\lambda}\Big)^{k-1}d\lambda=$$

$$=\frac{c_1^{k+1}}{(k-1)!}\int_{\{x\in Q_0:\varphi(x)>1\}}dx\int_1^{\varphi(x)}\frac{\varphi(x)}{\lambda}\Big(\log\frac{\varphi(x)}{\lambda}\Big)^{k-1}d\lambda.$$

Let $\frac{\varphi(x)}{\lambda}=\tau$. Then $d\lambda=-\frac{\lambda^2}{\varphi(x)}d\tau=-\frac{\varphi(x)}{\tau^2}d\tau$. Hence

$$|\{x\in Q_0 : M^{k+1}f(x)>t\}|\geq$$

$$\geq \frac{c_1^{k+1}}{(k-1)!}\int_{\{x\in Q_0:\varphi(x)>1\}}\varphi(x)dx\int_1^{\varphi(x)}\frac{(\log\tau)^{k-1}}{\tau}d\tau=$$

$$=\frac{c_1^{k+1}}{k!}\int_{\{x\in Q_0:\varphi(x)>1\}}\varphi(x)(\log\varphi(x))^k dx=$$

$$=\frac{c_1^{k+1}}{k!}\int_{\{x\in Q_0:|f(x)|>t\}}\frac{|f(x)|}{t}\Big(\log\frac{|f(x)|}{t}\Big)^k dx.$$

Thus we have obtained inequality (2). ■

Proof of Theorem 2. The proof for sufficiency of the condition A_1 almost repeats that of Theorem 1 from [1], but we cite it here for the completeness of our discussion.

Inequality (3) is proved by induction. This result for $k=1$ belongs to B. Muckenhoupt [3]. (It should be only noted that for $k=1$ in the left-hand side of inequality (3) we may write $2t$ instead of $4t$).

Let inequality (3) be valid for some $k\geq 1$. Consider the maximal operator M^{k+1}. Let f be any function from $\mathcal{R}_k\cap\mathcal{R}_k(w)$. Denote

$$f^{2t}(x)=\begin{cases} f(x) & \text{if } |f(x)|\geq 2t\\ 0 & \text{if } |f(x)|<2t\end{cases}\quad\text{and}\quad f_{2t}=f-f^{2t}.$$

Then $f = f^{2t} + f_{2t}$ and

$$M^{k+1}f(x) \le M^{k+1}f^{2t} + 2t.$$

Consequently,

$$\{x \in R^n : M^{k+1}f(x) > 4t\} \subset \{x \in R^n : M^{k+1}f^{2t}(x) > 2t\}.$$

Since $w \in A_1$

$$\begin{aligned} w\{x \in R^n : M^{k+1}f(x) > 4t\} &\le \\ \le w\{x \in R^n : M(M^k f^{2t})(x) > 2t\} &\le \\ \le \frac{c_1}{t} \int\limits_{\{x:M^k f^{2t}(x)>t\}} M^k f^{2t}(x)w(x)dx &= \\ = c_1 \int\limits_{\{x:M^k\varphi(x)>1\}} M^k\varphi(x)w(x)dx, \end{aligned} \tag{6}$$

where $\varphi(x) = \frac{|f^{2t}(x)|}{t}$. The right-hand side of (6) is equal to

$$c_1 \int\limits_0^\infty w\{x \in R^n : M^k\varphi(x) > 1, M^k\varphi(x) > \lambda\}d\lambda =$$

$$= cw\{x \in R^n : M^k\varphi(x) > 1\} + c_1 \int\limits_1^\infty w\{x \in R^n : M^k\varphi(x) > \lambda\}d\lambda. \tag{7}$$

By our induction hypothesis, the second term in the right-hand side of (7) can be estimated from above by:

$$4c_1c_k \int\limits_1^\infty \int\limits_{\{x\in R^n:4\varphi_{(x)}>\lambda\}} \frac{\varphi(x)}{\lambda}\Big(\log\frac{4\varphi(x)}{\lambda}\Big)^{k-1} w(x)dxd\lambda =$$

$$= c \int\limits_{\{x\in R^n:4\varphi(x)>1\}} \varphi(x)w(x)dx \int\limits_1^{4\varphi(x)} \frac{(\log\frac{4\varphi(x)}{\lambda})^{k-1}d\lambda}{\lambda} =$$

$$= \frac{c}{k}\int\limits_{R^n} \varphi(x)(\log^+ 4\varphi(x))^k w(x)dx. \tag{8}$$

