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ON UPPER AND LOWER DERIVATIVES OF LEBESGUE INTEGRAL

A mapping B defined on $\mathbb{R}^n$ is called differentiation basis in $\mathbb{R}^n$, if for every $x \in \mathbb{R}^n$ $B(x)$ is a collection of bounded open subsets of $\mathbb{R}^n$ containing x such that there is a sequence $\{R_k\} \subset B(x)$ with diam $R_k \to 0$ $(k \to \infty)$.

For $f \in L(\mathbb{R}^n)$ numbers

$$\overline{D}_B\Big(\int f, x\Big) = \lim_{r\to 0+}\Big[\sup_{R\in B(x), \operatorname{diam} R<r} \frac{1}{|R|}\int\limits_R f\Big]$$

and

$$\underline{D}_B\Big(\int f, x\Big) = \lim_{r\to 0+}\Big[\inf_{R\in B(x), \operatorname{diam} R<r} \frac{1}{|R|}\int\limits_R f\Big]$$

are called respectively upper and lower derivatives of the integral of f at a point x. If the upper and lower derivatives are equal, then their common meaning is called a derivative of the integral of f at a point x and it is denoted by $D_B\big(\int f, x\big)$. The basis B is said to differentiate the integral of f, if for almost every x

$$D_B\Big(\int f, x\Big) = f(x).$$

The basis B is called:

1) Buseman–Feller basis if for every $R \in \bigcup_{x\in\mathbb{R}^n} B(x)$, we have that $R \in B(y)$ for any $y \in R$;

2) density basis, if it differantiates an integral of characteristic function of any measurable set;

3) translation invariant, if $B(x) = \{x + R : R \in B(0)\}$ $(x \in \mathbb{R}^n)$, where 0 is the origin of the coordinate system in $\mathbb{R}^n$;

4) homothety invariant if for every $x \in \mathbb{R}^n$, $R \in B(x)$ and a homothety H with the center in $x - H(R) \in B(x)$;

5) formed of sets from a cartain class K if $\bigcup_{x\in\mathbb{R}^n} B(x) \subset K$;

6) convex if it is formed of convex sets.

Note that any Buseman–Feller and homothety invariant basis is translation invariant too.

Let I is the basis in $\mathbb{R}^2$, for which $I(x)(x \in \mathbb{R}^2)$ consists of all two-dimentional intervals containing x.

Besicovitch [1] prooved that for every function $f \in L(\mathbb{R}^2)$ both of sets

$$\Big\{x : f(x) < \overline{D}_I\Big(\int f, x\Big) < \infty\Big\}, \quad \Big\{x : -\infty < \underline{D}_I\Big(\int f, x\Big) < f(x)\Big\}$$

have a measure equal to zero.

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Note that the following simple assertion is true: if convex basis B is translation or homothety invariant then for every $f \in L$

$$\underline{D}_B\Big(\int f, x\Big) \le f(x) \le \overline{D}_B\Big(\int f, x\Big) \qquad (*)$$

for almost every x.

Thus by virtue of Besicovitch's result the integral means taken over two-dimentional intervals may diverge only unboundedly.

Introduce the following

Definition 1. Say that a basis B in $\mathbb{R}^n$ has the Besicovitch's property if for every function $f \in L(\mathbb{R}^n)$ both of sets

$$\Big\{x : f(x) < \overline{D}_B\Big(\int f, x\Big) < \infty\Big\}, \qquad (1)$$

$$\Big\{x : -\infty < \underline{D}_B\Big(\int f, x\Big) < f(x)\Big\} \qquad (2)$$

have a measure equal to zero.

Guzmán and Menarges [2] generalized Besicovitch's result. They proved that every Buseman–Feller homothety invariant density convex basis, which is formed of central-symmetric sets, has the Besicovitch's property.

We have established that in Guzmán–Menarges' assertion the condition of central-symmetrity is unnecessary, in particular it's just

Theorem 1. *Every Buseman–Feller homothety invariant density convex basis has the Besicovitch's property.*

Within this theorem there arise questions:

1) whether every Buseman–Feller translation invariant density convex basis has the Besicovitch's property?

2) whether every homothety invariant density convex basis has the Besicovitch's property?

Both of questions have the negative answer. Moreover the following theorems are true:

Theorem 2. *There is a Buseman–Feller translation invariant basis B, formed of two-dimentional intervals, for which there is a function $f \in L(\mathbb{R}^2)$, $f \ge 0$, such that*

$$f(x) < \overline{D}_B\Big(\int f, x\Big) < \infty$$

almost everywhere on $\mathbb{R}^2$.

Theorem 3. *There is a homothety invariant basis B, formed of two-dimentional intervals, for which there is a function $f \in L(\mathbb{R}^2)$, $f \ge 0$, such that*

$$f(x) < \overline{D}_B\Big(\int f, x\Big) < \infty$$

almost everywhere on $\mathbb{R}^2$.

Finally we note that $(*)$ inequalities did not extend on any convex basis. In particular, we have

Theorem 4. *There is a Buseman–Feller basis B, formed of two-dimentional intervals, for which there is a function $f \in L(\mathbb{R}^2)$, $f \geq 0$, such that*

$$D_B\left(\int f, x\right) = \infty$$

almost everywhere on $\mathbb{R}^2$.

References

1. A. S. Besicovitch, On differentiation of Lebesgue double integrals. *Fund. Math.* **25**(1935), 209–216.

2. M. de Guzmán, Differentiation of integrals in $\mathbb{R}^n$, *Lecture Notes in Math.* **481**, Springer, 1975.

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