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METRIC PROPERTIES OF FOURIER SERIES IN WEIGHTED FUNCTION CLASSES

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In the present note we give the necessary and sufficient conditions on Orlicz class generating function φ and weight w for the validity of strong type one-weight inequalities of several kinds for partial sums and Cesaro means in weighted Orlicz class $\varphi_w(L)$. At the same time we study the one-weight problem for maximal functions with respect to the continuous Borel measures on the line. Using these results we obtain the criteria of one-weight strong type Orlicz class inequality for the Abel–Poisson means of Fourier series with respect to the ultraspherical (Gegenbauer) polynomials. Based on these results we investigated the convergence and summability problems of Fourier series in Orlicz weighted classes. In particular we prove that the mean convergence of Fourier series in Orlicz classes is equivalent by the norm to that in the corresponding Orlicz spaces. The criteria of mean convergence and summability are derived.

Now we introduce some definitions required to state our results.

In what follows the symbol Φ will be used to denote the set of all functions $\varphi : R \to R$ which are non-negative, even and increasing on $[0,\infty)$, such that $\varphi(0+) = 0$, $\lim_{u\to\infty} \varphi(u) = \infty$. If w is a weight function on $(-\pi,\pi)$ (positive almost everywhere and summable function), then by $\varphi_w(L)$ we denote a set of measurable functions for which $\int_{-\pi}^{\pi} \varphi(f(x))w(x)dx < \infty$.

A function φ is called quasi-convex if there exist a Young function ω and a constant $c > 1$ such that

$$\omega(x) \le \varphi(x) \le \omega(cx), \quad x \in \mathbb{R}. \tag{1}$$

If the condition (1) is fulfilled for $|x| > |x_0| > 0$, then we say that φ is quasi-convex in a neighborhood of ∞.

To each quasi-convex function φ we can put into accordance its complementary function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(t) = \sup_{s\ge 0}(st - \varphi(s).$$

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By definition, the function φ satisfies Δ_2 condition ($\varphi \in \Delta_2$) if there exist numbers $c > 0$ and $x_0 > 0$ such that

$$\varphi(2x) \le c\varphi(x), \quad \text{when} \quad x > x_0. \tag{2}$$

If (2) is fulfilled for all x, then they say that φ satisfies the global Δ_2 condition.

Suppose that $1 < p < \infty$. By definition, a weight function $w \in A_p$ (B. Muckenhoupt) if

$$\sup_{I\subset[-\pi,\pi]} \Big(\frac{1}{|I|}\int_I w\Big)\Big(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\Big) < \infty,$$

where the supremum is taken over all intervals from $(-\pi, \pi)$.

In the subsequent theorems by $S_n f$ we denote the partial sums of trigonometric Fourier series and by $\sigma_n f$ the Cesaro means.

Now we are ready to formulate main results of our work.

Theorem 1. *Let $\varphi \in \Phi$ and w be a weight function. The following conditions are equivalent:*

(i) *there is $c > 0$ such that the inequalities*

$$\int_{-\pi}^{\pi} \varphi(S_n f(x))w(x)dx < c\int_{-\pi}^{\pi} \varphi(f(x))w(x)dx, \quad n = 1, 2, \dots,$$

hold for every $f \in \varphi_w(L)$;

(ii) *φ satisfies the global Δ_2 condition, φ^α is quasi-convex for some α, $0 < \alpha < 1$, and $w \in A_{p(\varphi)}$, where*

$$\frac{1}{p(\varphi)} = \inf\{\alpha : \alpha > 0, \quad \varphi^\alpha \quad \textit{is quasi-convex}\}. \tag{3}$$

Theorem 2. *Let $\varphi \in \Phi$ and w be a weight function. The following conditions are equivalent:*

(i) *there is $c > 0$ such that the inequalities*

$$\int_{-\pi}^{\pi} \varphi(S_n f(x)w(x))dx < c\int_{-\pi}^{\pi} \varphi(f(x)w(x))dx, \quad n = 1, 2, \dots,$$

hold for every integrable f;

(ii) *φ satisfies the global Δ_2 condition, φ^α is quasi-convex for some α, $0 < \alpha < 1$, and $w^{p(\varphi)} \in A_{p(\varphi)}$, $w^{-p(\widetilde{\varphi})} \in A_{p(\widetilde{\varphi})}$.*

Theorem 3. *Let $\varphi \in \Phi$ and w be a weight function. The following conditions are equivalent:*

(i) *there is $c > 0$ such that the inequalities*

$$\int_{-\pi}^{\pi} \varphi\Big(\frac{S_n f(x)}{w(x)}\Big)w(x)dx < c\int_{-\pi}^{\pi} \varphi\Big(\frac{f(x)}{w(x)}\Big)w(x)dx, \quad n = 1, 2, \dots,$$

hold for every integrable f;

(ii) φ *satisfies the global* Δ_2 *condition,* φ^α *is quasi-convex for some* α, $0 < \alpha < 1$, *and* $w \in A_{p(\widetilde{\varphi})}$.

Theorem 4. *Let* $\varphi \in \Phi$ *and* w *be a weight function. The following conditions are equivalent:*

(i) *there is* $c > 0$ *such that for any* $f \in \varphi_w(L)$ *and* $\lambda > 0$ *the following inequalities*

$$\varphi(\lambda) w\{x : |S_n f(x)| > \lambda\} < c \int_{-\pi}^{\pi} \varphi(f(x)) w(x) dx, \quad n = 1, 2, \dots ,$$

hold;

(ii) φ *satisfies the global* Δ_2 *condition,* φ^α *is quasi-convex for some* α, $0 < \alpha < 1$, *and* $w \in A_{p(\varphi)}$.