Since $\log^+(ab) \le \log^+ a + \log^+ b$ and $(a+b)^k \le 2^k(a^k + b^k)$ for all $a, b \ge 0$, we can estimate (8) from above by:

$$A\int\limits_{R^n}\varphi(x)w(x)dx + B\int\limits_{R^n}\varphi(x)(\log^+\varphi(x))^k w(x)dx \le$$

$$\le A\int\limits_{\{x:|f(x)|>2t\}}\frac{|f(x)|}{t}w(x)dx +$$

$$+\beta\int\limits_{\{x:|f(x)|>2t\}}\frac{|f(x)|}{t}(\log^+\frac{|f(x)|}{t})^k w(x)dx \le$$

$$\le \frac{A}{(\log 2)^k}\int\limits_{\{x:|f(x)|>2t\}}\frac{|f(x)|}{t}(\log\frac{|f(x)|}{t})^k w(x)dx +$$

$$+B\int\limits_{\{x:|f(x)|>2t\}}\frac{|f(x)|}{t}\Big(\log\frac{|f(x)|}{t}\Big)^k w(x)dx \le$$

$$\le \Big(\frac{A}{(\log 2)^k} + B\Big)\int\limits_{R^n}\frac{|f(x)|}{t}(\log^+\frac{|f(x)|}{t})^k w(x)dx. \tag{9}$$

As for the first term in the right-hand side of (7), we have

$$c_1 w\{x \in R^n : M^k\varphi(x) > 1\} \le$$

$$\le 4c_1c_k\int\limits_{\{x:4\varphi(x)>1\}}\varphi(x)(\log 4\varphi(x))^{k-1}w(x)dx =$$

$$= c\int\limits_{R^n}\varphi(x)(\log^+ 4\varphi(x))^{k-1}w(x)dx \le$$

$$\le A\int\limits_{R^n}\varphi(x)w(x)dx +$$

$$+B\int\limits_{R^n}\varphi(x)(\log^+\varphi(x))^{k-1}w(x)dx \le$$

$$\le \frac{A}{(\log 2)^k}\int\limits_{R^n}\frac{|f(x)|}{t}(\log^+\frac{|f(x)|}{t})^k w(x)dx +$$

$$+\frac{B}{\log 2}\int\limits_{R^n}\frac{|f(x)|}{t}(\log^+\frac{f(x)}{t})^k w(x)dx. \tag{10}$$

From (9) and (10) we obtain inequality (3).

Now let us prove that inequality (3) implies $w \in A_1$. We fix an arbitrary cube $Q \subset R^n$ and the number $\varepsilon > 0$. Let

$$E_\varepsilon = \{x \in Q : w(x) < \operatorname*{ess\,inf}_{x \in Q} w(x) + \varepsilon\}.$$

Then $|E_\varepsilon| > 0$. Let E be some subset of E_ε. Obviously, E may be of arbitrarily small positive measure. The set E and its measure will be chosen specifically below. Taking $f(x) = \chi_E(x)$ for any point $y \in Q$, we have

$$M^k f(y) \geq \frac{1}{|Q|} \int_Q M^{k-1} f(x) dx = \frac{1}{|Q|} \int_0^\infty |\{x \in Q : M^{k-1} f(x) > t\}| dt \geq$$

$$\geq \frac{1}{|Q|} \int_{(M^{k-2} f)_Q}^\infty |\{x \in Q : M^{k-1} f(x) > t\}| dt.$$

By Theorem 1, this expression can be estimated from below by

$$\frac{c_k}{|Q|} \int_{(M^{k-2} f)_Q}^\infty dt \int_{\{x \in Q : |f(x)| > t\}} \frac{|f(x)|}{t} \Big(\log \frac{|f(x)|}{t} \Big)^{k-2} dx. \tag{11}$$

By Fubini's theorem, (11) is equal to

$$\frac{c_k}{|Q|} \int_{\{x \in Q : |f(x)| > (M^{k-2} f)_Q\}} f(x) dx \int_{(M^{k-2} f)_Q}^{|f(x)|} \frac{1}{t} \Big(\log \frac{|f(x)|}{t} \Big)^{k-2} dt =$$

$$= \frac{c_k}{k-1} \frac{1}{|Q|} \int_{\{x \in Q : |f(x)| > (M^{k-2} f)_Q\}} |f(x)| \Big(\log \frac{|f(x)|}{(M^{k-2} f)_Q} \Big)^{k-1} dx. \tag{12}$$

