Proceedings of A. Razmadze
Mathematical Institute
Vol. **168** (2015), 79–86

ON DUALLY c-MACKEY SPACES

E. MARTÍN-PEINADOR AND V. TARIELADZE

Abstract. A Hausdorff locally convex topological vector space X is called *a dually c-Mackey space* if its dual space X^* equipped with the topology of uniform convergence on compact subsets of X is a Mackey space. It is shown that a metrizable locally convex space X is a dually c-Mackey space iff X has the Schur property. It follows that a reflexive Banach space X is a dually c-Mackey space iff X is finite-dimensional.

რეზიუმე. ჰაუსდორფის ლოკალურად ამოზნექილ ტოპოლოგიურ ვექტორულ X სივრცეს ვუწოდებთ დუალურად c-მაკის სივრცეს, თუ მისი დუალური X^* სივრცე X-ის კომპაქტებზე თანაბარი კრებადობის ტოპოლოგიის მიმართ არის მაკის სივრცე. ნაჩვენებია რომ მეტრიზებადი ლოკალურად ამოზნექილი X სივრცე დუალურად c-მაკის სივრცეა მაშინ და მხოლოდ მაშინ, როდესაც X-ს გააჩნია შურის თვისება. აქდან როგორც შედეგი მიღებულია, რომ ბანახის რეფლექსური X სივრცე დუალურად c-მაკის სივრცეა მაშინ და მხოლოდ მაშინ, როდესაც X სასრულ განზომილებიანია.

1. Introduction

Let E be a vector space over $\mathbb{R}$ and let τ be a topology on E. Denote by $(E,\tau)^*$ the dual space of the topologized vector space (E,τ), i.e. the set of all τ-continuous linear forms $l : E \to \mathbb{R}$ endowed with its natural vector space structure. A topology η on E is said to be compatible with τ if $(E,\eta)^* = (E,\tau)^*$. For a topologized vector space (E,τ) write:

- $w(\tau)$ for the coarsest topology on E with respect to which all elements $l \in (E,\tau)^*$ are continuous. Then, $w(\tau)$ is a *locally convex vector space* topology on E, compatible with τ. The topology $w(\tau)$ is called the *weak topology* of (E,τ) and it is often denoted by $\sigma((E,\tau),(E,\tau)^*)$ or simply by $\sigma(E,E^*)$ when the topology τ is understood;

2010 *Mathematics Subject Classification.* Primary 46A03, 46A20; Secondary 46B10, 46B20.

Key words and phrases. Normed space, locally convex topology, compatible topology, Mackey topology, Mackey space.

- $m(\tau)$ for the *Mackey topology* of (E, τ), which is the finest *locally convex vector space* topology on E compatible with τ. It is well-known that such a topology exists. This topology is often denoted by $m((E, \tau), (E, \tau)^*)$ or simply by $m(E, E^*)$ when the topology τ is understood.

For a topological vector space (E, τ) let $\sigma((E, \tau)^*, (E, \tau))$ be the topology of point-wise convergence (equivalently, the w^*-topology) on $(E, \tau)^*$.

A costructive description of the Mackey topology is provided by the Mackey-Arens Theorem. Namely, $m(\tau)$ coincides on E with the topology of uniform convergence on *absolutely convex* $\sigma((E, \tau)^*, (E, \tau))$-compact subsets of $(E, \tau)^*$.

A topological vector space (E, τ) is called *a Mackey space* provided $m(\tau) = \tau$.

It seems that the term "a Mackey space" first appeared in [9], where it is also shown that every first countable locally convex topological vector space is a Mackey space [9, Corollary 22.3] (see also, [11, (IV.3.4)]); this term is supported also in [12], where many properties of the class of Mackey spaces are surveyed as well.

The following characterization of Mackey spaces follows from Mackey-Arens Theorem.

Theorem 1.1. *For a locally convex topological vector space* (E, τ) *TFAE:*

(i) (E, τ) *is a Mackey space.*
(ii) *Every absolutely convex* $\sigma((E, \tau)^*, (E, \tau))$*-compact subset of* $(E, \tau)^*$ *is equicontinuous on* (E, τ).

