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TWO-WEIGHT INEQUALITIES FOR THE HARDY OPERATOR ON THE LORENTZ SPACES DEFINED ON HOMOGENEOUS GROUPS

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Let G be a homogeneous group representing a connected, simply connected nilpotent Lie group on whose Lie algebra g is given one-parameter group of extensions $\delta_t = \exp(A \ln t)$, $t > 0$, where A is a diagonalized linear operator on g whose eigenvalues are positive (see, e.g., [1], p. 5).

For the homogeneous group G the mapping $\exp o\delta_t o \exp^{-1}$, $t > 0$, is the automorphism on G which is again denoted by δ_t. The number $Q = trA$ is said to be the homogeneous dimension of G.

On the homogeneous group we introduce the homogeneous norm representing a continuous function $r : G \to [0, \infty)$ which is smooth on $G \backslash \{e\}$ and satisfies the following conditions:

(1) $r(x) = r(x^{-1})$ for any $x \in G$;

(2) $r(\delta_t x) = tr(x)$ for any $t > 0$ and $x \in G$;

(3) $r(x) = 0 \iff x = e$;

(4) there exists a constant $c_0 > 0$ such that for the arbitrary x and y from G the inequality $r(xy^{-1}) \le c_0(r(x) + r(y))$ holds.

Let us fix on the group G the normed Haar measure so that the measure of the unit ball $B(e, 1) = \{y \in G : r(x) < 1\}$ is equal to 1. If $E \subset G$ is a measurable set, then the integral on E taken with respect to the given measure we denote by $\int_E f(x)dx$.

Almost everywhere positive, locally integrable function $\omega : G \to \mathbb{R}^1$ will be called the weight.

Denote by $L^{pq}_\omega(G)$ the Lorentz space with the weight ω which represents the class of all measurable functions $f : G \to \mathbb{R}^1$ for which

$$\|f\|_{L^{pq}_\omega(G)} = \left(q \int_0^\infty \lambda^{q-1} \left(\int_{\{x \in G : |f(x)| > \lambda\}} \omega(y)dy \right)^{\frac{q}{p}} d\lambda \right)^{\frac{1}{q}} < \infty,$$

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where $1 \le p \le \infty$, $1 \le q < \infty$, and

$$\|f\|_{L^{p\infty}_{\omega}(G)} = \sup_{\lambda>0} \lambda \Bigg(\int\limits_{\{x\in G:|f(x)|>\lambda\}} \omega(y)dy \Bigg)^{\frac{1}{p}} < \infty,$$

where $1 \le p < \infty$, and dy is the Haar measure on G.

Let $a(x)$, $b(x)$, $v(x)$, and $\omega(x)$ be the weight functions on G. Denote by H the Hardy's operator defined as:

$$Hf(x) = H_{a,b}f(x) = a(x) \int\limits_{r(y)\le r(x)} f(y)b(y)\omega(y)dy.$$

Define similarly the conjugate operator H^*:

$$H^*f(x) = H^*_{a,b}(x) = b(x) \int\limits_{r(y)\ge r(x)} f(y)a(y)\omega(y)dy.$$

The following theorems are valid.

Theorem 1. *Let $r = s = 1$ or $r = s = \infty$ or $1 < r < \infty$ and $1 < s < \infty$, $p = q = 1$ or $p = q = \infty$ or $1 < p < \infty$ and $1 < q < \infty$, provided $\max(r,s) \le \min(p,q)$. For the validity of the inequality*

$$\|Hf\|_{L^{pq}_v(G)} \le c_1 \|f\|_{L^{rs}_\omega(G)},$$

where the constant c_1 does not depend on f, it is necessary and sufficient to fulfil the condition

$$\sup_{t>0} \big\|a(\cdot)\chi_{\{r(y)>t\}}(\cdot)\big\|_{L^{pq}_v(G)} \big\|b(\cdot)\chi_{\{r(y)<t\}}(\cdot)\big\|_{L^{r's'}_\omega(G)} < \infty$$
$$\Big(r' = \frac{r}{r-1}, \quad s' = \frac{s}{s-1}\Big).$$

Theorem 2. *Let the numbers r, s, p, q satisfy the conditions of Theorem 1. The inequality*

$$\|H^*f\|_{L^{pq}_v(G)} \le c_2 \|f\|_{L^{rs}_\omega(G)},$$

where the constant c_2 does not depend on f, can be fulfilled if and only if

$$\sup_{t>0} \big\|b(\cdot)\chi_{\{r(y)<t\}}(\cdot)\big\|_{L^{pq}_v(G)} \big\|a(\cdot)\chi_{\{r(y)>t\}}(\cdot)\big\|_{L^{r's'}_\omega(G)} < \infty.$$

Criteria for the pairs of weights ensuring the boundedness of classical Hardy's operators $Hf(x) = \int_0^x f(t)dt$, $H^*f(x) = \int_x^\infty f(t)dt$ from L^p_ω to L^q_v have been found by V. Kokilashvili [2], J. Bradley [3], V. Maz'ya [4] for $p < q$ and by B. Mucknhoupt [5], G. Talenti [6] and G. Tomaselli [7] for $1 \le p = q < \infty$, while analogous problems for the Lorentz spaces were studied by E. Sawyer [8], D. Edmunds, P. Gurka and L. Pick [9] (see also [10]).

Corollary. *Let* $1 < r,\ s < \infty$, $1 < p,\ q < \infty$, *provided* $\max(r,s) \leq \min(p,q)$. *If* $c < -\frac{Q}{p}$, $\beta = cr + r(\frac{Q}{r'} + \frac{Q}{p})$, *then the inequality*

$$\left\| r(\cdot)^c \int\limits_{r(y)\leq r(\cdot)} f(y)dy \right\|_{L^{pq}(G)} \leq c_3 \|f\|_{L^{rs}_{r^{\beta}(\cdot)}(G)},$$

holds, where the constant c_3 *does not depend on* f.

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