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SOME REMARKS ON UNCONDITIONAL CONVERGENCE OF SERIES IN BANACH SPACES

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Abstract. In this paper a sufficient and a necessary condition for unconditional convergence of a series in a Banach space with an unconditional basis are analyzed.

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1. Introduction

The first draft of this paper was prepared in collaboration with Professor Nicholas Vakhania (1930–2014). N. Vakhania was always deeply interested in questions of unconditional convergence and his challenging ideas were helping us to understand the topic better.

In what follows X will stand for an infinite-dimensional real Banach space with the norm $\|\cdot\|_X$ and X^* will be the continuous dual space of X with the (dual) norm $\|\cdot\|_{X^*}$.

A sequence $(\varphi_i)_{i\in\mathbb{N}}$ of elements of X is a (Schauder) basis of X if for each $x \in X$ there exists a unique sequence $(t_i)_{i\in\mathbb{N}}$ of real numbers such that

$$\lim_n \|x - \sum_{i=1}^{n} t_i\varphi_i\|_X = 0\,.$$

A basis $(\varphi_i)_{i\in\mathbb{N}}$ has its biorthogonal sequence $(\varphi_j^*)_{j\in\mathbb{N}}$, i.e. a sequence $\varphi_j^* \in X^*$, $j = 1, 2, \ldots$, such that $\langle\varphi_j^*, \varphi_i\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker symbol.

We will use the following known statement (cf. [1, Proposition V.4.2 (p. 302)] and [1, Theorem II.3.3 (p. 118)]):

Proposition 1.1. *For a sequence (a_k) of elements of a Banach space X the following statements are equivalent:*

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(i) *The series* $\sum\limits_{k=1}^{\infty} a_{\pi(k)}$ *converges for every permutation* π *of the positive integers.*

(ii) *The series* $\sum\limits_{k=1}^{\infty} \vartheta_k a_k$ *converges for every choice of signs* ϑ_k *(i.e.* $\vartheta_k = \pm 1$*).*

(iii) *The sequence* (a_k) *is weakly absolutely summable , i.e.,*

$$\sum_{k=1}^{\infty} |\langle x^*, a_k\rangle| < \infty \qquad \textit{for every} \qquad x^* \in X^*, \tag{1}$$

and

$$\lim_{n\to\infty} \sup_{\|x^*\|_{X^*}\le 1} \sum_{k=n}^{\infty} |\langle x^*, a_k\rangle| = 0\,. \tag{2}$$

It is known that if X does not contain a subspace isomorphic to c_0, then in Proposition 1.1 (iii) the second condition follows from the first one [2].

A series $\sum\limits_{k=1}^{\infty} a_k$ which satisfies one, and thus all of the conditions of Proposition 1.1, is said to be unconditionally convergent.

A basis $(\varphi_i)_{i\in\mathbb{N}}$ of X is called unconditional if for every $x \in X$ the series $\sum\limits_{i=1}^{\infty} \langle \varphi_i^*, x\rangle \varphi_i$ converges in X unconditionally.

We need the following statement [3, p. 19]: for an unconditional basis $(\varphi_i)_{i\in\mathbb{N}}$ of X there exists a finite constant $K > 0$ such that for every choice of scalars (α_i) for which the series $\sum\limits_{i=1}^{\infty} \alpha_i\varphi_i$ converges, and for every choice of a bounded sequence (β_i) of scalars (the series $\sum\limits_{i=1}^{\infty} \beta_i\alpha_i\varphi_i$ converges in X and) the inequality

$$\Big\| \sum_{i=1}^{\infty} \beta_i\alpha_i\varphi_i \Big\|_X \le K \sup_{i\ge 1} |\beta_i| \Big\| \sum_{i=1}^{\infty} \alpha_i\varphi_i \Big\|_X \tag{3}$$

holds.

Definition 1.2. Let $0 < s < +\infty$. We say that a sequence $(a_k)_{k\in\mathbb{N}}$ of elements of a Banach space X with a basis (φ_i) *satisfies the condition* V_s if

$$\sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k\rangle|^s < \infty \quad \forall i \in \mathbb{N} \tag{4}$$

and

$$\text{the series } \sum_{i=1}^{\infty} \Big(\sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k\rangle|^s \Big)^{1/s} \varphi_i \text{ converges in } X\,. \tag{5}$$

Note that if $0 < s_1 < s_2 < +\infty$ and a sequence $(a_k)_{k\in\mathbb{N}}$ of elements of a Banach space X with an *unconditional* basis (φ_i) satisfies the condition V_{s_1}, then it satisfies the condition V_{s_2} too.