In [13] a locally convex topological vector space (E, τ) is defined to have *the convex strong Mackey property* (CSMP) if absolutely convex $\sigma((E, \tau)^*, (E, \tau))$-*countably compact* subsets of $(E, \tau)^*$ are equicontinuous on (E, τ). Clearly if (E, τ) has (CSMP) then it is a Mackey space (the converse is not true [13, Example 3.7]).

For a completely regular Hausdorff topological (=Tikhonov) space X let $C_c(X)$ be the vector space $C(X)$ of all continuous functions $f : X \to \mathbb{R}$ endowed with the compact-open (=compact-convergence) topology. From [13, Corollary 2.8] it follows in particular that if X is first-countable, then $C_c(X)$ has (CSMP), hence it is a Mackey space too. Note that in general $C_c(X)$ may not be a Mackey space, even if X is a pseudo-compact locally compact space ([13, (modified J. B. Conway's) Example 2.4]).

We call a Hausdorff topological vector space X *a dually c-Mackey space* if its dual space X^* equipped with the topology induced from $C_c(X)$ is a Mackey space (in other words, X is a dually c-Mackey space if the dual space X^* endowed with the compact open topology $c(X^*, X)$ is a Mackey space). In the present paper we give a complete characterization of dually c-Mackey spaces among metrizable locally convex topological vector spaces.

To formulate this result, let us recall that a topologized vector space (E, τ) is said to have *the Schur property* if every $w(\tau)$-convergent sequence of elements of E is τ-convergent.

Theorem 1.2. *For a metrizable locally convex topological vector space X TFAE:*

(i) *X is a dually c-Mackey space.*
(ii) *X has the Schur property.*

From this theorem we get

Corollary 1.3. *For a reflexive Banach space X TFAE:*

(i) *X is a dually c-Mackey space.*
(ii) *X is finite-dimensional.*

Proof. (i) $\Longrightarrow$(ii). From (ii) by Theorem 1.2 we get that X has the Schur property. This and reflexivity of X imply in a standard way that X is finite-dimensional.

(ii) $\Longrightarrow$ (i) is easy. ☐

Corollary 1.4. *Let X be an infinite-dimensional reflexive Banach space. Then the Mackey space $C_c(X)$ has a closed vector subspace which is not a Mackey space (hence, $C_c(X)$ is not a hereditarily Mackey space).*

Proof. Clearly X^* is a closed vector subspace of $C_c(X)$. Since X is an infinite-dimensional reflexive Banach space, X^* with the topology induced from $C_c(X)$ is not a Mackey space, according to the implication (i) $\Longrightarrow$ (ii) of Corollary 1.3. ☐

2. Auxiliary Results

In this section we collect few results needed for the proof of Theorem 1.2. They might be known, but in any case they have an independent interest.

Lemma 2.1. *Let (E, τ) be a first countable topological vector space, $Y \subset E$ a τ-dense vector subspace.*

(a) *For any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of E, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of Y, such that $(y_n - x_n)_{n\in\mathbb{N}}$ converges to zero in (E, τ).*

(b) *If $(Y, \tau|_Y)$ has the Schur property, then (E, τ) has the Schur property too.*

Proof. (a) Fix a fundamental sequence $(V_n)_{n\in\mathbb{N}}$ of τ-neighborhoods of zero such that $V_n \supset V_{n+1}$, $n = 1, 2, \ldots$. Pick $y_n \in (x_n + V_n) \cap Y$ for every $n \in \mathbb{N}$. Clearly: $y_n - x_n \in V_n$, $n = 1, 2, \ldots$, and hence the sequence $(y_n - x_n)_{n\in\mathbb{N}}$ converges to zero in (E, τ).