Obviously, if $|E|$ is small, then $(M^{k-2} f)_Q = (M^{k-2} \chi_E)_Q < 1$. Hence $\{x \in Q : |f(x)| > (M^{k-2} f)_Q\} = E$, and (12) takes the form

$$\frac{c}{|Q|} \int_E \Big(\log \frac{1}{(M^{k-2} \chi_E)_Q} \Big)^{k-1} dx = c \frac{|E|}{|Q|} \Big(\log \frac{1}{(M^{k-2} \chi_E)_Q} \Big)^{k-1}.$$

This means that for any $x \in Q$

$$M^k \chi_E(x) > c \frac{|E|}{|Q|} \Big(\log \frac{1}{(M^{k-2} \chi_E)_Q} \Big)^{k-1} \equiv 4\lambda.$$

Thus
$$Q \subset \{x \in R^n : M^k \chi_E(x) > 4\lambda\},$$
and by condition (3),
$$wQ \le w\{x \in R^n : M^k \chi_E(x) > 4\lambda\} \le$$
$$\le c_k \int\limits_{\{x:\chi_E(x)>\lambda\}} \frac{\chi_E(x)}{\lambda}(\log \frac{\chi_E(x)}{\lambda})^{k-1} w(x)dx.$$

Clearly, for any $x \in Q : \lambda < M^k \chi_E(x) \le 1$. Therefore we have
$$wQ \le c_k \int\limits_E \frac{1}{\lambda}\Big(\log \frac{1}{\lambda}\Big)^{k-1} w(x)dx \le$$
$$\le c_k \frac{1}{\lambda}\Big(\log \frac{1}{\lambda}\Big)^{k-1} |E|(\operatorname*{ess\,inf}_{x\in Q} w(x) + \varepsilon) =$$
$$= c \frac{1}{\frac{|E|}{|Q|}(\log \frac{1}{(M^{k-2}\chi_E)_Q})^{k-1}} \times$$
$$\times \Bigg(\log \frac{c'}{\frac{|E|}{Q}(\log \frac{1}{(M^{k-2}\chi_E)_Q})^{k-1}}\Bigg)^{k-1} |E|(\operatorname*{ess\,inf}_{x\in Q} w(x) + \varepsilon) =$$
$$= c \frac{1}{(\log \frac{1}{(M^{k-2}\chi_E)_Q})^{k-1}} \times$$
$$\times \Bigg(\log \frac{c'}{\frac{|E|}{Q}(\log \frac{1}{(M^{k-2}\chi_E)_Q})^{k-1}}\Bigg)^{k-1} |Q|(\operatorname*{ess\,inf}_{x\in Q} w(x) + \varepsilon). \tag{13}$$

Let us show that if $|E|$ is sufficiently small, then
$$\frac{\log \frac{c'}{\frac{|E|}{|Q|}(\log \frac{1}{(M^{k-2}\chi_E)_Q})^{k-1}}}{\log \frac{1}{(M^{k-2}\chi_E)_Q}} \le 1. \tag{14}$$

Estimate first $(M^{k-2}\chi_E)_Q$. If $k = 2$, then
$$(M^{k-2}\chi_E)_Q = (\chi_E)_Q = \frac{|E|}{|Q|} = \frac{|E|}{|Q|}\Big(\log \frac{|Q|}{|E|}\Big)^{k-2}.$$
If $k \ge 3$, then
$$(M^{k-2}\chi_E)_Q = \frac{1}{|Q|}\int\limits_Q M^{k-2}\chi_E(x)dx =$$

$$= \frac{4}{|Q|} \int\limits_0^\infty |\{x \in Q : M^{k-2}\chi_E(x) > 4t\}| dt =$$

$$= \frac{4}{|Q|} \Bigg(\int\limits_0^{(\chi_E)_Q} + \int\limits_{(\chi_E)_Q}^\infty \Bigg) |\{x \in Q : M^{k-2}\chi_E(x) > 4t\}| dt \le$$

$$\le 4(\chi_E)_Q + \frac{4c_{k-2}}{|Q|} \int\limits_{(\chi_E)_Q}^1 dt \int\limits_E \frac{\chi_E(x)}{t} \Big(\log^+ \frac{\chi_E(x)}{t} \Big)^{k-3} dx =$$

$$= 4\frac{|E|}{|Q|} + 4c_{k-2} \frac{1}{|Q|} \int\limits_E dx \int\limits_{\frac{|E|}{|Q|}}^1 \frac{1}{t} \Big(\log \frac{1}{t} \Big)^{k-3} dt =$$

$$= 4\frac{|E|}{|Q|} + \frac{4c_{k-2}}{k-2} \frac{1}{|Q|} \int\limits_E \Bigg(\log \frac{1}{\frac{|E|}{|Q|}} \Bigg)^{k-2} dx \le$$

$$\le c \Bigg(\frac{|E|}{|Q|} + \frac{|E|}{|Q|} \Big(\log \frac{|Q|}{E} \Big)^{k-2} \Bigg). \qquad (15)$$

