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TWO-WEIGHT INEQUALITIES FOR SINGULAR INTEGRALS ON THE LORENTZ SPACES DEFINED ON HOMOGENEOUS GROUPS

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In the present note, in some particular cases we have found the optimal sufficient conditions for pairs of weights ensuring the validity of two-weight inequalities for singular integrals on the Lorentz spaces defined on homogeneous groups.

In what follows we shall need the notions of a homogeneous group G and weighted Lorentz spaces $L^{pq}_\omega(G)$ defined on this groups. All these notions the reader can find in the previous report.

Let $k : G \to \mathbb{R}$ be a measurable function such that:

(1) $|k(x)| \le \frac{c}{r(x)^Q}$ for an arbitrary element $x \neq e$;

(2) there exists a constant $c_1 > 0$ such that for arbitrary $x, y \in G$, provided $r(xy^{-1}) < \frac{1}{2}r(x)$, the following inequality is fulfilled:

$$|k(x) - k(y)| \le c_1 \omega\Big(\frac{r(xy^{-1})}{r(x)}\Big)\frac{1}{r(x)^Q},$$

where $\omega : [0,1] \to \mathbb{R}^1$ is the non-decreasing function such that $\omega(0) = 0$, $\omega(2t) \le c_2\omega(t)$ for any $t > 0$, and $\int_0^1 \frac{\omega(t)}{t}dt < \infty$.

We shall also assume that k, together with the above conditions, satisfies the condition: a singular integral

$$Tf(x) = p \cdot v \cdot \int\limits_G k(xy^{-1})f(y)dy \overset{def}{=} \lim_{\varepsilon\to 0} \int\limits_{G\setminus B(x,\varepsilon)} k(xy^{-1})f(y)dy$$

defines bounded operator in $L^2(G)$, where dy is the Haar measure on G, and $B(x,\varepsilon) = \{y \in G : r(xy^{-1}) < \varepsilon\}$.

Basic Theorem. *Let $1 < p,\ s < \infty$, $1 < q \le \infty$, $s \le p \le q$, $u_1(x)$, $u_2(x)$, $v(x)$ and $w(x)$ be weight functions on G satisfying the following conditions:*

(1) $\sup\limits_{t>0} \big\|u_2(\cdot)r^{-Q}(\cdot)\chi_{\{r(y)>t\}}(\cdot)\big\|_{L^{pq}_v(G)} \Big\|\dfrac{1}{u_1(\cdot)\omega(\cdot)}\chi_{\{r(y)<t\}}(\cdot)\Big\|_{L^{p's'}_\omega(G)} < \infty;$

(2) $\sup\limits_{t>0} \big\|u_2(\cdot)\chi_{\{r(y)<t\}}(\cdot)\big\|_{L^{pq}_v(G)} \Big\|\dfrac{1}{u_1(\cdot)\omega(\cdot)} \cdot r^{-Q}(\cdot)\chi_{\{r(y)>t\}}(\cdot)\Big\|_{L^{p's'}_\omega(G)} < \infty;$

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(3) *there exists a positive constant b such that for any $t > 0$ the inequality*

$$\sup_{\frac{t}{c_0} < r(x) \le 8c_0 t} (u_2(x) v^{\frac{1}{p}}(x)) \le b \inf_{\frac{t}{c_0} < r(x) \le 8c_0 t} \big(u_1(x) \omega^{\frac{1}{p}}(x)\big)$$

holds. Then we have

$$\|u_2(\cdot) Tf(\cdot)\|_{L^{pq}_v(G)} \le c \|u_1(\cdot) f(\cdot)\|_{L^{ps}_\omega(G)}, \tag{1}$$

where the constant c does not depend on f. If for the Hilbert transform inequality (1) *holds, then the conditions* (1) *and* (2) *are fulfilled.*

Corollary 1. *Let $1 < p$, $s < \infty$, $1 < q \le \infty$, $s \le p \le q$, $u_1(t)$, $u_2(t)$, $v(t)$ be positive, increasing on $(0, \infty)$ functions satisfying the condition*

$$\sup_{t>0} \big\|u_2(r(\cdot)) r^{-Q}(\cdot) \chi_{\{r(y)>t\}}(\cdot)\big\|_{L^{pq}_{v(r(\cdot))}(G)} \Big\| \frac{1}{u_1(r(\cdot))} \chi_{\{r(y)<t\}}(\cdot) \Big\|_{L^{p's'}(G)} < \infty.$$

Then there exists a positive constant c such that for an arbitrary f the inequality

$$\|u_2(r(\cdot)) Tf(\cdot)\|_{L^{pq}_{v(r(\cdot))}(G)} \le c \|u_1(r(\cdot)) f(\cdot)\|_{L^{ps}(G)} \tag{2}$$

holds.

Corollary 2. *Let $1 < p$, $s < \infty$, $1 < q \le \infty$, $s \le p \le q$, $u_1(t)$, $u_2(t)$, $v(t)$ be positive, decreasing on $(0, \infty)$ functions satisfying the condition*

$$\sup_{t>0} \big\|u_2(r(\cdot)) \chi_{\{r(y)<t\}}(\cdot)\big\|_{L^{pq}_{v(r(\cdot))}(G)} \Big\| \frac{1}{u_1(r(\cdot))} r^{-Q}(\cdot) \chi_{\{r(y)>t\}}(\cdot) \Big\|_{L^{p's'}(G)} < \infty.$$

Then the inequality (2) *holds.*

Corollary 3. *Let $1 < p < \infty$, $1 < q \le \infty$, $p \le q$, $v(t)$ and $\omega(t)$ be positive, increasing on $(0, \infty)$ functions satisfying the condition*

$$\sup_{t>0} \|r^{-Q}(\cdot) \chi_{\{r(y)>t\}}(\cdot)\|_{L^{pq}_{v(r(\cdot))}(G)} \Big\| \frac{1}{\omega(r(\cdot))} \chi_{\{r(y)<t\}}(\cdot) \Big\|_{L^{p'}_{\omega(r(\cdot))}(G)} < \infty.$$

Then the following inequality holds:

$$\|Tf(\cdot)\|_{L^{pq}_{v(r(\cdot))}(G)} \le c \|f(\cdot)\|_{L^{p}_{\omega(r(\cdot))}(G)},$$

where the constant $c > 0$ does not depend on f.

In the case of monotone weights the similar results for the Hilbert transform on the Lebesgue spaces have been obtained in [1] and those for the singular integrals defined on the Euclidean spaces can be found in [2]. This last result for the singular integrals defined on the Heisenberg groups is generalized in [3]. In some particular cases the optimal conditions for pairs of weights ensuring the validity of two-weight inequalities for Calderon–Zygmund singular integrals are found in [4], while analogous problems for singular integrals defined on homogeneous groups are considered in [5].

Example. *Let* $1 < r < \infty$, $1 < q < \infty$, $r < q$, $v(t) = t^{r-1}$, $\omega(t) = t^{r-1} \ln^{\gamma} \frac{1}{t}$, *where* $r-1 < \gamma < r$, *and* $\gamma = \frac{r}{q} + r - 1$. *Then the following inequality holds:*

$$\|\widetilde{f}(\cdot)\|_{L^{rq}_{v(|\cdot|)}} \leq c\|f(\cdot)\|_{L^{r}_{\omega(|\cdot|)}},$$

where $\widetilde{f}$ *is the conjugate function and the constant* c *does not depend on* f.

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