2. Results

In this section we shall show that V_1 is a sufficient condition and V_2 is a necessary condition for the unconditional convergence of a series in Banach spaces with *unconditional basis* and the requirements under which this result is proved are analyzed.

Theorem 2.1. *Let X be a Banach space with an unconditional basis (φ_i) and (φ_j^*) be its biorthogonal sequence. For a sequence $(a_k)_{k\in\mathbb{N}}$ of elements of X consider the conditions:*

(a) *The sequence $(a_k)_{k\in\mathbb{N}}$ satisfies the condition V_1.*

(b) *The series $\sum\limits_{k=1}^{\infty} a_k$ converges unconditionally.*

(c) *The series $\sum\limits_{k=1}^{\infty} a_k$ converges almost unconditionally (i.e. the series $\sum\limits_{k=1}^{\infty} a_k \mathfrak{r}_k(\omega)$ converges in X for λ-almost every $\omega \in [0,1]$, where λ is the Lebesgue measure and $(\mathfrak{r}_k)$ is the sequence of Rademacher functions).*

(d) *The sequence $(a_k)_{k\in\mathbb{N}}$ satisfies the condition V_2.*
Then the following implications are true: $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d)$.

Proof. $(a) \Longrightarrow (b)$. It is sufficient to derive from (a) that the sequence (a_k) satisfies the conitions (1) and (2) of Proposition 1.1. Since by (a) V_1 is satisfied, we have

$$t_i = \sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k\rangle| < \infty, \quad i = 1, 2, \dots$$

and the series $\sum\limits_{i=1}^{\infty} t_i\varphi_i$ converges unconditionally. So, by Proposition 1.1 we can write:

$$\sum_{i=1}^{\infty} t_i |\langle x^*, \varphi_i\rangle| < \infty \qquad \text{for every} \qquad x^* \in X^*, \tag{6}$$

and

$$\lim_{j\to\infty} \sup_{\|x^*\|_{X^*}\le 1} \sum_{i=j}^{\infty} t_i |\langle x^*, \varphi_i\rangle| = 0\,. \tag{7}$$

For a fixed $k \in \mathbb{N}$ we can express a_k as the convergent in X series

$$a_k = \sum_{i=1}^{\infty} \langle \varphi_i^*, a_k \rangle \varphi_i \,,$$

so, for $x^* \in X^*$,

$$\langle x^*, a_k \rangle = \sum_{i=1}^{\infty} \langle \varphi_i^*, a_k \rangle \langle x^*, \varphi_i \rangle \,, \quad |\langle x^*, a_k \rangle| \le \sum_{i=1}^{\infty} |\langle \varphi_i^*, a_k \rangle||\langle x^*, \varphi_i \rangle|$$

and

$$\sum_{k=n}^{\infty} |\langle x^*, a_k \rangle| \le \sum_{i=1}^{\infty} \left(\sum_{k=n}^{\infty} |\langle \varphi_i^*, a_k \rangle| \right) |\langle x^*, \varphi_i \rangle|, \quad n = 1, 2, \dots .$$

From this inequality and (6) we get

$$\sum_{k=1}^{\infty} |\langle x^*, a_k \rangle| \le \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k \rangle| \right) |\langle x^*, \varphi_i \rangle| = \sum_{i=1}^{\infty} t_i |\langle x^*, \varphi_i \rangle| < \infty$$

for every $x^* \in X^*$. Consequently the sequence (a_k) satisfies conditions (1). To verify conditions (2) fix $\varepsilon > 0$. Using (7) we can find $j_\varepsilon \in \mathbb{N}$ such that

$$\sup_{\|x^*\|_{X^*} \le 1} \sum_{i=j_\varepsilon}^{\infty} t_i |\langle x^*, \varphi_i \rangle| < \frac{\varepsilon}{2} \,. \tag{8}$$

Since $t_i < \infty$, $i = 1, \dots, j_\varepsilon$, we have:

$$\sum_{k=1}^{\infty} \left(\sum_{i=1}^{j_\varepsilon} |\langle \varphi_i^*, a_k \rangle| \|\varphi_i\|_X \right) = \sum_{i=1}^{j_\varepsilon} t_i \|\varphi_i\|_X < \infty \,.$$