Since $|E|$ may be taken arbitrarily small, we may assume that $\log |Q|/|E| > 1$, and therefore (15) takes the form

$$(M^{k-2}\chi_E)_Q \le c \frac{|E|}{|Q|} \Big(\log \frac{|Q|}{|E|} \Big)^{k-2} = c \frac{(\log t)^{k-2}}{t}, \qquad (16)$$

where $t = \frac{|Q|}{|E|}$. If $|E| \to 0$, then $t \to \infty$. But $\lim\limits_{t\to\infty} \frac{(\log t)^{k-2}}{t} = 0$. Hence we may write

$$(M^{k-2}\chi_E)_Q \le c \frac{|E|}{|Q|} \Big(\log \frac{|Q|}{|E|} \Big)^{k-2} \le 1.$$

Taking into account (16), the left-hand side of (14) is estimated from above by the quantity

$$\log \frac{ct}{(\log \frac{t}{c(\log t)^{k-2}})^{k-1}} : \log \frac{t}{c(\log t)^{k-2}}.$$

Therefore, to prove (14) it suffices to show that for large t

$$\frac{c(\log t)^{k-2}}{(\log \frac{t}{c(\log t)^{k-2}})^{k-1}} < 1.$$

We have

$$\frac{c(\log t)^{k-2}}{(\log \frac{t}{c(\log t)^{k-2}})^{k-1}} = c\frac{(\log \frac{t}{c(\log t)^{k-2}} + \log c(\log t)^{k-2})^{k-2}}{(\log \frac{t}{c(\log t)^{k-2}})^{k-1}} =$$
$$= c2^{k-2}\frac{(\log tc(\log t)^{k-2})^{k-2} + (\log c(\log t)^{k-2})^{k-2}}{(\log \frac{t}{c(\log t)^{k-2}})^{k-1}} =$$
$$= \frac{c}{\log \frac{t}{c(\log t)^{k-2}}}\Big(1 + \frac{(\log c(\log t)^{k-2})^{k-2}}{(\log \frac{t}{c(\log t)^{k-2}})^{k-2}}\Big). \tag{17}$$

But $\lim\limits_{t\to\infty} \frac{t}{c(\log t)^{k-2}} = \infty$ and

$$\frac{c(\log t)^{k-2}}{\frac{t}{c(\log t)^{k-2}}} = c^2\frac{(\log t)^{2k-4}}{t} \to 0 \quad \text{as} \quad t \to 0.$$

Consequently, (17) tends to zero for t tending to infinity which implies that taking $|E|$ sufficiently small, we obtain inequality (14).

According to (14), inequality (13) takes the form

$$wQ \le c|Q|(\operatorname*{ess\,inf}_{x\in Q} w(x) + \varepsilon).$$

Since ε is arbitrary, we obtain that $w \in A_1$. ■

Proof of Theorem 3. If $w \in A'_\infty(I_0)$, then according to the result of [5], the operator M acts weakly in $L^1_w(I_0)$. Taking this fact into account, the proof of implication (5)⇒(4) is carried out quite similarly to that of Theorem 1.

Prove that (4)⇒(5). Let I be an arbitrary fixed interval from I_0. Let E be some measurable subset of I and $f(x) = \chi_E(x)$.

Note first the general fact that if φ is non-increasing and non-negative function on $(a, +\infty)$, then $M\varphi$ is the same. Indeed, let $x_1, x_2 \in (a, +\infty)$ and $x_1 < x_2$. Then

$$M\varphi(x_2) = \sup_{I\ni x_2}\frac{1}{|I|}\int\limits_I \varphi(x)dx = \max\Big\{\sup_{I\ni x_2,x_1}\varphi_I; \sup_{\substack{I\ni x_2\\ I\not\ni x_1}}\varphi_I\Big\} \le$$
$$\le \max\Big\{M\varphi(x_1), \sup_{\substack{I\ni x_2\\ I\not\ni x_1}}\varphi_I\Big\}.$$

