Proc. A. Razmadze Math. Inst. 117(1998), 126–128

A. Meskhi

BOUNDEDNESS AND COMPACTNESS WEIGHTED CRITERIA FOR RIEMANN–LIOUVILLE AND ONE-SIDED MAXIMAL OPERATORS

(Reported on November 20, 1997)

Let for Lebesgue–measurable function $f : (0,\infty) \to \mathbb{R}^1$

$$R_\alpha f(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad (\alpha > 0)$$

be its Riemann-Liouville transform.

For $0 < \beta \le \alpha \le 1$ define maximal operators:

$$M^+_{\alpha,\beta} f(x) = \sup_{0<x<c} \frac{1}{(c-x)^\beta} \int_x^c \frac{|f(t)|}{(t-x)^{1-\alpha}} dt$$

and

$$M^-_{\alpha,\beta} f(x) = \sup_{0<c<x} \frac{1}{(x-c)^\beta} \int_c^x \frac{|f(t)|}{(x-t)^{1-\alpha}} dt.$$

It is clear, that if $0 < \gamma < 1$ then $M^+_{1,1-\gamma} f(x)$ and $M^-_{1,1-\gamma} f(x)$ are one-sided fractional maximal functions defined on $\mathbb{R}^1_+$.

We assume, that ν is a non-negative σ -finite Borel measure on $(0,\infty)$.

Let $1 < p < \infty$. Denote by L^p_ν the class of all ν– measurable functions g, for which

$$\|g\|_{L^p_\nu} = \Big(\int_0^\infty |g(x)|^p d\nu \Big)^{1/p} < \infty.$$

If ν is absolute continuous ($d\nu = vdx$, where v is Lebesgue- measurable, a.e. positive function on $(0,\infty)$), then instead of L^p_ν we use the symbol L^p_v.

Let $L^{p\infty}_\nu$ be a set of all ν–measurable functions g, satisfying the condition

$$\|g\|_{L^{p\infty}_\nu} = \sup_{\lambda>0} \lambda(\nu\{x : |g(x)| > \lambda\})^{1/p} < \infty.$$

The following theorems were proved:

Theorem 1. *Let $1 < p \le q < \infty$, $\alpha > \frac{1}{p}$. The following statements are equivalent:*

(a) *R_α is bounded from L^p to L^q_ν ;*

(b) *R_α is bounded from L^p to $L^{q\infty}_\nu$;*

1991 *Mathematics Subject Classification.* 42B20, 42B25.

Key words and phrases. Riemann-liouville operator, boundedness, compactness, weight.

(c)

$$A = \sup_{t>0} A(t) = \sup_{t>0} \Big(\int_t^\infty \frac{1}{x^{(1-\alpha)q}} d\nu \Big)^{\frac{1}{q}} t^{\frac{1}{p'}} < \infty;$$

(d)

$$A_1 = \sup_{k \in Z} (\nu[2^k, 2^{k+1}))^{1/q} 2^{k(\alpha - 1/p)} < \infty.$$

Moreover, $\|R_\alpha\|_{L^p \to L^q_\nu} \approx \|R_\alpha\|_{L^p \to L^{q\infty}_\nu} \approx A \approx A_1$.

Theorem 2. *Let* $1 < p < \infty$, $\alpha > \frac{1}{p}$. R_α *is compact from* L^p *to* L^q_ν *if and only if* $A < \infty$ *and* $\lim\limits_{a \to 0} A_a = \lim\limits_{b \to \infty} A_b = 0$, *where*

$$A_a = \sup_{0<t<a} \Big(\int_t^a \frac{1}{x^{(1-\alpha)q}} d\nu \Big)^{\frac{1}{q}} t^{\frac{1}{p'}}$$

and

$$A_b = \sup_{t>b} \Big(\int_t^\infty \frac{1}{x^{(1-\alpha)q}} d\nu \Big)^{\frac{1}{q}} (t-b)^{\frac{1}{p'}}$$

Necessary and sufficient conditions for weight v ensuring the boundedness and compactness of R_α from L^p to L^q_v in the case when $1 < q < p < \infty$ and $\alpha > 1/p$ were found in [1].