So we can find $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=n_\varepsilon}^{\infty} \left(\sum_{i=1}^{j_\varepsilon} |\langle \varphi_i^*, a_k \rangle| \|\varphi_i\|_X \right) < \frac{\varepsilon}{2} \,. \tag{9}$$

Fix then $x^* \in X^*$ with $\|x^*\|_{X^*} \le 1$ and $n \in \mathbb{N}$ with $n > n_\varepsilon$; clearly,

$$\sum_{k=n}^{\infty} |\langle x^*, a_k \rangle| \le \sum_{i=1}^{\infty} \left(\sum_{k=n}^{\infty} |\langle \varphi_i^*, a_k \rangle| \right) |\langle x^*, \varphi_i \rangle| =$$

$$= \sum_{i=1}^{j_\varepsilon} \left(\sum_{k=n}^{\infty} |\langle \varphi_i^*, a_k \rangle| \right) |\langle x^*, \varphi_i \rangle| + \sum_{i=j_\varepsilon+1}^{\infty} \left(\sum_{k=n}^{\infty} |\langle \varphi_i^*, a_k \rangle| \right) |\langle x^*, \varphi_i \rangle| =$$

$$= \sum_{k=n}^{\infty} \left(\sum_{i=1}^{j_\varepsilon} |\langle \varphi_i^*, a_k \rangle||\langle x^*, \varphi_i \rangle| \right) + \sum_{i=j_\varepsilon+1}^{\infty} \left(\sum_{k=n}^{\infty} |\langle \varphi_i^*, a_k \rangle| \right) |\langle x^*, \varphi_i \rangle| \le$$

$$\le \sum_{k=n}^{\infty} \left(\sum_{i=1}^{j_\varepsilon} |\langle \varphi_i^*, a_k \rangle| \|\varphi_i\|_X \right) + \sum_{i=j_\varepsilon}^{\infty} t_i |\langle x^*, \varphi_i \rangle| \le$$

$$\leq \sum_{k=n_\varepsilon}^{\infty} \left(\sum_{i=1}^{j_\varepsilon} |\langle \varphi_i^*, a_k \rangle| \|\varphi_i\|_X \right) + \sum_{i=j_\varepsilon}^{\infty} t_i |\langle x^*, \varphi_i \rangle| \,,$$

i.e.,

$$\sum_{k=n}^{\infty} |\langle x^*, a_k \rangle| \leq \sum_{i=1}^{\infty} \left(\sum_{k=n}^{\infty} |\langle \varphi_i^*, a_k \rangle| \right) |\langle x^*, \varphi_i \rangle| \leq$$

$$\leq \sum_{k=n_\varepsilon}^{\infty} \left(\sum_{i=1}^{j_\varepsilon} |\langle \varphi_i^*, a_k \rangle| \|\varphi_i\|_X \right) + \sum_{i=j_\varepsilon}^{\infty} t_i |\langle x^*, \varphi_i \rangle| \,.$$

From this inequality, after taking into account (9) and (8), we obtain for every $n \in \mathbb{N}$ with $n > n_\varepsilon$

$$\sup_{\|x^*\|_{X^*} \leq 1} \sum_{k=n}^{\infty} |\langle x^*, a_k \rangle| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \,.$$

Consequently, the sequence (a_k) satisfies conditions (2) too and (a)$\Longrightarrow$ (b) is proved.

(b)$\Longrightarrow$ (c) follows from Proposition 1.1.

(c)$\Longrightarrow$ (d). Let us recall first that for a sequence (t_k) of real numbers the series $\sum_{k=1}^{\infty} t_k$ converges almost unconditionally if and only if $\sum_{k=1}^{\infty} t_k^2 < \infty$ and the following Khinchin-Szarek inequality holds:

$$\left(\sum_{k=1}^{\infty} t_k^2 \right)^{\frac{1}{2}} = \left(\int_0^1 \left| \sum_{k=1}^{\infty} t_k \mathfrak{r}_k \right|^2 d\lambda \right)^{\frac{1}{2}} \leq$$

$$\leq \sqrt{2} \int_0^1 \left| \sum_{k=1}^{\infty} t_k \mathfrak{r}_k \right| d\lambda \quad \forall (t_k) \in l_2 \,. \tag{10}$$

