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TWO-WEIGHT COMPACTNESS CRITERIA FOR POTENTIAL TYPE OPERATORS

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In the present note we give a characterization of couples of weights which guarantee the compactness of potential type operators without using the capacity notion.

For the classical results in the unweighted case we refer to [1] and for the weighted case in terms of capacity to [2].

Here we consider the model case of Riesz potentials

$$(I_\gamma f)(x) = \int\limits_{R^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy, \ \ 0<\gamma<n,$$

although all the following problems are investigated in a more general case of integral transforms with positive kernels under some restrictions imposed on them.

Let now w be a weight, i. e., an almost everywhere positive locally integrable on R^n function. For any Borel set E we define the measure

$$wE = \int\limits_E w(x)dx.$$

In the sequel $L^p_w(R^n)$ $(1 \le p < \infty)$ denotes a set of all measurable functions $f: R^n \to R^1$ for which

$$\|f\|_{L^p_w} = \Big(\int\limits_{R^n} |f(x)|^p w(x)dx\Big)^{1/p} < \infty$$

and $L^{p\infty}_w(R^n)$ - a set of all measurable functions $f:\ R^n \to R^1$ by the condition

$$\|f\|_{L^{p\infty}_w} = \sup_{\lambda>0} \lambda(w\{x\in R^n:\ |f(x)|>\lambda\})^{1/p} < \infty.$$

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The main sources of our investigation are the solutions of the two-weight boundedness problems for the above-mentioned operator (see, for example, [3]).

Theorem A. *Let $1<p<q<\infty$. The operator I_γ is bounded from $L^p_w(R^n)$ into $L^{q\infty}_v(R^n)$ iff*

$$\sup_{\substack{x\in R^n\\ r>0}} A_1(x,r) < \infty, \tag{1}$$

where

$$A(x,r) = \Big(\int\limits_{B(x,r)} v(y)dy\Big)^{1/q} \Big(\int\limits_{R^n\setminus B(x,r)} \frac{w^{1-p'}(y)}{|x-y|^{(n-\gamma)p'}} dy\Big)^{1/q},$$

$B(x,r)$ is a ball in R^n with a center in x and radii r.

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Theorem B. *Let $1 < p < q < \infty$. The operator I_γ is bounded from $L^p_w(R^n)$ into $L^q_v(R^n)$ iff* (1) *and the condition*

$$\sup_{\substack{x\in R^n\\ r>0}} A_2(x,r) < \infty \tag{2}$$

are fulfilled, where

$$A_2(x,r) = \Big(\int\limits_{B(x,r)} w^{1-p'}(y)dy\Big)^{1/p'}\Big(\int\limits_{R^n\setminus B(x,r)} \frac{v(y)}{|x-y|^{(n-\gamma)q}}dy\Big)^{1/q}$$

For the domain $\Omega \subset R^n$ we introduce the quantities

$$A_1(x,r,\Omega) = \Big(\int\limits_{B(x,r)\cap\Omega} v(y)dy\Big)^{1/q}\Big(\int\limits_{\Omega\setminus B(x,r)} \frac{w^{1-p'}(y)}{|x-y|^{(n-\gamma)p'}}dy\Big)^{1/p'} \tag{3}$$

and

$$A_2(x,r,\Omega) = \Big(\int\limits_{B(x,r)\cap\Omega} w^{1-p'}(y)dy\Big)^{1/p'}\Big(\int\limits_{\Omega\setminus B(x,r)} \frac{v(y)}{|x-y|^{(n-\gamma)q}}dy\Big)^{1/q} \tag{4}$$

Theorem 1. *Let $1 < p < q < \infty$, and Ω be a bounded domain. The operator I_γ is compact from $L^p_w(\Omega)$ into $L^q(\Omega)$ iff the conditions*

$$\sup_{x\in\Omega}\sup_{r>0} A_1(x,r,\Omega) < \infty, \tag{5}$$

and

$$\lim_{s\to 0}\sup_{x\in\Omega}\sup_{0<r<1} A_1(x,r,\Omega) = 0 \tag{6}$$

are fulfilled.

Theorem 2. *Let $1 < p < q < \infty$, and Ω be a bounded domain. The operator I_γ is compact from $L^p_w(\Omega)$ into $L^q_v(\Omega)$ iff the conditions* (5), (6),

$$\sup_{x\in\Omega}\sup_{r>0} A_2(x,r,\Omega) < \infty, \tag{7}$$

and

$$\lim_{s\to 0}\sup_{x\in\Omega}\sup_{0<r<1} A_2(x,r,\Omega) = 0 \tag{8}$$

are satisfied.

Theorem 3. *Let $1 < p < q < \infty$. The operator is compact from $L^p_w(R^n)$ into $L^q_v(R^n)$ iff it is bounded from $L^p_w(R^n)$ into $L^{q\infty}_v(R^n)$ and the following conditions are fulfilled:*

$$\lim_{s\to 0}\sup_{\substack{x\in R^n\\ 0<r<1}} A_i(x,r) = 0 \quad (i = 1,\, 2), \tag{9}$$

and

$$\lim_{|x|\to\infty}\sup_{0<r<1} A_i(x,r) = 0 \quad (i = 1,\, 2) \tag{10}$$

Theorem 4. *Let $1 < p < q < \infty$. For the Riesz potential I_γ to be compact from $L^p_w(R^n)$ into $L^{q\infty}_v$ it is necessary and sufficient that it be bounded from $L^p_w(R^n)$ into $L^{q\infty}_v(R^n)$ and for $i = 1$ conditions* (9) *and* (10) *be satisfied.*

Corollary 1. *If* $1 < p < q < \infty$ *and* Ω *is a bounded domain, then the operator* I_γ *is compact from* $L^p(\Omega)$ *into* $L^q_v(\Omega)$ *iff*

$$D(s) = \sup_{\substack{0<r<1 \\ x\in\Omega}} \gamma^{\frac{q(\gamma p - n)}{p}} \int\limits_{B(x,r)\cap\Omega} v(y)dy < \infty$$

for some $s > 0$ *and*

$$\lim_{s\to 0} D(s) = 0.$$

Definition 1. A measure ν satisfies the reverse doubling condition ($\nu \in (RD)$) if there exist constants $\beta_1 > 1$ and $\beta_2 < 1$ such that

$$\nu B(x,r) \leq \beta_2 \nu B(x, \beta_1 r)$$

for all $x \in R^n$ and $r > 0$.

It is well known that the (RD) condition is weaker than the doubling condition.

In the sequel we give easier criteria for compactness under additional assumptions for a couple of weights.

Theorem 5. *If* $1 < p < q < \infty$, Ω *is a bounded domain and the weights satisfy the* (RD) *condition, then for* $I\gamma$ *to be compact from* $L^p_w(\Omega)$ *into* $L^q_v(\Omega)$ *it is necessary and sufficient that the following two conditions be satisfied:*

i) $M(c) = \sup\limits_{\substack{x\in\Omega \\ 0<r<c}} |B(x,r)\cap\Omega|^{\frac{\gamma}{n}-1} (vB(x,r)\cap\Omega)^{1/q} \Big(\int\limits_{B(x,r)\cap\Omega} w^{1-p'}(y)dy \Big)^{1/p'} < \infty$

and

ii) $\lim\limits_{c\to 0} M(c) = 0.$

An corresponding result for R^n *holds as well.*

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