(b) Fix a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of E, which converges to zero in $(E, w(\tau))$. By (a) we can take a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of Y, such that $(y_n - x_n)_{n\in\mathbb{N}}$ converges to zero in (E,τ). In particular, $(y_n - x_n)_{n\in\mathbb{N}}$ converges to zero in $(E, w(\tau))$. Since the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n - x_n)_{n\in\mathbb{N}}$ converge to zero in $(E, w(\tau))$, $(y_n)_{n\in\mathbb{N}}$ also converges to zero in $(E, w(\tau))$. From this, as $y_n \in Y$, $n = 1, 2, \ldots$ and $w(\tau)|_Y = w(\tau|_Y)$, we get that $(y_n)_{n\in\mathbb{N}}$ converges to zero in $(Y, w(\tau|_Y))$. By the Schur property of Y, $(y_n)_{n\in\mathbb{N}}$ converges to zero in $(Y, \tau|_Y)$ and hence, in (E,τ) too. Finally, since the sequences $(y_n)_{n\in\mathbb{N}}$ and $(y_n - x_n)_{n\in\mathbb{N}}$ converge to zero in (E,τ), $(x_n)_{n\in\mathbb{N}}$ also converges to zero in (E,τ). ☐

Remark 2.2. We do not know whether Lemma 2.1(b) remains true without the first countability assumption.

Proposition 2.3. *Let (E,τ) be a metrizable locally convex topological vector space. TFAE:*

(i) *All $w(\tau)$-compact subsets of E are τ-compact.*
(ii) *(E,τ) has the Schur property.*

Proof. (i) $\Longrightarrow$ (ii) is standard.

(ii) $\Longrightarrow$ (i). Fix a $w(\tau)$-compact subset K of E. According to Shmulian's theorem [6, Theorem 1 (p.207)], the metrizability of (E,τ) implies that K is sequentially $w(\tau)$-compact. This and *(ii)* yield that K is sequentially τ-compact. Leaning again on the metrizability of (E,τ), we obtain that K is τ-compact. ☐

Remark 2.4. In general, for a non-metrizable Mackey space (E,τ) the implication (ii) $\Longrightarrow$ (i) of Proposition 2.3 may fail [14, Example 6 (p. 267)] (see also [5, Example 19.19 (p. 106)]).

Notation 2.5. *For a Hausdorff topological vector space E and a vector subspace $X \subset E$, let $c(E^*, X)$ be the topology on E^* of uniform convergence on compact subsets of E, contained in X.*

Proposition 2.6. *Let E be a metrizable topological vector space, $X \subset E$ a dense vector subspace. Then*

$$c(E^*, X) = c(E^*, E).$$

Proof. Clearly, $c(E^*, X) \leq c(E^*, E)$. To show the reverse inequality $c(E^*, E) \leq c(E^*, X)$, fix a non-empty compact $K \subset E$ and let us find a non-empty compact subset $C \subset X$, such that $C^\circ \subset K^\circ$. According to [8, Theorem 9.4.2 (p. 182)] there exists a (countable) compact $C \subset X$ whose closed convex hull C_1 contains K. From $C_1 \supset K$ we have: $(C_1)^\circ \subset K^\circ$. Since C_1 is the closure of the convex hull of C, we can write: $C^\circ = (C_1)^\circ$. Hence, $C^\circ \subset K^\circ$. ☐

Remark 2.7. Proposition 2.6 for a metrizable *locally convex* topological vector space E coincides with [2, Lemma 17.4 (p.154)]; its analogue remains true when E is a metrizable topological abelian group, $X \subset E$ a dense subgroup and E^* is replaced by the dual group $E^\wedge$ [1, 3] (see also [4]).

The following examples show that the equality $c(E^*, X) = c(E^*, E)$ in Proposition 2.6 may fail for a non-metrizable Hausdorff locally convex topological vector space E.

Example 1 (pointed out to us by Xabier Domínguez)**.** Let I be an uncountably infinite set, $E = \mathbb{R}^I$ endowed with the product topology $\mathcal{P}$ and $E = \mathbb{R}^{(I)}$ (recall that a function $x \in \mathbb{R}^I$ belongs to $\mathbb{R}^{(I)}$ iff the set $\{i \in I : x(i) \neq 0\}$ is finite). Then X is dense in E, but $c(E^*, X) \neq c(E^*, E)$ (see [7, Theorem 4.5]).

