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ATOMIC AND SUBATOMIC DECOMPOSITIONS IN ANISOTROPIC FUNCTION SPACES

If $1 < p < \infty$ and $(s_1, \dots, s_n)$ is an n- tuple of natural numbers then

$$W_p^{s,a}(\mathbb{R}^n) = \Big\{ f \in S'(\mathbb{R}^n) \,:\, \|f \mid L_p(\mathbb{R}^n)\| + \sum_{k=1}^n \Big\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \mid L_p(\mathbb{R}^n) \Big\| < \infty \Big\}$$

is the classical anisotropic Sobolev space on $\mathbb{R}^n$. In contrast to the usual (isotropic) Sobolev space ($s_1 = \cdots = s_n$) the smoothness properties of an element from $W_p^{s,a}(\mathbb{R}^n)$ depend on the chosen direction in $\mathbb{R}^n$. The number s defined by

$$\frac{1}{s} = \frac{1}{n}\Big(\frac{1}{s_1} + \cdots + \frac{1}{s_n} \Big)$$

is usually called the "mean smoothness" and $a = (a_1, \dots, a_n)$, where $a_1 = s/s_1, \dots, a_n = s/s_n$, characterises the anisotropy. Anisotropic Bessel potential spaces, or fractional Sobolev spaces, defined by

$$H_p^{s,a}(\mathbb{R}^n) = \Big\{ f \in S'(\mathbb{R}^n) \,:\, \Big\| \Big(\sum_{k=1}^n (1 + \xi_k^2)^{s/(2a_k)} \widehat{f} \Big)^{\vee} \mid L_p(\mathbb{R}^n) \Big\| < \infty \Big\}$$

where $1 < p < \infty$, $s \in \mathbb{R}$ and $a = (a_1, \dots, a_n)$ is a given anisotropy, generalise in a natural way the above spaces (as usual, $S'(\mathbb{R}^n)$ is the space of tempered distributions and $\widehat{f}$, $\check{f}$ are respectively the Fourier and the inverse Fourier transform of f).

Similar to the isotropic case, the study of anisotropic Bessel potential spaces $H_p^{s,a}(\mathbb{R}^n)$ for a fixed anisotropy $a = (a_1, \dots, a_n)$, is a part of the more general theory of the spaces of $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ type. Spaces of that type (or on domains in $\mathbb{R}^n$) have been studied in great detail by S. M. Nikol'skij in [1] and by O. V. Besov, V. P. Il'in and S. M. Nikol'skij in [2], and it is well known that this theory has a more or less full counterpart to the basic facts (definitions, elementary properties, embeddings for different metrics, interpolation) of isotropic spaces $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ as it was presented in the works of H. Triebel (see [3] and [4]).

In the theory of isotropic function spaces there is a well known type of decomposition in simple building blocks, the so-called atoms. They have a history of some twenty years and in [4], 1.9, cf. also [5], a historical report was given on this topic; we do not repeat it here. We only want to mention that the (smooth) atoms in isotropic $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ spaces as they were defined by M. Frazier and B. Jawerth in [6], [7] (cf. also [8]), proved to be a powerful tool in the theory of function spaces. We also wish to emphasise that there exist many other types of atomic decompositions in isotropic spaces but we will not discuss this point here. More information about this subject is given in [7], [4] and [5] where one can find many modifications and applications as well as comprehensive references extending the subject.

Several authors were concerned in the last years with the problem of obtaining useful decompositions of anisotropic function spaces in simple building blocks: a construction

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of unconditional bases in $B^{s,a}_{pq}(\mathbb{R}^n)$ and $F^{s,a}_{pq}(\mathbb{R}^n)$ spaces using Meyer wavelets was done by M. Z. Berkolaiko and I. Ya. Novikov in [9] (and used then in [10]). P. Dintelmann obtained in [11] (Theorem 1) a decomposition for anisotropic function spaces which is the counterpart of the characterisation of isotropic function spaces with the help of the φ-transform of M. Frazier and B. Jawerth (see [7] and the survey [8]) and used it in connection with the theory of Fourier- multipliers for anisotropic function spaces. Our approach is different, especially from the point of view of the localization of the building blocks.