Using Theorem 1 we obtain a convergence test in weighted Orlicz classes of trigonometric Fourier series.

Theorem 5. *Let us suppose that* $\varphi \in \Phi$, φ *satisfies the* Δ_2 *condition,* φ^α *is quasi-convex in the neighborhood of* ∞ *for some* α, $0 < \alpha < 1$, *and* $w \in A_{p(\varphi)}$. *Then*

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} \varphi\big(f(x) - S_n f(x)\big) w(x) dx = 0. \tag{4}$$

The following theorem is a partially reversion of Theorem 5.

Theorem 6. *If for any* $f \in \varphi_w(L)$ *the equality* (4) *is fulfilled, then* φ *satisfies the* Δ_2 *condition and* φ *is quasi-convex in the neighborhood of* ∞.

The proof of this theorem is based on the following lemma which at the same time generalizes a result of P. Oswald's.

Lemma 1. *Let* E *be some set from* $(-\pi, \pi)$ *with positive Lebesgue measure,* w *be some weight function and suppose that for any* $f \in \varphi_w(L)$, $\operatorname{supp} f \subset E$ *we have*

$$\lim_{n \to \infty} \int_{E} \varphi\big(f(x) - S_n f(x)\big) w(x) dx = 0.$$

Then $\varphi \in \Delta_2$ *and* φ *is quasi-convex in the neighborhood of* ∞.

A behavior of the means of Fourier series corresponding to certain regular summability methods are studied. We have the following criteria.

Theorem 7. *Let $\varphi \in \Phi$ and w be a weight function. The following conditions are equivalent:*

(i) *there is $c > 0$ such that the inequalities*

$$\int_{-\pi}^{\pi} \varphi(\sigma_n f(x)) w(x) dx < c \int_{-\pi}^{\pi} \varphi(cf(x)) w(x) dx, \quad n = 1, 2, \dots,$$

hold for any $f \in \varphi_w(L)$;

(ii) *φ is quasi-convex and $w \in A_{p(\varphi)}$.*

Theorem 8. *Assume that $\lambda > 0$, $f(\theta) \sin^{2\lambda} \theta$ is integrable on $(0, \pi)$, $\sum \alpha_n P_n^\lambda(\cos\theta)$ is the Gegenbauer expansion of f and for $0 \le r < 1$, $f(r, \theta) = \sum \alpha_n r^n P_n^\lambda(\cos\theta)$. Then the following inequalities are equivalent:*

(i) *there is $c > 0$ such that the inequalities*

$$\int_0^{\pi} \varphi(f(r, \theta)) w(\theta) d\theta < c \int_0^{\pi} \varphi(cf(\theta)) w(\theta) d\theta, \quad 0 \le r < 1,$$

hold for any $f \in \varphi_w(L)$ on $(0, \pi)$;

(ii) *φ is quasi-convex and for every subinterval I of $(0, \pi)$,*

$$\Big(\int_I w(\theta) d\theta\Big)\Big(\int_I w(\theta)^{-\frac{1}{p(\varphi)-1}} d\theta\Big)^{p(\varphi)-1} < K\Big(\int_I (\sin\theta)^{2\lambda} d\theta\Big)^{p(\varphi)},$$

where K is independent of I.

Theorem 8 is a consequence of the criterion of one weighted strong type inequality in $\varphi_w(L)$ for the maximal functions with respect to the continuous Borel measure on the line.

Let μ be a non-negative continuous Borel measure ($\mu\{\alpha\} = 0$ for any $\alpha \in \mathbb{R}$). For any $f \in L_{loc}(\mu)$ we define a maximal function

$$M_\mu f(x) = \sup \frac{1}{\mu I} \int_I |f(x)| d\mu(x),$$

where the supremum is taken over all intervals I and the ratio is considered to be zero when $\mu I = 0$.

Theorem 9. *Let $\varphi \in \Phi$ and w be a weight function. The following statements are equivalent:*

(i) *there is $c > 0$ such that*

$$\int_{-\infty}^{\infty} \varphi(Mf(x)) w(x) d\mu(x) < c \int_{-\infty}^{\infty} \varphi(cf(x)) w(x) d\mu(x)$$

for any $f \in \varphi_w(L)$;

(ii) φ^α *is quasi-convex for some* α, $0 < \alpha < 1$, *and* $w \in A_{p(\varphi)}$, *and* $w \in A_{p(\mu)}$ *means that*

$$\sup \Big(\frac{1}{\mu I} \int\limits_I w(x) d\mu(x)\Big) \Big(\frac{1}{\mu I} \int\limits_I (w(x))^{-\frac{1}{p-1}} d\mu(x)\Big)^{p-1} < \infty,$$

where the supremum is taken over all intervals I.

It should be mentioned that Theorems 1–3 are valid for the Fourier– Vilenkin series as well.

Finally we note that L^p variants of the above-mentioned results for trigonometric Fourier series were proved by R.A. Hunt, B. Muckenhoupt and Wheeden [1] and for Fourier–Vilenkin series by Wo-Sang Young [2]. The complete solution of weight problems for maximal functions and singular integrals in Orlicz classes were obtained by A. Gogatishvili and V. Kokilashvili [3], [4].

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