Example 2. Let X be an infinite-dimensional reflexive Banach space, $X_\sigma := (X, \sigma(X, X^*))$ and let E be the algebraic dual of X^* equipped with the topology $\mathfrak{p}$ induced from $(\mathbb{R}^{X^*}, \mathcal{P})$. Then X can be identified with a dense vector subspace of E in such a way that $(X, \mathfrak{p}|_X) = X_\sigma$ and $c(E^*, X_\sigma) \neq c(E^*, E)$.

Proof. Let us identify $x \in X$ with the linear form $x^* \mapsto x^*(x)$ defined on X^*. The density of X into E follows from the known fact that $E^* = X^*$ and from Hahn-Banach Theorem. Since the compact subsets of E contained in X are precisely the weakly compact subsets of X and X *is a reflexive Banach space*, we get that $c(E^*, X_\sigma)$ is the Banach-space topology of X^*. Consequently, there are $c(E^*, X_\sigma)$ -convergent sequences in $E^* = X^*$ with ranges not contained in a finite-dimensional subspace of E^*. Since $c(E^*, E)$ is finer than $\sigma(E^*, E) = \sigma(X^*, E)$, the range of each $c(E^*, E)$ -convergent sequence in $E^* = X^*$ *is contained in a finite-dimensional subspace of* E^*. Consequently, $c(E^*, X_\sigma) \neq c(E^*, E)$. □

For a Hausdorff locally convex topological vector space E we denote by $m(E^*, E)$ the topology on E^* of uniform convergence on weakly compact *absolutely convex* subsets of E. (Note that, as $(E^*, \sigma(E^*, E))^*$ can be naturally identified with E, the topology $m(E^*, E)$ is the Mackey topology of the locally convex topological vector space $(E^*, \sigma(E^*, E))$.)

A topological vector space E is said to have *the convex compactness property* (for short, *the ccp*) if the closure of the absolutely convex hull of every compact subset of E is again compact [10].

Proposition 2.8. *For a Hausdorff locally convex topological vector space E the following statements are equivalent:*

(i) *E has the ccp.*
(ii) $c(E^*, E) \leq m(E^*, E)$.

Proof. (i) $\Longrightarrow$ (ii). Fix a non-empty compact $K \subset E$ and let us find a non-empty weakly compact convex $C \subset E$, such that $C^\circ \subset K^\circ$. Let C be the closure in E of the convex hull of K. By (i), C is compact in E and in particular, it is also weakly compact. Since C is the closure of the convex hull of K, we can write: $C^\circ = K^\circ$.

(ii) $\Longrightarrow$ (i). Fix a non-empty compact $K \subset E$ and let K_1 be the closure in E of the absolutely convex hull of K. We need to show that K_1 is compact in E. Since K° is a $c(E^*, E)$-neighborhood of zero and (ii) is satisfied, we can find a non-empty weakly compact absolutely convex $C \subset E$, such that $C^\circ \subset K^\circ = (K_1)^\circ$. From this by the bi-polar theorem we get: $K_1 \subset C$. This inclusion, as K_1 is weakly closed too, implies that K_1 is weakly compact. In particular, the set K_1 is $\sigma(E, E^*)$-*complete* and using again the local convexity of E, we get that K_1 *is a complete subset of* E. On the other hand K_1 is precompact in E, as it is the closed absolutely convex hull of a compact subset of E ([6, Proposition 35 (p.90)]). Thus, K_1 is compact in E. □

Corollary 2.9. *Let E be a Hausdorff locally convex topological vector space. If $(E, \sigma(E, E^*))$ has the ccp, then E also has the ccp.*

Proof. Let $E_\sigma = (E, \sigma(E, E^*))$. Clearly, $(E_\sigma)^* = E^*$. As E_σ has the ccp, applying (i) $\Longrightarrow$ (ii) of Proposition 2.8 to E_σ gives: $c(E^*, E_\sigma) \leq m(E^*, E_\sigma) = m(E^*, E)$. From $c(E^*, E) \leq c(E^*, E_\sigma)$, it follows that $c(E^*, E) \leq m(E^*, E)$. Now, according to (ii) $\Longrightarrow$ (i) in the same Proposition, we obtain that E has the ccp. □

Remark 2.10. The above given proof of Corollary 2.9 was pointed out to us by Xabier Domínguez.