Hence it suffices to show that

$$\sup_{\substack{I\ni x_2\\ I\not\ni x_1}} \frac{1}{|I|}\int\limits_I \varphi(x)dx \le M\varphi(x_1) \tag{18}$$

Let I_{x_2} be an arbitrary interval such that $x_2 \in I_{x_2}$ and $x_1 \notin I_{x_2}$. Consider the shift of the interval I_{x_2} to the left so that for the shifted interval I_{x_1} the inclusions $x_1 \in I_{x_1}$ and $a \notin I_{x_1}$ hold. Obviously,

$$\frac{1}{|I_{x_2}|}\int\limits_{I_{x_2}} \varphi(x)dx \le \frac{1}{|I_{x_1}|}\int\limits_{I_{x_1}} \varphi(x)dx \le M\varphi(x_1)$$

which proves (18) and hence the fact that $M\varphi$ is non-increasing on $(a,+\infty)$.

Analogously we can prove that if φ is non-decreasing non-negative function on $(-\infty,a)$, then $M\varphi$ is the same.

Applying these facts to the function χ_E, we conclude that $M^k\chi_E(x)$ does not decrease on the left of I and does not increase on the right of I $(k\ge 1)$.

Let now $I=(a,b)$, $2I=(a-\frac{b-a}{2}, b+\frac{b-a}{2})=(a',b')$ and let φ be non-negative function non-decreasing on $(-\infty,a)$ and non-increasing on $(b,+\infty)$. Let us show that under these conditions for any point $x\notin 2I$

$$M\varphi(x) \le \frac{4}{|I|}\int\limits_{2I} \varphi(x)dx. \tag{19}$$

Taking the above arguments into account, it is sufficient to prove (19) for $x=a'$ and $x=b'$.

Let $x=b'$, $I_1=(a',a)$, $I_2=(b,b')$ and let J be some interval containing the point b'.

Consider several cases:

(1) Let $J\supset 2I$. Denote $I'=(-\infty,a')\cap J$ and $I''=(b',+\infty)\cap J$. Then

$$\varphi_J = \frac{\varphi(I')+\varphi(2I)+\varphi(I'')}{|I'|+|2I|+|I''|} \le \varphi_{I'}+\varphi_{2I}+\varphi_{I''} \le \varphi_{I_1}+\varphi_{2I}+\varphi_{I_2} =$$
$$= \frac{2}{|I|}\int\limits_{I_1}\varphi(x)dx + \frac{1}{2|I|}\int\limits_{2I}\varphi(x)dx + \frac{2}{|I|}\int\limits_{I_2}\varphi(x)dx \le \frac{5}{2}\frac{1}{|I|}\int\limits_{2I}\varphi(x)dx.$$

(2) Let $J \not\supset 2I$. Denote $I''' = 2I \cap J$ and $I'' = (b', +\infty) \cap J$. Then

$$\varphi_J = \frac{\varphi(I''') + \varphi(I'')}{|I'''| + |I''|} \le \varphi_{I'''} + \varphi_{I''} \le \varphi_{I'''} + \varphi_{I_2}.$$

If $I''' \subset I_2$, then

$$\varphi_{I'''} \le \varphi_{I_2} \quad \text{and} \quad \varphi_J \le 2\varphi_{I_2} \le \frac{4}{|I|} \int\limits_{2I} \varphi(x)dx.$$

If $I_2 \subset I'''$, then

$$\varphi_{I'''} = \frac{1}{|I'''|} \int\limits_{I'''} \varphi(x)dx \le \frac{1}{|I_2|} \int\limits_{2I} \varphi(x)dx = \frac{2}{|I|} \int\limits_{2I} \varphi(x)dx$$

and hence in this case

$$\varphi_J \le \frac{4}{|I|} \int\limits_{2I} \varphi(x)dx.$$

Combining cases (1) and (2), we conclude that $\varphi_J \le \frac{4}{|I|} \int_{2I} \varphi(x)dx$ for any interval $J \ni b'$. Hence (19) takes place for $x = b'$. The case $x = a'$ is proved similarly.

Proceeding from this general fact, we obtain that for every $k \ge 1$

$$M^k \chi_E(x) \le \frac{4}{|I|} \int\limits_{2I} M^{k-1} \chi_E(y)dy \tag{20}$$

for any $x \notin 2I$.

