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# THE CONVERGENCE ALMOST EVERYWHERE OF FOURIER SERIES

Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a given non-decreasing function, $d \in N$. Set $T^d = [0,2\pi)^d$ and denote by $\varphi(L)(T^d)$ the set of measurable functions $f$ on $T^d$ such that $\varphi(|f(x)|) \in L$. Let $S_n(f,x)$ denote the $n$th partial sum of Fourier series of the function $f$ at the point $x$.

Let $d = 1$. Introduce the function

$$Mf(x) = \sup_{n\geq 1} |S_n(f,x)|, \quad x \in T.$$

L. Carleson [1] proved that the Fourier series of any function $f \in L^2(T)$ converges almost everywhere (a.e.). This fact was extended by Hunt [2]. He proved a.e. convergence of the Fourier series for functions $f \in L^p(T)$, $p > 1$, and for $f \in L(\log^+ L)^2(T)$. The basic result of [2] is the following estimate for the characteristic function $\chi_F$ of an arbitrary set $F \subset T$

$$m\{x \in T : M\chi_F(x) > y\} \leq (B_p)^p y^{-p} mF, \quad y > 0, \quad 1 < p < \infty, \tag{1}$$

where $B_p \leq const.\ p^2/(p-1)$. It follows from (1) (see [3, p. 563]) that

$$m\{x \in T : M\chi_F(x) > y\} \leq C\frac{1}{y}\log(\frac{1}{y})mF \quad \text{for} \quad 0 < y < \frac{1}{2}, \tag{2}$$

with some constant $C$. Using (2) Sjölin [3] proved the almost everywhere convergence of Fourier series for functions $f$ from the class $L(\log^+ L)(\log^+\log^+ L)$. We obtained from (2) following result.

**Theorem 1 ([4]).** *If $f \in L(\log^+ L)(\log^+\log^+\log^+ L)$, then the Fourier series of $f$ converges almost everywhere.*

Let $d \geq 2$. N. Tevzadze [5] proved the a.e. convergence of the Fourier series of functions $f(x) \in L^2(T^2)$. Then P. Sjölin [6] showed that the Fourier series of any function from $f \in L(\log^+ L)^d(\log^+\log^+ L)(T^d)$ converges a.e. over cubes. To obtain this result Sjölin used the corresponding one-dimensional theorem [3].

We proved (theorem 2 below) that the convergence a.e. of Fourier series of some class $\varphi(L)(T)$ in one-dimensional case implies convergence over cubes of multiple Fourier series for functions from corresponding classes $\varphi_d(L)(T^d)$.

Let $\psi : [0,+\infty) \to [0,+\infty)$ be a non-decreasing function and the function $\psi(u)u^{-\delta}$ (for any $\delta > 0$) decreases for large $u$.

Let functions $\varphi_d(u) = u\psi(u)(\log(u+1))^{d-1}$, $d \in N$, satisfy conditions

$$\begin{cases} \varphi(u) & \text{is convex and increasing on } [u_0,\infty), \quad u_0 \geq 0, \\ \varphi(u^{\frac{1}{2}}) & \text{is concave function of } u, \ \ u_0 \leq u < \infty \end{cases}$$

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**Theorem 2.** *Assume that for any $f \in \varphi_1(L)(T)$*

$$\lim_{n\to\infty} S_n(f,x) = f(x) \quad a.e.$$

*Then for any $d \geq 1$ the condition $f \in \varphi_d(L)(T^d)$ implies the a.e. convergence over cubes for the Fourier series of $f$ and each of its conjugate series.*

The next result follows from theorems 1 and 2.

**Theorem 3.** *If $f \in L(\log^+ L)^d(\log^+ \log^+ \log^+ L)(T^d)$, $d \in N$, then the Fourier series of $f$ and each of its conjugate series converges over cubes almost everywhere.*

It is known ($d = 1$ – Körner [7], see also Heladze [8], $d \geq 2$ – Konyagin [9]) that for every function $\psi_d : [0, +\infty) \to [0, +\infty)$ satisfying condition

$$\psi_d(u) = 0\Big((\log u)^{d-1} \log\log u\Big) \quad (u \to \infty)$$

there exists a function $f_d \in L\psi_d(L)(T^d)$ with its Fourier series diverging over cubes everywhere.

**Problem 1.** Is the following statement true: if for some class $\varphi(L)(T)$ there exists a functions $f \in \varphi(L)(T)$ such that Fourier series of $f$ diverges a.e., then for every $d \geq 2$ there exists $f \in \varphi(L)(\log^+ L)^{d-1}(T^d)$ such that multiple Fourier series of $f$ diverges a.e. over cubes?

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