Since the series $\sum_{k=1}^{\infty} a_k$ converges in X almost unconditionally, we can introduce a measurable mapping $\mathfrak{a} : [0,1] \to X$ such that

$$\mathfrak{a} = \sum_{k=1}^{\infty} a_k \mathfrak{r}_k \ \lambda \text{ a.e.}$$

Clearly,

$$t_i := \left(\sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k \rangle|^2 \right)^{\frac{1}{2}} = \left(\int_0^1 |\langle \varphi_i^*, \mathfrak{a} \rangle|^2 d\lambda \right)^{\frac{1}{2}} < \infty, \quad i = 1, 2, \ldots \,.$$

Hence the sequence (a_k) satisfies (4) for $s = 2$. It remains to show that for our sequence (a_k) the condition (5) for $s = 2$ is satisfied as well. For this we will use Kahane's integrability theorem ([1, Corollary 2(b) of Theorem V. 3. 1 (p. 287)]; see also [6]):

$$\int_0^1 \|\mathfrak{a}\|_X d\lambda < \infty \,. \tag{11}$$

Fix $m, n \in \mathbb{N}$, $n < m$. Using (3) we can write:

$$\Big\| \sum_{i=n}^{m} |\langle \varphi_i^*, \mathfrak{a}\rangle| \varphi_i \Big\|_X \le K \Big\| \sum_{i=1}^{\infty} \langle \varphi_i^*, \mathfrak{a}\rangle \varphi_i \Big\|_X = K \|\mathfrak{a}\|_X \,. \tag{12}$$

Since

$$t_i \le \sqrt{2} \int_0^1 |\langle \varphi_i^*, \mathfrak{a}\rangle| d\lambda \,, \quad i = 1, 2, \dots$$

we can use again (3) and write:

$$\Big\| \sum_{i=n}^{m} t_i \varphi_i \Big\|_X \le \sqrt{2} K \Big\| \sum_{i=n}^{m} \Big(\int_0^1 |\langle \varphi_i^*, \mathfrak{a}\rangle| d\lambda \Big) \varphi_i \Big\|_X \,.$$

As

$$\Big\| \sum_{i=n}^{m} \Big(\int_0^1 |\langle \varphi_i^*, \mathfrak{a}\rangle| d\lambda \Big) \varphi_i \Big\|_X \le \int_0^1 \Big\| \sum_{i=n}^{m} |\langle \varphi_i^*, \mathfrak{a}\rangle| \varphi_i \Big\|_X d\lambda \,,$$

we obtain

$$\Big\| \sum_{i=n}^{m} t_i \varphi_i \Big\|_X \le \sqrt{2} K \int_0^1 \Big\| \sum_{i=n}^{m} |\langle \varphi_i^*, \mathfrak{a}\rangle| \varphi_i \Big\|_X d\lambda \,. \tag{13}$$

Now since

$$\lim_{n,m} \Big\| \sum_{i=n}^{m} |\langle \varphi_i^*, \mathfrak{a}\rangle| \varphi_i \Big\|_X = 0 \,,$$

the inequality (12) together with (11) allows to apply Lebesgue's dominated convergence theorem, and we get:

$$\lim_{n,m} \int_0^1 \Big\| \sum_{i=n}^{m} |\langle \varphi_i^*, \mathfrak{a}\rangle| \varphi_i \Big\|_X d\lambda = 0 \,.$$

The last relation and (13) imply:

$$\lim_{n,m} \Big\| \sum_{i=n}^{m} t_i \varphi_i \Big\|_X = 0 \,.$$

Hence the series $\sum_{i=1}^{\infty} t_i \varphi_i$ converges in X and $(c) \Longrightarrow (d)$ is proved. □

Let's make several comments about Proposition 2.1.

Remark 2.2. During this remark X will stand for a Banach space with an unconditional basis (φ_i).