Take now $\varepsilon > 0$ to be arbitrary. Let

$$E_\varepsilon = \{x \in I : w(x) \ge \operatorname*{ess\,sup}_{x \in I} w(x) - \varepsilon\}.$$

Then $|E_\varepsilon| > 0$. Let $E \subset E_\varepsilon$. As we have mentioned above, E may be taken to be of arbitrarily small measure. The set E and its measure will be chosen specifically below. First we assume $\frac{|E|}{|I|} < \frac{1}{4}$.

From inequalities (1) and (20) for any $y \notin 2I$ we get

$$M^k \chi_E(y) \le \frac{4}{|I|} \int\limits_{2I} M^{k-1} \chi_E(x)dx =$$

$$= \frac{4}{|I|} \int\limits_0^\infty |\{x \in 2I : M^{k-1} \chi_E(x) > t\}|dt =$$

$$= \frac{4}{|I|} \int_0^{4(\chi_E)_I} + \frac{4}{|I|} \int_{4(\chi_E)_I}^{\infty} =$$

$$= 32(\chi_E)_I + \frac{16}{|I|} \int_{(\chi_E)_I}^{\infty} |\{x \in 2I : M^{k-1}\chi_E(x) > 4t\}| dt =$$

$$= 32\frac{|E|}{|I|} + \frac{16}{|I|} \int_{\frac{|E|}{|I|}}^{\infty} |\{x \in 2I : M^{k-1}\lambda_E(x) > 4t\}| dt.$$

But as it is clear $M^{k-1}\chi_E(x) \le 1 < 4$ for any $x \in R^1$. Therefore

$$M^k\chi_E(y) \le c\frac{|E|}{|I|} + c\frac{1}{|I|} \int_{\frac{|E|}{|I|}}^{1} |\{x \in R^1 : M^{k-1}\chi_E(x) > 4t\}| dt \le$$

$$\le c\frac{|E|}{|I|} + c\frac{1}{|I|} \int_{\frac{|E|}{|I|}}^{1} dt \int_{R^1} \frac{\chi_E(x)}{t} \Big(\log^+ \frac{\chi_E(x)}{t} \Big)^{k-2} dx =$$

$$= c\frac{|E|}{|I|} + c\frac{1}{|I|} \int_{\frac{|E|}{|I|}}^{1} dt \int_{E} \frac{1}{t} \Big(\log \frac{1}{t} \Big)^{k-2} dx =$$

$$= c\frac{|E|}{|I|} + c\frac{|E|}{|I|} \int_{\frac{|E|}{|I|}}^{1} \frac{1}{t} \Big(\log \frac{1}{t} \Big)^{k-2} dt =$$

$$= c\frac{|E|}{|I|} + \frac{c}{k-1} \frac{|E|}{|I|} \Big(\log \frac{|I|}{|E|} \Big)^{k-1}. \tag{21}$$

Since $\frac{|E|}{|I|} < \frac{1}{4}$, we have $\log \frac{|I|}{|E|} > 1$. Thus (21) takes the form

$$M^k\chi_E(y) \le c' \frac{|E|}{|I|} \Big(\log \frac{|I|}{|E|} \Big)^{k-1}, \quad \forall y \notin 2I.$$

By (16), there exists a constant c'' such that

$$(M^k\chi_E)_{I_0} \le c'' \frac{|E|}{|I_0|} \Big(\log \frac{|I_0|}{|E|} \Big)^{k-1}.$$

Consider the function $\frac{(\log t)^{k-1}}{t}$. Then $\lim\limits_{t\to 0}\frac{(\log t)^{k-1}}{t}\to 0$. Moreover, it can be easily verified that this function strictly decreases in the neighborhood of infinity so that if $|E|$ is sufficiently small, then

$$\frac{|E|}{|I_0|}\Big(\log\frac{|I_0|}{|E|}\Big)^{k-1}<\frac{|E|}{|I|}\Big(\log\frac{|I|}{|E|}\Big)^{k-1}.$$

Let $c^*=\max\{c',c''\}$ and $\lambda=c^*\frac{|E|}{|I|}\Big(\log\frac{|I|}{|E|}\Big)^{k-1}$. Then we have

$$\forall y\notin 2I: M^k\chi_E(y)\leq\lambda \quad\text{and}\quad (M^{k-1}\chi_E)_{I_0}<\lambda. \tag{22}$$

Moreover, further decrease of $|E|$ results to the fact that $\lambda<1$.

