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TWO-WEIGHT INEQUALITIES FOR POTENTIALS ON THE LORENTZ SPACES DEFINED ON HOMOGENEOUS GROUPS

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In the present note in some particular cases the necessary and sufficient conditions are found for pairs of weights ensuring the validity of two-weight inequalities for potentials in the Lorentz spaces defined on homogeneous groups.

The two-weight problems for potentials in the Lebesgue spaces are already solved (see [1–6]).

In the sequel we shall need the notions of a homogeneous group G and of weighted Lorentz spaces $L^{pq}_{\omega}(G)$ defined on it. All these notions can be found in the previous report.

For the measurable function $f : G \to \mathbb{R}$ the integral transform

$$T_\alpha f(x) = \int\limits_G \frac{f(y)}{r(xy^{-1})^{Q-\alpha}}\, dy, \quad 0 < \alpha < Q,$$

is referred to as the potential defined on G, where dy is the Haar measure on G.

Basic Theorem. *Let* $0 < \alpha < Q$, $1 < r < \frac{Q}{\alpha}$, $\frac{1}{r^*} = \frac{1}{r} - \frac{\alpha}{Q}$, $1 < s < \infty$, $1 < p \le r^*$, $1 < q \le \infty$, *provided* $\max(r, s) \le \min(p, q)$. *If the weight functions* $v(x)$, $\omega(x)$, $u_1(x)$, $u_2(x)$ *satisfy the following conditions:*

(1) $\sup\limits_{t>0} \big\|u_2(\cdot) r^{\alpha-Q}(\cdot)\chi_{\{r(y)>t\}}(\cdot)\big\|_{L^{pq}_v(G)} \Big\|\frac{1}{u_1(\cdot)\omega(\cdot)}\chi_{\{r(y)<t\}}(\cdot)\Big\|_{L^{r's'}_\omega(G)} < \infty;$

(2) $\sup\limits_{t>0} \big\|u_2(\cdot)\chi_{\{r(y)<t\}}(\cdot)\big\|_{L^{pq}_v(G)} \Big\|\frac{1}{u_1(\cdot)\omega(\cdot)} r^{\alpha-Q}(\cdot)\chi_{\{r(y)>t\}}\Big\|_{L^{r's'}_\omega(G)} < \infty;$

(3) *there exists a positive constant b such that for an arbitrary $t > 0$ the inequality*

$$t^{Q[\frac{\alpha}{Q} - \frac{1}{r} + \frac{1}{p}]} \sup_{\frac{t}{c_0} < r(x) \le 8c_0 t} \big(u_2(x) v^{\frac{1}{p}}(x)\big) \le b \inf_{\frac{t}{c_0} < r(x) \le 8c_0 t} \big(u_1(x) w^{\frac{1}{r}}(x)\big)$$

holds.

Then the inequality

$$\|u_2(\cdot) T_\alpha f(\cdot)\|_{L^{pq}_v(G)} \le c\|u_1(\cdot) f(\cdot)\|_{L^{rs}_\omega(G)}, \tag{1}$$

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where the constant $c > 0$ does not depend on f, is fulfilled.

If inequality (1) *holds, then conditions* (1) *and* (2) *are fulfilled.*

Corollary 1. *Let the numbers α, r, s, p, q satisfy the conditions of the theorem and let $u_1(t)$, $u_2(t)$, $v(t)$ be positive, increasing on $(0,\infty)$ functions. The inequality*

$$\|u_2(r(\cdot))T_\alpha f(\cdot)\|_{L^{pq}_{v(r(\cdot))}(G)} \le c\|u_1(r(\cdot))f(\cdot)\|_{L^{rs}(G)}, \tag{2}$$

where the constant $c > 0$ does not depend on f, is fulfilled if and only if

$$\sup_{t>0} \big\|u_2(r(\cdot))r^{\alpha-Q}(\cdot)\chi_{\{r(y)>t\}}(\cdot)\big\|_{L^{pq}_{v(r(\cdot))}(G)} \Big\|\frac{1}{u_1(r(\cdot))}\chi_{\{r(y)<t\}}(\cdot)\Big\|_{L^{r's'}(G)} < \infty$$
$$\Big(r' = \frac{r}{r-1}, \quad s' = \frac{s}{s-1}\Big).$$

Corollary 2. *Let the numbers α, r, s, p, q satisfy the conditions of the theorem and let $u_1(t)$, $u_2(t)$, $v(t)$ be positive, decreasing on $(0,\infty)$ functions. Inequality* (2) *is fulfilled if and only if*

$$\sup_{t>0} \big\|u_2(r(\cdot))\chi_{\{r(y)<t\}}(\cdot)\big\|_{L^{pq}_{v(r(\cdot))}(G)} \Big\|\frac{1}{u_1(r(\cdot))}r^{\alpha-Q}(\cdot)\chi_{\{r(y)>t\}}(\cdot)\Big\|_{L^{r's'}(G)} < \infty.$$

Corollary 3. *Let $0 < \alpha < Q$, $1 < r < \frac{Q}{\alpha}$, $\frac{1}{r^*} = \frac{1}{r} - \frac{\alpha}{Q}$, $1 < p \le r^*$, $1 < q \le \infty$ provided $r \le \min(p,q)$, and let $v(t)$ and $\omega(t)$ be positive, increasing on $(0,\infty)$ functions. The inequality*

$$\|T_\alpha f(\cdot)\|_{L^{pq}_{v(r(\cdot))}(G)} \le c\|f(\cdot)\|_{L^r_{\omega(r(\cdot))}}$$

holds if and only if the condition

$$\sup_{t>0} \|r^{\alpha-Q}(\cdot))\chi_{\{r(y)>t\}}(\cdot)\|_{L^{pq}_{v(r(\cdot)}(G)} \Big\|\frac{1}{\omega(r(\cdot))}\chi_{\{r(y)<t\}}(\cdot)\Big\|_{L^{r'}_{\omega(r(\cdot))}} < \infty$$

is fulfilled.

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