(I.1) Without the assumption of the unconditionality of the basis (φ_i) the implication $(a) \Longrightarrow (b)$ of Theorem 2.1 may fail; moreover the condition (a) may not imply even ordinary (conditional) convergence of the series $\sum_{k=1}^{\infty} a_k$. In fact, let $X = c_0$ and (e_i) be the unit vector basis of it. Let

$$\varphi_i = (-1)^i (e_1 + \cdots + e_i), \quad i = 1, 2, \ldots .$$

It is known that the sequence (φ_i) forms a (conditional) basis of X with biorthogonal sequence (φ_j^*), where $\varphi_j^* = (-1)^j (e_j^* - e_{j+1}^*)$, $j = 1, 2, \ldots$. Note that for any integer $N > 0$ and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_N$ the following equality holds:

$$\sum_{i=1}^{N} \alpha_i \varphi_i = \sum_{i=1}^{N} \left(\sum_{k=i}^{N} (-1)^k \alpha_k \right) e_i \,. \tag{14}$$

Now for a fixed $k \in \mathbb{N}$ take $\alpha_{2k} = -\alpha_{2k-1} = \frac{1}{k}$ and $a_k = \alpha_k \varphi_k$, $k = 1, 2, \ldots$. Using equality (14) it is easy to show, that for the sequence (a_k) the condition (a) is satisfied, but the series $\sum_{k=1}^{\infty} a_k$ does not converge.

(I.2) For X the implication (b)$\Longrightarrow$ (a) holds if and only if X is isomorphic to c_0 [4].

(II) The implication (c)$\Longrightarrow$ (b) is not true. As we have already mentioned for a sequence (t_k) of real numbers the series $\sum_{k=1}^{\infty} t_k$ converges almost unconditionally if and only if $\sum_{k=1}^{\infty} t_k^2 < \infty$.

(III. 1) From (c) it follows that together with (d) the following inequality is satisfied:

$$\left\| \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k \rangle|^2 \right)^{1/2} \varphi_i \right\|_X \le \sqrt{2} K \int_0^1 \left\| \sum_{k=1}^{\infty} a_k \mathfrak{r}_k \right\|_X d\lambda \,. \tag{15}$$

This follows from (13). An analogue of (13) for Banach lattices was obtained by B. Maurey [7, Lemma 5 (p.14)].

(III.2) If X does not contain l_∞^n's uniformly, then for it implication (d)$\Longrightarrow$ (c) holds.

This follows from implication $(1^\circ) \Longrightarrow (3^\circ)$ of [5, Corollary 2.2].

(III.3) If for X implication (d)$\Longrightarrow$ (c) holds, then X does not contain l_∞^n's uniformly.

This can be derived from implication (d)$\Longrightarrow$ (a) of [7, Theoreme 4 (p. 21)] and [7, Corollaire 1 (p.22)]. The validity of (III.3) can be established also

by using of the following steps.
(III.3.1) If for X implication (d)$\Longrightarrow$ (c) holds, then there exists a finite constant $C_X > 0$ such that for every sequence (a_k) satisfying (d) the following inequality is valid:

$$\int_0^1 \left\| \sum_{k=1}^{\infty} a_k \mathfrak{r}_k \right\|_X d\lambda \le C_X \left\| \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k \rangle|^2 \right)^{1/2} \varphi_i \right\|_X . \tag{16}$$

This can be established by using the closed graph theorem arguments.
(III.3.2) If for X there exists a finite constant $C_X > 0$ such that for every $N \in \mathbb{N}$ every finite sequence a_j, $j = 1, \dots, N$ of elements of X the inequality

$$\int_0^1 \| \sum_{j=1}^{N} a_j \mathfrak{r}_j \|_X d\lambda \le C_X \| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{N} |\langle \varphi_i^*, a_j \rangle|^2 \right)^{1/2} \varphi_i \|_X \tag{17}$$

holds and $(\mathfrak{g}_k)_{k\in\mathbb{N}}$ is a sequence of stochastically independent standard Gaussian random variables given on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then for X there exists a finite constant $C'_X > 0$ such that for every $n \in \mathbb{N}$ every finite sequence $x_k, k = 1, \dots, n$ of elements of X the inequality

$$\mathbb{E} \| \sum_{k=1}^{n} x_k \mathfrak{g}_k \|_X \le C'_X \| \sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} |\langle \varphi_i^*, x_k \rangle|^2 \right)^{1/2} \varphi_i \|_X . \tag{18}$$

holds as well.