From (22) and (4) we now obtain

$$\begin{aligned} w(2I\cap I_0)\geq w\{x\in I_0: M^k\chi_E(x)>\lambda\}\geq \\ \geq c\int\limits_{\{x\in I_0:\chi_E(x)>\lambda\}}\frac{\chi_E(x)}{\lambda}\Big(\log\frac{\chi_E(x)}{\lambda}\Big)^{k-1}w(x)dx= \\ =c\int\limits_E\frac{1}{\lambda}\Big(\log\frac{1}{\lambda}\Big)^{k-1}w(x)dx\geq \\ \geq\frac{c}{\lambda}\Big(\log\frac{1}{\lambda}\Big)^{k-1}|E|(\operatorname*{ess\,sup}_{x\in I}w(x)-\varepsilon)= \\ =c\frac{1}{(\log\frac{|I|}{|E|})^{k-1}}\Big(\log\frac{|I|}{|E|}(\log\frac{|I|}{E})^{k-1}\Big)^{k-1}|I|(\operatorname*{ess\,sup}_{x\in I}w(x)-\varepsilon). \end{aligned} \tag{23}$$

If we denote $\frac{|I|}{|E|}=t$, then (23) takes the form

$$\frac{w(2I\cap E_0)}{|I|}\geq c\frac{(\log\frac{t}{c^*(\log t)^{k-1}})^{k-1}}{(\log t)^{k-1}}(\operatorname*{ess\,sup}_{x\in I}w(x)-\varepsilon).$$

When $|E|\to 0$, we have that $t\to\infty$. Therefore let us calculate

$$\lim_{t\to\infty}\frac{(\log\frac{t}{c^*(\log t)^{k-1}})^{k-1}}{(\log t)^{k-1}}=\lim_{t\to\infty}\left(1+\frac{\log\frac{1}{c^*(\log t)^{k-1}}}{\log t}\right)^{k-1}=1.$$

Hence, if $|E|$ is sufficiently small, then

$$\log\frac{1}{(\log\frac{|I|}{|E|})^{k-1}}\Big(\log\frac{|I|}{c^*|E|(\log\frac{|I|}{|E|})^{k-1}}\Big)^{k-1}\geq\frac{1}{2}.$$

Therefore (23) implies

$$w(2I \cap I_0) \geq \frac{c}{2}|I|(\operatorname*{ess\,sup}_{x \in I} w(x) - \varepsilon).$$

Since c does not depend on ε, we obtain (5). ■

3. Remarks and Other Results

The following two theorems indicate that the global versions of Theorems 1 and 3 are valid.

Theorem 1′. *Let $k \geq 1$. Then there exists a constant $c_k > 0$ such that for any function $f \in \mathcal{R}_{k-1}$ and any $t > 0$*

$$\{x \in R^n : M^k f(x) > t\} \geq$$
$$\geq c_k \int\limits_{\{x \in R^n : |f(x)| > t\}} \frac{|f(x)|}{t} \Big(\log \frac{|f(x)|}{t} \Big)^{k-1} dx. \tag{24}$$

Theorem 3′. *Let $k \geq 1$. Then the inequality*

$$w\{x \in R^1 : M^k f(x) > t\} \geq c \int\limits_{\{x \in R^1 : |f(x)| > t\}} \frac{|f(x)|}{t} \Big(\log \frac{|f(x)|}{t} \Big)^{k-1} w(x) dx$$

holds for any $f \in \mathcal{R}_{k-1(w)}$ and $t > 0$ iff

$$w \in A'_\infty.$$

The proofs of these theorems are similar to those of Theorems 1 and 3. We have only to apply the corresponding well-known results (mentioned in the first section) to the case $k = 1$.

By means of inequalities (1) and (24) we can obtain the following characteristic of Zygmund's class $L(1 + \log^+ L)^k$. (This characteristic has been obtained by M. Guzman [6] and E.M. Stein [2]).

Theorem 4. *Let $f \in L(1 + \log^+ L)^{k-1}$, $k \geq 1$. Then $f \in L(\log^+ L)$ if and only if*