The first aim of this note is to introduce smooth anisotropic atoms and to obtain a decomposition theorem which extends the atomic decomposition theorem of M. Frazier and B. Jawerth, see [6-7], to the anisotropic function spaces $B^{s,a}_{pq}(\mathbb{R}^n)$ and $F^{s,a}_{pq}(\mathbb{R}^n)$. Roughly speaking, we will show that for any $g \in F^{s,a}_{pq}(\mathbb{R}^n)$ it is possible to find a decomposition (convergence in $S'(\mathbb{R}^n)$)

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, \rho^a_{\nu m}, \tag{1}$$

where $\rho^a_{\nu m}$ are the anisotropic atoms and $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$ belongs to an appropriate sequence space f^a_{pq}, such that

$$\| g \,|\, F^{s,a}_{pq}(\mathbb{R}^n)\| \sim \|\lambda \,|\, f^a_{pq}\| \tag{2}$$

(and a similar assertion for $B^{s,a}_{pq}(\mathbb{R}^n)$ spaces). Hence the study of function spaces can be done with the help of some sequence spaces in an analogous way as it is done in the isotropic case in the above cited works of M. Frazier and B. Jawerth.

However in (1) no information is given about the possibility to obtain atomic decompositions in which the atoms are constructed with the help of (anisotropic) dilatations and translations from one smooth function ρ having compact support, cf. also [9] (Comment 2). For isotropic function spaces this was already done by M. Frazier and B. Jawerth (see [7], 4.2) and W. Sickel (see [12]). It might be possible to extend the technique of W. Sickel, at least for large values of the smoothness parameter, using the characterisation of anisotropic function spaces via oscillation from the work of A. Seeger [13]. But to construct such a basic function ρ for the atoms having all required properties seems to be not very easy, at least at the first glance, see the above cited papers.

We arrive at the second aim of this paper, the subatomic (or quarkonial) decomposition theorem which states that given $g \in F^{s,a}_{pq}(\mathbb{R}^n)$ (with s sufficiently large) it is possible to obtain the decomposition

$$g = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m}\, (\beta qu)^a_{\nu m}, \tag{3}$$

convergence being in $S'(\mathbb{R}^n)$, with

$$\| g \,|\, F^{s,a}_{pq}(\mathbb{R}^n)\| \sim \sup_{\beta \in \mathbb{N}_0^n} 2^{ra\beta}\, \|\lambda^\beta \,|\, f^a_{pq}\|, \tag{4}$$

where $r > 0$ is large enough, $a\beta = a_1\beta_1 + \cdots + a_n\beta_n$ if β is the multi-index $(\beta_1, \ldots, \beta_n)$, $\lambda^\beta = \{\lambda^\beta_{\nu m} : \nu \in \mathbb{N}_0\,, m \in \mathbb{Z}^n\}$ and where each $(\beta qu)^a_{\nu m}$ is an extremely simple building block (in particular an anisotropic atom without moment conditions), called anisotropic quark, compactly supported and which can be obtained starting from one smooth function (and a corresponding assertion for $B^{s,a}_{pq}(\mathbb{R}^n)$). Of course in (3) there are infinitely many sums over $(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n$ but this is well compensated by (4) with r large. Furthermore, it turns out from the proof that the dependence of the coefficients $\lambda^\beta_{\nu m}$ on g is linear.

Isotropic quarks were recently introduced by H. Triebel in [14] and the subatomic (quarkonial) decomposition theorem he obtained in [14] (Chapter 14) proved to be a very useful ingredient for the estimation of entropy numbers of compact embeddings between function spaces on fractals. Compared with the results in [14] (Chapter 14) our Theorem is in fact the extension of quarkonial decompositions to anisotropic function spaces in the case of large values of the smoothness parameter, in particular for $s > 0$ if $p \geq 1$ and $q \geq 1$. If one wishes to extend the result to all $s \in \mathbb{R}$ then a lifting argument would be needed. But while the lift operator $(id - \Delta)^{\frac{L+1}{2}}$ between isotropic function spaces causes no problem in keeping the localisation of the (isotropic) quarks (and this fact was essentially used in [14], 14.4) the situation becomes difficult in case of the anisotropic lift operator. It is well known that if $\sigma \in \mathbb{R}$ then the operator

$$I_\sigma(f) = \Big(\Big(\sum_{k=1}^{n} (1 + \xi_k^2)^{\frac{1}{2a_k}} \Big)^{\sigma} \widehat{f} \Big)^{\vee}$$

maps $F_{pq}^{s,a}(\mathbb{R}^n)$ isomorphically onto $F_{pq}^{s-\sigma,a}(\mathbb{R}^n)$ and $\|I_\sigma(\cdot) \mid F_{pq}^{s-\sigma,a}(\mathbb{R}^n)\|$ is an equivalent quasi-norm on $F_{pq}^{s,a}(\mathbb{R}^n)$ (similar result for $B_{pq}^{s,a}(\mathbb{R}^n)$), see [15]; it is clear that I_σ causes a lot of troubles in keeping the localization of the anisotropic quarks and this is the reason why we will restrict ourselves to large values of the smoothness parameter.

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