Proof of (III.3.2). Fix $n \in \mathbb{N}$ and $x_k \in X$, $k = 1, \dots, n$. Fix moreover $m \in \mathbb{N}$, $m > 1$ and put

$$\xi_{m,k} = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \mathfrak{r}_{(k-1)m+i}, \quad k = 1, 2, \dots$$

$$a_{(k-1)m+i} = \frac{1}{\sqrt{m}} x_k, \quad i = 1, \dots, m, \quad k = 1, \dots, n .$$

Observe that

$$\sum_{k=1}^{n} x_k \xi_{m,k} = \sum_{j=1}^{nm} a_j \mathfrak{r}_j .$$

So,

$$\int_0^1 \left\| \sum_{k=1}^{n} x_k \xi_{m,k} \right\|_X d\lambda = \int_0^1 \left\| \sum_{i=1}^{nm} a_i \mathfrak{r}_i \right\|_X d\lambda \tag{19}$$

From (17) applied to $N = nm$ we get:

$$\int_0^1 \left\| \sum_{i=1}^{nm} a_i \mathfrak{r}_i \right\|_X d\lambda \le C_X \left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{nm} |\langle \varphi_i^*, a_j \rangle|^2 \right)^{1/2} \varphi_i \right\|_X \tag{20}$$

Observe now that

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{nm}|\langle\varphi_i^*,a_j\rangle|^2\right)^{1/2}\varphi_i=\sum_{i=1}^{\infty}\left(\sum_{k=1}^{n}|\langle\varphi_i^*,x_k\rangle|^2\right)^{1/2}\varphi_i \tag{21}$$

From (19), (20) and (21) we get:

$$\int_0^1\Big\|\sum_{k=1}^{n}x_k\xi_{m,k}\Big\|_X d\lambda\le C_X\Big\|\sum_{i=1}^{\infty}\left(\sum_{k=1}^{n}|\langle\varphi_i^*,x_k\rangle|^2\right)^{1/2}\varphi_i\Big\|_X. \tag{22}$$

Note now that by the classical central limit theorem the random sequence

$$(\xi_{m,1},\dots,\xi_{m,n}),\quad m=1,2,\dots$$

converges in distribution to the standard Gaussian random vector $(\mathfrak{g}_1,\dots,\mathfrak{g}_n)$. This implies that

$$\lim_m\int_0^1\Big\|\sum_{k=1}^{n}x_k\xi_{m,k}\Big\|_X d\lambda=\mathbb{E}\Big\|\sum_{k=1}^{n}x_k\mathfrak{g}_k\Big\|_X \tag{23}$$

Clearly, (23) and (22) imply (18) and (III.3.2) *is proved.* □

(III.3.3) If for X there exists a finite constant $C'_X>0$ such that for every $n\in\mathbb{N}$ every finite sequence $x_k, k=1,\dots,n$ of elements of X the inequality (18) holds, then X does not contain l_∞^n's uniformly according to implication $(3^\circ)\Longrightarrow(1^\circ)$ of [5, Corollary 2.2].

From (III.3.1), (III.3.2) and (III.3.3) we get that (III.3) is true.

(III.4) The implication $(d)\Longrightarrow(c)$ is not true if $X=c_0$ and (φ_i) is the natural basis of c_0. (This follows from (III.3).

Now we shall describe a class of series for which the condition (a) of Theorem 2.1 provides also a necessary condition for unconditional convergence.

Proposition 2.3. *Let X be a Banach space with an unconditional basis (φ_i), (t_k) be a sequence of real numbers such that the series $\sum_{i=1}^{\infty}t_i\varphi_i$ converges in X and let*

$$a_k=\sum_{i=k}^{\infty}t_i\varphi_i,\quad k=1,2,\dots.$$

Then for the sequence (a_k) the following assertions are equivalent:

(i) *The series $\sum_{k=1}^{\infty}a_k$ converges weakly to an element of X.*

(ii) *The sequence $(a_k)_{k\in\mathbb{N}}$ satisfies the condition V_1.*

(iii) *The series $\sum_{i=1}^{\infty}i|t_i|\varphi_i$ converges in X.*

(iv) *The series $\sum_{k=1}^{\infty}a_k$ converges unconditionally.*

Proof. Observe that

$$\langle \varphi_i^*, a_k \rangle = t_i \quad \text{if} \quad k \le i \text{ and } \quad \langle \varphi_i^*, a_k \rangle = 0 \quad \text{if} \quad k > i \quad \forall i, k \in \mathbb{N}\,. \tag{24}$$

