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EXISTENCE OF LINEAR EXTENSION OPERATOR FOR TRACES OF FUNCTION SPACES ON IRREGULAR DOMAINS

Let H_p^s ($1 < p < \infty$, $s > 0$) denote the Bessel potential space on $\mathbb{R}^n$, otherwise known as the fractional Sobolev space (it coincides with the usual Sobolev space for $s \in \mathbb{N}$). Let $\Omega \in \mathbb{R}^n$ be an open set. Denote by $H_p^s(\Omega)$ the linear space whose elements f are restrictions of functions $g \in H_p^s$ from $\mathbb{R}^n$ to Ω. The norm in $H_p^s(\Omega)$ is given by

$$\|f\|_{H_p^s(\Omega)} = \inf\Big\{ \|g\|_{H_p^s} : g|_\Omega = f \Big\}.$$

With this norm $H_p^s(\Omega)$ becomes a Banach space, which is called the *trace space* of H_p^s on Ω. The *trace operator* $tr_\Omega : f \mapsto f|_\Omega$ is then a linear bounded operator from H_p^s to $H_p^s(\Omega)$. A linear bounded operator $ext : H_p^s(\Omega) \to H_p^s$ is called *linear extension operator* for $H_p^s(\Omega)$, if $tr_\Omega \circ ext = id$ in $H_p^s(\Omega)$. We are interested in conditions on Ω under which

$$\text{there exists a linear extension operator for } H_p^s(\Omega). \tag{1}$$

Seeger [1] proved that (1) is true if Ω satisfies the (ε, δ) condition of Jones [2]. The class of such Ω is strictly larger than the class of domains with Lipschitz boundary. For Ω from the latter class (1) was obtained earlier by Strichartz [3].

Let $B(x, R) = \{y \in \mathbb{R}^n : |x - y| < R\}$. We say that Ω satisfies the *ball condition* if there is an $\varepsilon > 0$ such that for every $x \in \Omega$ and $0 < R < 1$ there exists a point $y = y(x, R)$ such that $B(y, \varepsilon R) \subset B(x, R) \cap \Omega$. The ball condition is more general than the (ε, δ) condition, and therefore the following theorem generalizes Seeger's result.

Theorem 1. *If Ω satisfies the ball condition, then* (1) *is true.*

Let $aE = \{ax : x \in E\}$ for $E \subset \mathbb{R}^n$ and $a > 0$. We say that Ω satisfies the *H_p^s-capacity condition* if there is an $\varepsilon > 0$ such that for every $x \in \Omega$ and $0 < R < 1$

$$cap(R^{-1}(B(x, R) \cap \Omega), H_p^s) \geq \varepsilon.$$

Here the *H_p^s-capacity* $cap(E, H_p^s)$ of a set $E \subset \mathbb{R}^n$ (see e.g. Maz′ya [4]) is defined by

$$cap(E, H_p^s) = \inf\Big\{ \|f\|_{H_p^s}^p : f \geq 1 \text{a. e. in a neighborhood of } E \Big\}.$$

Note that by the embedding $H_p^s \subset L_p$ we have $cap(E, H_p^s) \geq C\, mes\, E$ for any measurable E, whereas sets of measure 0 can have a positive capacity. For example, the capacity of a point $cap(\{x_0\}, H_p^s) > 0$ if $s > \frac{n}{p}$, and the capacity of a continuous curve $cap(\gamma, H_p^s) > 0$ if $s > \frac{n-1}{p}$. It follows that the H_p^s-capacity condition is more general than the ball condition.

Theorem 2. *Let Ω satisfy the $H_{p_0}^{s_0}$-capacity condition for some $p_0 \leq 2$ and $s_0 < 1$. Then* (1) *is true for all p, s such that $p > p_0$ and $s - [s] > s_0$, where $[s]$ is the integer part of s.*

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The proofs of the given results are based on the method of local polynomial approximation going back to Brudnyi and applied in various situations by Shvartsman, Jonsson-Wallin, and other authors. Actually, we operate in the framework of general Besov and Lizorkin-Triebel spaces and have counterparts of the above results for those spaces. An important role in the argument is played by the technique of Peetre's maximal functions and by atomic characterizations due to Frazier-Jawerth and Netrusov.

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