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SYMMETRIC DIFFERENTIABILITY OF FUNCTIONS OF TWO VARIABLES

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Everywhere bellow, the function to be considered will be assumed to take only finite values.

The function $f(x)$ is said to possess at an inner point x_0 of the interval (a, b) a symmetric derivative if there exists the limit

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}, \tag{$*$}$$

sometimes we denote it by the symbol $f^{(\prime)}(x_0)$. This definition was introduced by P. bu Buis-Reymond in 1875 ([1]; [2], p. 216) and $(*)$ - limit was used by M. V. Ostrogradsky ([3], p. 166; [2], p. 216).

Definition 1. A function $f(x, y)$ is said to possess at a point (x_0, y_0) a symmetric partial derivative with respect to x [with respect to y] if there exists the limit

$$\lim_{h\to 0} \frac{f(x_0+h, y_0)-f(x_0-h, y_0)}{2h} \equiv f_x{}^{(\prime)}(x_0, y_0)$$

[correspondingly

$$\left.\lim_{k\to 0} \frac{f(x_0, y_0+k)-f(x_0, y_0-k)}{2k} \equiv f_y{}^{(\prime)}(x_0, y_0)\right].$$

Definition 2. A function $f(x, y)$ will be called symmetrically totally differentiable, or simply, symmetrically differentiable at the point (x_0, y_0) if there exist two finite numbers $A = A(x_0, y_0)$ and $B = B(x_0, y_0)$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k)-f(x_0-h, y_0-k)-2hA-2kB}{|h|+|k|} = 0.$$

Theorem 1. *If the function $f(x, y)$ is symmetrically differentiable at the point (x_0, y_0), then there exist $f_x{}^{(\prime)}(x_0, y_0)$ and $f_y{}^{(\prime)}(x_0, y_0)$, and vice versa.*

Analogue of Theorem 1 does not take place for approximate differentiability ([4], p.300).

Theorem 2. *If the function $f(x, y)$ is Stolz differentiable at the point (x_0, y_0), then $f(x, y)$ is symmetrically differentiable at (x_0, y_0), and not vice versa.*

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Definition 3. A function $f(x,y)$ is said to possess at a point (x_0,y_0) a symmetric strong partial derivative with respect to x [with respect to y] if there exists the limit

$$\lim_{(h,k)\to(0,0)} \frac{f\,(x_0+h,y_0+k)-f\,(x_0-h,y_0+k)}{2h} \equiv f_{[1]}{}^{(\prime)}\,(x_0,y_0)$$

[correspondingly,

$$\lim_{(h,k)\to(0,0)} \frac{f\,(x_0+h,y_0+k)-f\,(x_0+h,y_0-k)}{2k} \equiv f_{[2]}{}^{(\prime)}\,(x_0,y_0)].$$

Theorem 3. *If there exist $f_{[1]}{}^{(\prime)}\,(x_0,y_0)$ and $f_{[2]}^{(1)}\,(x_0,y_0)$, then the function $f(x,y)$ is symmetrically differentiable at the point (x_0,y_0), and not vice versa.*

Theorem 4. *Differentiability of the function $f(x,y)$ in the sense of Stolz at the given point and the existence of strong symmetric partial derivatives at the same point are incompatible properties.*

Definition 4. A function $f(x,y)$ is said to possess at the point (x_0,y_0) a symmetric angular partial derivative with respect to x [with respect to y] if for every constant $C>0$ there exists C -independent limit

$$\lim_{\substack{(h,k)\to(0,0),\\ |k|\le C|h|}} \frac{f\,(x_0+h,y_0+k)-f\,(x_0-h,y_0+k)}{2h} \equiv f_{\widehat{1}}{}^{(\prime)}\,(x_0,y_0)$$

(correspondingly,

$$\lim_{\substack{(h,k)\to(0,0),\\ |h|\le C\ |k|}} \frac{f\,(x_0+h,y_0+k)-f\,(x_0+h,y_0-k)}{2h} \equiv f_{\widehat{2}}{}^{(\prime)}\,(x_0,y_0))$$

Theorem 5. *Symmetric differentiability and the existence of symmetric angular partial derivatives of the function $f(x,y)$ are incompatible properties.*

Remark 1. It is known that differentiability in the sense of Stolz is equivalent to the finiteness of angular partial derivatives [5].

Theorem 6. *If the function $f(x,y)$ is symmetrically differentiable at the point (x_0,y_0) and there exist finite $f_{[1]}{}^{(\prime)}\,(x_0,y_0)$ [or $f_{[2]}{}^{(\prime)}\,(x_0,y_0)$], then there exists finite $f_{\widehat{2}}{}^{(\prime)}\,(x_0,y_0)$, [correspondingly, $f_{\widehat{1}}{}^{(\prime)}\,(x_0,y_0)$].*

References

1. P. Du Buis-Reimond, *Z. Math. und Phys., His-lit. Abt.* **20**(1875), 121–129.
2. F. A. Medvedev, The guess and development of integral. (Russian) *Publishing House, Nauka, Moscow,* 1974.
3. M. V. Ostrogradsky, Complete works. *The Publishing House of Academy of Sciences of Ukrainian SSR, Kiev* **3**(1961).
4. S. Saks, Theory of the integral. *Monografie Matematyczne, Warszawa-Lwo'w* **7**(1937).
5. O. Dzagnidze, *Proc. A.Razmadze Math. Inst.* **106**(1993), 7–48.

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