The reverse implication in Corollary 2.9 does not hold in general: a Mackey space E may have the ccp, but $(E, \sigma(E, E^*))$ may not have it [14, (16.4.10)].

3. Additional Results and Proof of Theorem 1.2

Proposition 3.1. *For a Hausdorff locally convex topological vector space E for which $(E, \sigma(E, E^*))$ has the ccp TFAE:*

(i) *E is dually c-Mackey.*
(ii) *All $\sigma(E, E^*)$-compact subsets of E are compact in E.*
(iii) *$c(E^*, E) = m(E^*, E)$.*

Proof. (i) $\Longrightarrow$ (ii). Fix a $\sigma(E, E^*)$-compact subset K of E and let us show that it is compact in E. *Since $(E, \sigma(E, E^*))$ has the ccp, we can suppose that K is convex.* Write $E_c^* = (E^*, c(E^*, E))$. We can view K as a $\sigma((E_c^*)^*, E_c^*)$-compact convex subset of $(E_c^*)^*$. By (i) E_c^* is a Mackey space, and according Theorem 1.1 we can conclude that K is equi-continuous on E_c^*. So, we can find and fix a compact $C \subset E$ such that $K \subset {}^\circ(C^\circ)$. Taking into account Corollary 2.9, we can suppose that C is absolutely convex. So, by the bipolar theorem, $C = {}^\circ(C^\circ)$ and K, as a closed subset of the compact set C is itself compact in E.

(ii) $\Longrightarrow$ (iii). The inequality $c(E^*, E) \leq m(E^*, E)$ holds by Proposition 2.8. The inequality $m(E^*, E) \leq c(E^*, E)$ follows directly from (ii).

(iii) $\Longrightarrow$ (i). By Mackey-Arens theorem $(E^*, m(E^*, E))$ is always a Mackey space, hence (iii) gives that $E_c^* = (E^*, c(E^*, E))$ is a Mackey space. Consequently, E is a dually c-Mackey space. □

Theorem 3.2. *For a complete metrizable locally convex topological vector space E TFAE:*

(i) *E is a dually c-Mackey space.*
(ii) *E has the Schur property.*
(iii) $c(E^*, E) = m(E^*, E)$.

Proof. In the considered case, according to Krein's theorem [6, Theorem 4 (p. 211)], $(E, \sigma(E, E^*))$ has the ccp. So, the result follows from Propositions 2.3 and 3.1. □

Proof of Theorem 1.2. If X is complete, then the result is true by Theorem 3.2. Otherwise, denote by E the completion of X. The vector space E^* can be identified with the vector space X^*. By Proposition 2.6 we have: $c(E^*, X) = c(E^*, E)$.

(i) $\Longrightarrow$ (ii). By (i) we have that $(X^*, c(X^*, X)) = (E^*, c(E^*, X))$ is a Mackey space. Hence $(E^*, c(E^*, E))$ is a Mackey space. So, by the implication (i) $\Longrightarrow$ (ii) of Theorem 3.2, E has the Schur property. Clearly, its subspace X has this property too.