Theorem 3. a) *Let* $1 < p \le q < \infty$, $\beta + \frac{1}{p} < \alpha \le 1$. *The following statements are equivalent:*

i) $M^-_{\alpha,\beta}$ *is bounded from* L^p *to* L^q_ν;

ii) $M^-_{\alpha,\beta}$ *is bounded from* L^p *to* $L^{q\infty}_\nu$;

iii)

$$D = \sup_{t>0} \Big(\int_t^\infty \frac{1}{x^{(1-\alpha+\beta)q}} d\nu \Big)^{1/q} t^{1/p'} < \infty.$$

Moreover, $\|M^-_{\alpha,\beta}\|_{L^p \to L^q_\nu} \approx \|M^-_{\alpha,\beta}\|_{L^p \to L^{q\infty}_\nu} \approx D$.

b) *Let* $1 < p \le q < \infty, \beta + \frac{q-1}{q} < \alpha \le 1$, w *be the positive Lebesgue-measurable function on* $\mathbb{R}^1_+$. *The following statements are equivalent:*

i) $M^+_{\alpha,\beta}$ *is bounded from* L^p_w *to* L^q;

ii) $M^+_{\alpha,\beta}$ *is bounded from* L^p_w *to* $L^{q\infty}$;

iii)

$$D_1 = \sup_{t>0} \Big(\int_t^\infty \frac{w^{1-p'}(x)}{x^{(1-\alpha+\beta)p'}} dx \Big)^{1/p'} t^{1/q} < \infty.$$

Moreover, $\|M^+_{\alpha,\beta}\|_{L^p_w \to L^q} \approx \|M^+_{\alpha,\beta}\|_{L^p_w \to L^{q\infty}} \approx D_1$.

A complete description of the weight pairs (v, w) guaranteeing the boundedness of operator R_α from L^p_w to L^q_v when $1 < p < q < \infty$ and $0 < \alpha < 1$ was given in [2], while two- weight problem for this operator in the case when $1 < p \le q < \infty$ and $\alpha > 1$ was solved in [3] (for the case $\alpha = 1$ see [4-6]).

Necessary and sufficient conditions for pairs of weights, which provides the compactness of operator R_α for $1 < p \le q < \infty$ and $\alpha > 1$ were obtained in [7] (analogous problem was solved in [8] for $\alpha = 1$).

Two-weight inequalities of weak type for operators $M^+_{\alpha,\beta}$ and $M^-_{\alpha,\beta}$ defined on $\mathbb{R}^1$ have been studied in [9] (see also [2]).

For $1 < p = q < \infty$ and $0 < \alpha < 1$ two-weight problem for operator R_α remains unsolved and in this context the results presented here are interesting.

References

1. A. Meskhi, Solution of some weight problems for the Riemann-Liouville and Weyl operators. *Georgian Math. J.*, **5**(1998), No. 6, 565-574.

2. I. Genebashvili, A. Gogatishvili and V. Kokilashvili, Solution of two-weight problems for integral transforms with positive kernels, *Georgian Math. J.* **3**(1996), No.1, 319-342.

3. K. F. Andersen and H. P. Heinig, Weighted norm inequalities for certain integral operators. *SIAM J.Math.Anal.* **14** (1983), No.4, 834-844.

4. V. M. Kokilashvili, On Hardy's inequalities in weighted spaces. (Russian) *bull. Acad. Sci. Georgian SSR* **96**(1979), 37-40.

5. J. S. Bradley, Hardy's inequality with mixed norms. *Canad. Math. Bull.* **21**(1978), No. 1, 405–408.

6. V. G. Maz'ya, Sobolev spaces. *Spinger, Berlin,* 1985.

7. V. D. Stepanov, Weighted inequalities of Hardy type for higher derivatives and their applications. *Soviet Math. Dokl.,* **37**(1989), 389-393.

8. D. E. Edmunds, W. D. Evans and D. J. Harris, Approximation numbers of certain Volterra integral operators, *J. London Math. Soc.* **(2)37**(1988), 471-489.

9. F. J. Martin- Reyes and A. de la Torre, Weights for general one-sided maximal operators. *Preprint*, 1995.

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia