Proc. A. Razmadze Math. Inst. 117(1998), 147–149

U. Goginava

ON THE CONVERGENCE AND SUMMABILITY OF MULTIPLE FOURIER-WALSH SERIES IN $L^p\left([0,1]^N\right)$, $p \in [1,+\infty]$ SPACES

In this paper we study the convergence and summability by negative order Cesaro method of the N-multiple Fourier-Walsh series in $L^p\left([0,1]^N\right)$, $p \in [1,+\infty]$ metrics. As special cases we obtain of the R.D. Getsadze Theorem and generalization of the F. Moricz Theorem.

Let $Q_N = [0,1]^N$ denote a cube in the N-dimensional Euclidean space $\mathbb{R}^N$. The elements of $\mathbb{R}^N$ are denoted by $x = (x_1, \dots, x_N)$.

Let $M = \{1, \dots, N\}$, $B = \{\ell_1, \dots, \ell_r\}$, $B \subset M$, $\ell_k < \ell_{k+1}$, $k = 1, \dots, r-1, B' = M \backslash B$, $\alpha = (\alpha_1, \dots, \alpha_N)$. For the integer n the vector $(n, \dots, n)$ of the space $\mathbb{R}^N$ is denoted by $\tilde{n}$. Let $\overline{k}(B)$ be a number of elements in B. For any $x = (x_1, \dots, x_N)$ and any $B \subset M$, let x_B denote a vector of the $\mathbb{R}^N$ space, whose coordinates with indices from the set B coincide with the coordinates of the vector x, and the coordinates with indices from the set B' are zero. Denote $x_M = x$, $x_{\{\emptyset\}} = \widetilde{0}$.

Denote by $C_\rho(Q_N)$ the space of uniformly ρ-continues (see [1], p. 50) on Q_N, 1-periodical relative to each variable functions with the following norms:

$$\|f\|_{C_\rho} = \sup_{x \in Q_N} |f(x)|,$$

we write L^∞ instead of C_ρ and set

$$\|f\|_p = \int\limits_{[0,1]^N} |f(x)|^p \, dx, \ 1 \le p < \infty.$$

Let $\{\omega_j(x) : j \ge 0\}$ are the Walsh orthonormal system defined on the unit interval $[0,1)$ in the Paley enumeration [2].

Given a function $f \in L^1(Q_N)$, we form its N-multiple Fourier-Walsh series

$$\sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_N=0}^{\infty} a_{\nu_1,\dots,\nu_N} \prod_{i \in M} \omega_{\nu_i}(x_i) \tag{1}$$

with

$$a_{\nu_1,\dots,\nu_N}(f) = \int\limits_{[0,1]^N} f(x) \prod_{i \in M} \omega_{\nu_i}(x_i) \, dx.$$

The rectangular partial sums of series (1) defined by

$$S_{m_M}(f,x) = \sum_{\nu_1=0}^{m_1-1} \cdots \sum_{\nu_N=0}^{m_N-1} a_{\nu_1,\dots,\nu_N} \prod_{i \in M} \omega_{\nu_i}(x_i)$$

1991 *Mathematics Subject Classification.* 42C10.

Key words and phrases. Multiple Fourier-Waish series, module of variation, uniform convergence.

and

$$\sigma_{m_M}^{\alpha}(f,x)=\frac{1}{\prod\limits_{i\in M}A_{m_i}^{\alpha_i}}\sum_{\nu_1=0}^{m_1}\cdots\sum_{\nu_N=0}^{m_N}\prod_{i\in M}A_{m_i-\nu_i}^{\alpha_i-1}S_{\nu_M}(f,x)$$

- Cesaro (C,α) -means of the partial sums S_{m_M}.

Let $\Delta^{\{\ell_i\}}\left(f,x,h_{\{\ell_i\}}\right)=f\left(x\dot{+}h_{\{\ell_i\}}\right)-f(x)$, $i=1,\ldots,r$, where $\dot{+}$ is the operation in the diadic group, the set of all sequences $(t_1,\ldots,t_n,\ldots)$, $t_n=0\vee 1$ for $n=1,2,\ldots,$ is addition modul 2 in each coordinate.

The expression we obtain by a successive application of operations $\Delta^{\{\ell_1\}}\left(f,x,h_{\{\ell_1\}}\right),...,\Delta^{\{\ell_r\}}\left(f,x,h_{\{\ell_r\}}\right)$ will be denoted by $\Delta^{B}(f,x,h_B)$.