(ii) $\Longrightarrow$ (i). By Lemma 2.1 (b), E has the Schur property and the implication (ii) $\Longrightarrow$ (i) of Theorem 3.2 applies to prove that $(E^*, c(E^*, E))$ is a Mackey space. Since $(E^*, c(E^*, E)) = (E^*, c(E^*, X)) = (X^*, c(X^*, X))$, we obtain that $(X^*, c(X^*, X))$ is a Mackey space. Consequently, X is a dually c-Mackey space. □

Remark 3.3. The implication (ii) $\Longrightarrow$ (iii) of Theorem 3.2 may fail for a non-complete space E. If fact, let E be a dense countable-dimensional vector subspace of the Banach space $(l_1, \|\cdot\|_1)$. Then E has the Schur property (as $(l_1, \|\cdot\|_1)$ has it). However from dim(E) $= \aleph_0$ it is not hard to derive that $m(E^*, E) = \sigma(E^*, E) \neq c(E^*, E)$.

Acknowledgements

The authors are grateful to Xabier Domínguez for careful reading of this note and for several useful suggestions.

This paper was initiated during the second author's stay at Interdisciplinary Mathematical Institute (IMI) of Complutense University of Madrid in October 2014. He is grateful to the Georgian Technical University (Tbilisi, Georgia) for covering travel expenses and to the IMI for lodging and nice conditions of work.

The first author was partially supported by Spanish Ministerio de Economía y Competitividad. Grant MTM 2013-42486-P.

The second author was supported by Shota Rustaveli National Science Foundation grant No. FR/539/5-100/13.

References

1. L. Aussenhofer, Contributions to the duality theory of abelian topological groups and to the theory of nuclear groups. *Dissertationes Math.* (*Rozprawy Mat.*) **384** (1999).
2. W. Banaszczyk, Additive subgroups of topological vector spaces. *Lecture Notes in Mathematics* **1466** (1991).
3. M. J. Chasco, Pontryagin duality for metrizable groups. *Arch. Math.* (*Basel*) **70** (1998), No. 1, 22–28.
4. W. W. Comfort, S. U. Raczkowski and F. J. Trigos-Arrieta, The dual group of a dense subgroup. Czechoslovak Math. J. **54** (2004), No. 2, 509–533.
5. X. Domínguez Pérez, Grupos abelianos topológicos y sumabilidad. *Diss. Universidad Complutense de Madrid, Servicio de Publicaciones*, 2004.
6. A. Grothendieck, Topological vector spaces. *Taylor-Francis*, 1973.
7. S. Hernández, S. Macario and F. J. Trigos-Arrieta, Uncountable products of determined groups need not to be determined. *J. Math. Alal. Appl.* **348** (2008), No. 2, 834–842.
8. H. Jarchow, Locally convex spaces. *Mathematical Textbooks, B. G. Teubner, Stuttgart*, 1981.
9. J. L. Kelley, I. Namioka, W. F. Donoghue Jr, K. R. Lucas, B. J. Pettis, E. T. Poulsen, G. B. Price, W. Robertson, W. R. Scott and K. T. Smith, Linear topological spaces. *Springer Berlin Heidelberg*, 1963; *Second corrected printing*, 1976.
10. E. G. Ostling and A. Wilansky, Locally convex topologies and the convex compactness property. *Proc. Cambridge Philos. Soc.* **75** (1974), 45–50.
11. H. A. Schaefer, Topological vector spaces. *The Macmillan Co., New York, Collier-Macmillan Ltd., London*, 1966.
12. P. Pérez Carreras and J. Bonet, Barrelled locally convex spaces. *Elsevier*, 1987.
13. R. F. Wheeler, The Mackey problem for the compact-open topology. *Trans. Amer. Math. Soc.* **222** (1976), 255–265.
14. A. Wilansky, Modern methods in topological vector spaces. *Courier Corporation*, 2013.

(Received 9.04.2015)

Authors' addresses:

E. Martín-Peinador
Instituto de Matemática Interdisciplinar y Departamento
de Geometría y Topología, Universidad Complutense de Madrid
28040 Madrid, Spain
E-mail: em_peinador@mat.ucm.es

V. Tarieladze
Niko Muskhelishvili Institute of Computational Mathematics
of the Georgian Technical University, 0160 Tbilisi, Georgia
E-mail: vajatarieladze@yahoo.com