This equality implies:

$$\sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k \rangle| = i|t_i| < \infty \quad \forall i \in \mathbb{N}\,. \tag{25}$$

(i) $\Longrightarrow$ (ii). The last inequality shows that (4) is satisfied for $s = 1$. Let $a \in X$ be an element to which the sequence $(\sum_{k=1}^{n} a_k)$ converges weakly. Fix $i \in \mathbb{N}$. Clearly,

$$\langle \varphi_i^*, a \rangle = \sum_{k=1}^{\infty} \langle \varphi_i^*, a_k \rangle\,.$$

From this equality and (24) we get:

$$\langle \varphi_i^*, a \rangle = \sum_{k=1}^{\infty} \langle \varphi_i^*, a_k \rangle = i t_i\,.$$

This equality and (25) imply:

$$|\langle \varphi_i^*, a \rangle| = \sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k \rangle|\,. \tag{26}$$

Clearly, $a = \sum_{k=1}^{\infty} \langle \varphi_i^*, a \rangle \varphi_i$, where the series converges unconditionally in X. Hence the series

$$\sum_{k=1}^{\infty} |\langle \varphi_i^*, a \rangle| \varphi_i$$

converges in X. This and the equality (26) imply that (5) is satisfied for $s = 1$ as well.

(ii)$\Longrightarrow$ (iii). Clearly (5) for $s = 1$ and equality (25) imply that (iii) is satisfied.

(iii)$\Longrightarrow$ (ii). (4) for $s = 1$ is satisfied thanks to (25). From (iii) and the equality (25) we obtain that (5) for $s = 1$ is satisfied.

(ii) $\Longrightarrow$ (iv) by implication (a)$\Longrightarrow$ (b) of Theorem 2.1.

(iv)$\Longrightarrow$ (i) is evident. □

Remark 2.4. In the case $X = l_1$ with the unconditional unit vector basis (φ_i) Proposition 2.3 fails to provide an example of an unconditionally convergent series in l_1 which does not converge absolutely. The existence (but not a concrete example) of such series was proved in [8], while in [9] it was proved the existence of such series in every infinite-dimensional Banach space. In [10] Mazur's question (Scottish book, Problem 88) was answered in the negative by proving the existence of an element $\mathbf{c} = (c_i)_{i\in\mathbb{N}} \in l_1$ with the following property: if

$$\mathbf{c}_n = (c_{i+n})_{i\in\mathbb{N}}, \quad n = 1, 2, \dots,$$

then the series $\sum_n \mathbf{c}_n$ converges in l_1 unconditionally but not absolutely. In [11] and in [12] a method of producing of concrete examples of unconditionally but not absolutely convergent series in l_1 was proposed.

Finally we will show that the sufficient condition for the unconditional convergence given by Theorem 2.1 is in a sense best possible.

Proposition 2.5. *Let X be a Banach space with an unconditional basis $(\varphi_i)_{i\in\mathbb{N}}$ and s be a fixed real number such that $1 < s \le 2$. Then there exists a sequence (a_k) of elements of X which satisfies the condition V_s, but for which the corresponding series $\sum_{k=1}^{\infty} a_k$ is not even weakly convergent in X.*

Proof. Using the closed graph theorem it is easy to prove that there exists a sequence (t_i) of positive real numbers such that the series $\sum_{i=1}^{\infty} i^{1/s} t_i \varphi_i$ is convergent but the series $\sum_{i=1}^{\infty} i\, t_i \varphi_i$ does not converge. Let

$$a_k = \sum_{i=k}^{\infty} t_i \varphi_i, \quad k = 1, 2, \dots .$$

From equality (24) we can write:

$$\sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k\rangle|^s = i t_i^s < \infty \quad \forall i \in \mathbb{N}\,. \tag{27}$$

Hence for (a_k) the condition (4) is satisfied. From (27) we have also

$$\left(\sum_{k=1}^{\infty} |\langle \varphi_i^*, a_k\rangle|^s\right)^{1/s} = i^{1/s} t_i\,,$$

so for (a_k) the condition (5) is satisfied too. The series $\sum_{k=1}^{\infty} a_k$ is not even weakly convergent in X, because such a convergence by implication (i)$\Longrightarrow$ (iii) of Proposition 2.3 would imply the convergence of the series $\sum_{i=1}^{\infty} i\, t_i \varphi_i$, in contradiction with our choice of the sequence (t_i). □

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