Let $f\in L^p(Q_N)$. The expression

$$\omega_B(\delta,f)_p=\sup_{|h_i|\le\delta_i,\ i\in B}\left\|\Delta^B(f,\cdot,h_B)\right\|_p,\ \delta_i\in(0,1],\ i\in B$$

is called a mixed or a partial modulus of continuity in L^p-norm, when $\overline{k}(B)\in[2,N]$ or $\overline{k}(B)=1$, respectively. Let f be bounded on Q_N. The partial modulus of variation of the function f is defined by

$$\begin{aligned}\nu_i(m_i,f)\quad=\quad&\sup_{x_{M\setminus\{i\}}}\sup_{\Pi_{m_i}}\sum_{k=0}^{m_i-1}\Big|f\left(x_1,\ldots,x_{i-1},x_i^{(2k+1)},x_{i+1},\ldots,x_N\right)\\&-f\left(x_1,\ldots,x_{i-1},x_i^{(2k)},x_{i+1},\ldots,x_N\right)\Big|,\end{aligned}$$

where Π_{m_i} is an arbitrary system of disjoint intervals $\left(x_i^{(2k)},x_i^{(2k+1)}\right)$, $(k=0,\ldots,m_i-1)$, of the interval $[0,1)$, (for $N=1$, see [3]).

We prove the following

Theorem 1. *Let* $f\in L^p(Q_N)$, $p\in[1,+\infty]$, $\forall B:B\subset M,\ B\ne\{\emptyset\}$, $\alpha_i\in(0,1)$, $i=1,\ldots,r,$

$$\prod_{i\in B}\delta_i^{-\alpha_i}\omega_B(\delta,f)_p\to 0\qquad(\delta_i\to 0,\ i\in B),$$

then

$$\left\|\sigma_{m_M}^{-\alpha}(f)-f\right\|_p\to 0\quad(m_i\to\infty,\ i\in M).$$

Theorem 2. *Let* $f\in C_\rho(Q_N)$ *and* $N\ge 2$. *If there is* i_0 $(i_0\in M)$ *such that the following conditions are fulfilled*

$$\min_{1\le\ell\le n-1}\left\{\sqrt[N]{\omega_{\{i_0\}}(1/n,f)}\log\ell+\sum_{k=\ell+1}^{n-1}\frac{\sqrt[N]{\nu_{i_0}(k,f)}}{k^{1+1/N}}\right\}=o(1)$$

and

$$\min_{1\le\ell\le n-1}\left\{\sqrt[N]{\omega_i(1/n,f)}\log\ell+\sum_{k=\ell+1}^{n-1}\frac{\sqrt[N]{\nu_i(k,f)}}{k^{1+1/N}}\right\}=O(1)$$

$\left(\forall i:i\in\overline{1,N},\ i\ne i_0\right)$, $(n\to\infty)$,
then

$$\|S_{m_M}(f)-f\|_{C_\rho}\to 0\quad(m_i\to\infty,\ i\in M).$$

Corollary 1 ([4], [5]). *Let* $f \in C_\rho(Q_N)$ *and* $N \geq 2$. *If there exists* i_0 $(i_0 \in M)$ *such that*

$$\omega_{\{i_0\}}(\delta, f)\left(\log\frac{1}{\delta_{i_0}}\right)^N \to 0 \quad (\delta_{i_0} \to 0+)$$

and

$$\omega_{\{i\}}(\delta, f)\left(\log\frac{1}{\delta_i}\right)^N < \infty, \quad \delta_i \in]0,1], \ \left(i = \overline{1,N}, \ i \neq i_0\right),$$

then

$$\|S_{m_M}(f) - f\|_{C_\rho} \to 0 \quad (m_i \to \infty, \ i \in M).$$

Corollary 2. *Let* $f \in C_\rho(Q_N)$ *and* $(\forall j : j \in M)$

$$\sum_{k=1}^{\infty}\frac{\sqrt[N]{\nu_j(k,f)}}{k^{1+1/N}} < \infty$$

then

$$\|S_{m_M}(f) - f\|_{C_\rho} \to 0 \quad (m_i \to \infty, \ i \in M).$$

Corollary 3. *Let* $f \in C_\rho(Q_N)$ *and*

$$\nu_j(m_j, f) < \infty, \ \ j = 1, \dots, N,$$

then

$$\|S_{m_M}(f) - f\|_{C_\rho} \to 0 \quad (m_i \to \infty, \ i \in M).$$

Example 1. Obviously, the fulfilment of the condition of Corollary 1 implaes that of the condition of Theorem 2, but not vice versa; that is, there exists continuous function f for which the condition of Theorem 2 is fulfilled and the condition of Corollary 1 does not.

References

1. B. I. Golubov, A. V. Efimov, and V. A. Skvortsov. Series and Transformations of Walsh. (Russian) *Moscow,* 1987.

2. A. A. Paley, A remarkable series of orthonormal functions. *Proc. London Math. Soc.* **34**(1932), 241–279.

3. Z. A. Chanturia, Function variation modulus and it's application in the theory of Fourier series. (Russian) *Dokl. AN SSSR*, **214**(1974), No. 1, 63–66.

4. R. D. Getsadze, On the convergence and divergence of multiple Fourier series with respect to the Walsh-Paley systems in the spaces C and L. *Anal. Math.* **13**(1987), No. 1, 29–36.

5. F. Moricz. On the uniform convergence and L^1 -convergence of double Walsh-Fourier series. *Studia Math.* **102**(1992), No. 3, 225–237.

Author's address:
Sukhumi Branch of I. Djavakhishvili State University
12, Djikia St., Tbilisi 380086
Georgia