$$\int\limits_{\{x \in R^n : M^k f(x) > 1\}} M^k f(x) dx < \infty.$$

Proof. Let $f \in L(1+\log^+ L)^{k-1}$ be such that

$$\int\limits_{\{x\in R^n: M^k f(x)>1\}} M^k f(x)dx < \infty.$$

Then using inequality (24), we obtain

$$\infty > \int\limits_{\{x\in R^n: M^k f(x)>1\}} M^k f(x)dx \geq \int\limits_1^\infty |\{x \in R^n : M^k f(x) > t\}|dt \geq$$
$$\geq c_k \int\limits_1^\infty dt \int\limits_{\{x\in R^n:|f(x)|>t\}} \frac{|f(x)|}{t}\Big(\log\frac{|f(x)|}{t}\Big)^{k-1} dx =$$
$$= c_k \int\limits_{\{x\in R^n:|f(x)|>1\}} |f(x)|dx \int\limits_1^{|f(x)|} \frac{1}{t}\Big(\log\frac{|f(x)|}{t}\Big)^{k-1} dt =$$
$$= \frac{c_k}{k} \int\limits_{\{x\in R^n:|f(x)|>1\}} |f(x)|(\log|f(x)|)^k dx.$$

Hence $f \in L(\log^+ L)^k$.

Conversely, let $f \in L(\log^+ L)^k$. By the condition $f \in L^1$ we get $f \in L(1+\log^+ L)^k$. Therefore, using inequality (1) we have

$$\int\limits_{\{x\in R^n: M^k f(x)>1\}} M^k f(x)dx \leq$$
$$\leq \int\limits_0^\infty |\{x \in R^n : M^k f(x) > 1, M^k f(x) > t\}|dt =$$
$$= \int\limits_0^4 + \int\limits_4^\infty \leq 4|\{x \in R^n : M^k f(x) > 1\}| +$$
$$+4\int\limits_1^\infty |\{x \in R^n : M^k f(x) > 4t\}|dt \leq$$
$$\leq 4c_k \int\limits_{\{x\in R^n: f(x)>\frac{1}{4}\}} |4f(x)|(\log|4f(x)|)^{k-1} dx +$$

$$+4c_k\int\limits_1^\infty dt \int\limits_{\{x\in R^n:|f(x)|>t\}} \frac{|f(x)|}{t}\Big(\log\frac{|f(x)|}{t}\Big)^{k-1}dx =$$

$$= 4c_k\int\limits_{R^n}|4f(x)|(\log^+|4f(x)|)^{k-1}+$$

$$+\frac{4c_k}{k}\int\limits_{R^n}|f(x)|(\log^+|f(x)|)^k dx. \tag{25}$$

The second summand in (25) is finite owing to the fact that $f\in L(1+\log^+ L)^k$. But since $L(1+\log^+ L)^k \subset L(1+\log^+ L)^{k-1}$ and $L(1+\log^+ L)^{k-1}$ is a linear space, the first summand in (25) will also be finite. ■

As for the characteristic of the weighted space $L(1+\log^+ L)^k_w$, we have the following

Theorem 5. *If $w\in A_1$, then for any function $f\in L(1+\log^+ L)^k_w$*

$$\int\limits_{\{x\in R^n:M^k f(x)>1\}} M^k f(x)w(x)dx<\infty. \tag{26}$$

However, if $w\in A'_\infty$ and the function $f\in L(1+\log^+ L)^{k-1}_w$ is such that (26) *holds, then $f\in L(1+log^+ L)^k_w$.*

The proof of this theorem is analogous to that of Theorem 4, therefore it is omitted here. We have only to take into account the note at the end of the first section.

Finally, let us tough upon some generalizations concerning the space of homogeneous type, i.e., the space X with quasimetric ρ and σ-finite measure μ satisfying the doubling condition with respect to the balls $B(x,r)=\{y:\rho(x,y)<r\}$

$$\mu B(x,2r)\le c\mu B(x,r).$$

Continuous functions with compact supports are assumed to be continuous in $L^1(x,\mu)$.

In the definition of a maximal function we will take balls from X instead of cubes.

As it is known, for such a maximal operator both the weak and the reverse weak inequalities hold either without or with the corresponding weights (see, for e.g., [7], [8]). Therefore for the iterated maximal functions, defined in the space of homogeneous type, inequality (1), and

Theorems 1, $1'$, 4 and 5 are valid. But the condition A'_∞ for the space of homogeneous type will look like

$$\frac{w(NB)}{\mu\beta} \geq c \operatorname*{ess\,sup}_{x \in B} w(x),$$

where $N = \eta(1+\eta^2)$ and η is the constant from the inequality of a triangle for the quasimetric ρ.

If, however, the homogeneous space does not contain atoms, i.e., points with positive measure, in this case Theorem 2 will also be valid. This can be easily verified if we take into consideration the fact that for any measurable set E_0 with a positive measure and for any $\delta > 0$ there exists a subset $E \subset E_0$ with the condition $0 < \mu E < \delta$ (see [8]).

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