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WEIGHTED INEQUALITIES FOR RIEMANN-LIOUVILLE TRANSFORM

Let ν be a non-negative σ -finite Borel measure on $(0,\infty)$.

For $1<r<\infty$ and $1<s\leq\infty$, denote by L^{rs}_ν the Lorentz space with measure ν, which is a class of all ν- measurable functions $g:(0,\infty)\to\mathbb{R}^1$, for which

$$\|g\|_{L^{rs}_\nu}=\Big(s\int\limits_0^\infty \lambda^{s-1}(\nu\{x:|g(x)|>\lambda\})^{s/r}d\lambda\Big)^{1/s}<\infty,$$

where $1<r<\infty$, $1<s<\infty$ and

$$\|g\|_{L^{r\infty}_\nu}=\sup_{\lambda>0}\lambda(\nu\{x:|g(x)|>\lambda\})^{1/r}<\infty,$$

where $1<r<\infty$.

If ν is absolute continuous ($d\nu=vdx$, where v is Lebesgue-measurable, a.e. positive function on $(0,\infty)$), then instead of L^{rs}_ν we use the symbol L^{rs}_v.

Note, that $L^{rr}_\nu=L^r_\nu$ is Lebesgue space.

Let μ be another non-negative σ-finite Borel measure on $(0,\infty)$ and let

$$R_{\alpha\phi\psi\mu}f(x)=\phi(x)\int\limits_0^x\frac{f(t)\psi(t)}{(x-t)^{1-\alpha}}d\mu,\qquad \alpha>0,$$

be Riemann-Liouville operator, where ϕ (respectively ψ) is ν–measurable (respectively μ–measurable) positive function on $(0,\infty)$.

If μ is Lebesgue measure, then we denote $R_{\alpha\varphi\psi\mu}$ by the symbol $R_{\alpha\varphi\psi}$.

The following theorem is valid.

Theorem 1. *Let $1<r,s,p<\infty$, $1<q\leq\infty$, $\max\{r,s\}\leq\min\{p,q\}$, $\alpha>0$. If the following conditions are satisfied:*

(i)

$$\sup_{t>0}\|\phi(x)x^{\alpha-1}\chi_{(2t,\infty)}(x)\|_{L^{pq}_\nu}\|\psi(\cdot)\chi_{(0,t)}(\cdot)\|_{L^{r's'}_\mu}<\infty;$$

(ii)

$$\sup_{t>0}\|\phi(x)\chi_{[t/2,2t)}(x)\|_{L^{pq}_\nu}\|\psi(\cdot)\chi_{[t/2,t)}(\cdot)(t-\cdot)^{\alpha-1}\|_{L^{r's'}_\mu}<\infty,$$

then operator $R_{\alpha\phi\psi\mu}$ is bounded from L^{rs}_μ to L^{pq}_ν. Conversely, if $R_{\alpha\phi\psi}$ is bounded from L^{rs}_μ to L^{pq}_ν, then the condition (i) *is satisfied.*

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Theorem 2. *Let r, s, p and q satisfy the condition of Theorem 1, $\alpha > \frac{1}{r}$. Suppose that ψ is positive decreasing function on $(0,\infty)$.*

(a) *For the boundedness of $R_{\alpha\phi\psi}$ from L^{rs} to L^{pq}_{ν} it is necessary and sufficient that*

$$A = \sup_{t>0} \|\phi(x)x^{\alpha-1}\chi_{(t,\infty)}(x)\|_{L^{pq}_{\nu}} \|\psi(\cdot)\chi_{(0,\frac{t}{2})}(\cdot)\|_{L^{r's'}} < \infty.$$

Moreover, $\|R_{\alpha\phi\psi}\| \approx A$.

(b) *$R_{\alpha\phi\psi}$ is compact from L^{rs} to L^{pq}_{ν} if and only if $A < \infty$ and $\lim\limits_{a\to 0} A_a = \lim\limits_{b\to 0} A_b = 0$, where*

$$A_a = \sup_{0<t<a} \|\phi(x)x^{\alpha-1}\chi_{(t,a]}(x)\|_{L^{pq}_{\nu}} \|\psi(\cdot)\chi_{(0,\frac{t}{2})}(\cdot)\|_{L^{r's'}}$$

and

$$A_b = \sup_{t>b} \|\phi(x)x^{\alpha-1}\chi_{(t,\infty)}(x)\|_{L^{pq}_{\nu}} \|\psi(\cdot)\chi_{(b/2,t/2)}(\cdot)\|_{L^{r's'}}.$$

Analogous theorem in the case, when $\phi \equiv \psi \equiv 1$, $d\nu = vdx$ in Lebesque spaces was proved in [1].

Let $R_{\alpha\phi\psi} = R_\alpha$, if $\phi \equiv \psi \equiv 1$. The following theorem is valid.

Theorem 3. *Let $1 < p < \infty$, $0 < \alpha < \frac{1}{p}$, v and w be positive increasing functions on $(0,\infty)$. Operator R_α is bounded from L^p_w to L^p_v if and only if*

$$A_1 = \sup_{t>0} \bigg(\int_t^\infty \frac{v(y)}{y^{(1-\alpha)p}} dy \bigg)^{\frac{1}{p}} \bigg(\int_0^{\frac{t}{2}} w^{1-p'}(y)dy \bigg)^{\frac{1}{p'}} < \infty.$$

Moreover, $\|R_\alpha\| \approx A_1$.

Theorem 4. *Let $1 < q < p < \infty$, $\alpha > 1/p$. Suppose that w is positive increasing function on $(0,\infty)$. The following statements are equivalent:*

(a) *R_α is bounded from L^p_w to L^q_v;*

(b) *R_α is compact from L^p_w to L^q_v;*

(c)

$$B = \bigg(\int_0^\infty \bigg(\int_{2x}^\infty \frac{v(t)}{t^{(1-\alpha)q}} dt \bigg)^{\frac{p}{p-q}} \bigg(\int_0^x w^{1-p'}(t)dt \bigg)^{\frac{(q-1)p}{p-q}} w^{1-p'}(x)dx \bigg)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|R_\alpha\| \approx B$.

Necessary and sufficient conditions for the weight pairs (v, w) which guarantee the boundedness of operator R_α $(0 < \alpha < 1)$ from L^p_w to L^q_v, when $1 < p < q < \infty$ was found in [2]. Similar problem for $1 < p \le q < \infty$ and $\alpha > 1$ was solve in [3] (for $\alpha = 1$ see [4-6]).

The compactness weighted criteria for operator R_α in the case, when $\alpha > 1$ was derived in [7]. In [8] analogous problem was solved for $\alpha = 1$.

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