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ON A LINEAR GENERALIZED CENTRAL SPLINE ALGORITHM OF COMPUTERIZED TOMOGRAPHY

D. UGULAVA AND D. ZARNADZE

Abstract. The worst case setting of linear problems, when the error is measured with the help of a metric, is studied. In [5], a linear generalized central spline algorithm is constructed for the approximate calculation of solution operators of equations, containing positive differential and integral operators. In this paper an algorithm is constructed for same form equations with the operators admitting a singular value decomposition. In particular, a linear generalized central spline algorithm for computerized tomography problem is constructed and studied.

რეზიუმე. შესწავლილია უარესი დასმის შემთხვევა წრფივი ამოცანებისათვის, როდესაც ცდომილება გაზომილია მეტრიკის საშუალებით. [5]-ში აგებულია წრფივი განზოგადებულად ცენტრალური სპლაინური ალგორითმი ისეთ განტოლებათა ამოხსნის ოპერატორების მიახლოებითი გამოთვლისათვის, რომლებიც შეიცავენ დადებით ინტეგრალურ და დიფერენციალურ ოპერატორებს. სტატიაში აგებულია ალგორითმი იმავე სახის განტოლებებისათვის, რომლებშიც შემავალი ოპერატორები უშვებენ სინგულარული დაშლას. კერძოდ, აგებულია წრფივი განზოგადებულად ცენტრალური სპლაინური ალგორითმი კომპიუტერული ტომოგრაფიის ამოცანისათვის.

Introduction

Throughout the paper, we use the terminology and notation mainly from [1]. Let F_1 be a linear space over the scalar field of real or complex numbers with a nonincreasing sequence of absolutely convex absorbed sets $\{V_n\}$. We denote by E a metrizable locally convex space (lcs) whose topology is generated by a nondecreasing sequence of seminorms $\{\|\cdot\|_n\}$ for which $V_n = \{f \in E,\ \|f\|_n \leq 1\}$ are unit balls. Let F be an absolutely convex set in E. We consider the linear operator $S : E \to G$, called the solution

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operator, where G is a metric lcs over the scalar field of real or complex numbers with some metric d. We call elements f from F the problem elements of the solution operator and $S(f)$ the solution elements. Our aim is to calculate $S(f)$ for f. The investigation of this problem in a metric lcs is connected with the well-known Schwartz theorem. According to this theorem, the Radon operator used in the computerized tomography maps one-to-one Schwartz space onto a same type space.

Let $U(f)$ be the computed approximation. The distance $d(S(f), U(f))$ between $S(f)$ and $U(f)$ is called an absolute error. To construct the computed approximation $U(f)$, we need some additional information on a problem element f. Let $y = I(f)$ be some nonadaptive computed information of cardinality m, i.e.,

$$I(f) = [L_1(f), \dots, L_m(f)] \ , \tag{1}$$

where $L_1, \dots, L_m$ are linear functionals on the space E. If $y = I(f)$ is known, then the approximation $U(f)$ is computed to get an element of G, which approximates $S(f)$. Thus we have $U(f) = \varphi(I(f))$, where $\varphi : I(E) \to G$ is a mapping which is called an algorithm.

The worst case error of U is defined by

$$e(\varphi, I) = \sup\{d(S(f), U(f)),\ f \in F\}. \tag{2}$$

Naturally, we are interested in an algorithm with a minimal error. An algorithm φ^* is called an optimal error algorithm if it realizes inf in (2), i.e., $e(\varphi^*, I) = \inf\{e(\varphi, I) :\ \varphi \in \Phi\}$, where Φ is a set of all algorithms.

A generalization of the solution operator defined in [1] is given in [2]. Namely, given the sets F and G, the operator $S : F \times \mathbb{R}^+ \to 2^G$ is considered, where $\mathbb{R}^+ = [0, \infty)$, $S(f, 0) \neq 0$, for which $S(f, \delta_1) \subset S(f, \delta_2)$ if $\delta_1 \leq \delta_2$, $\delta_1, \delta_2 \in \mathbb{R}^+$, $f \in F$. Let $\varepsilon \geq 0$ be a parameter which is a measure of the admissible uncertainty. An element x, such that $x \in S(f, \varepsilon)$, is called an ε-approximation. The problem consists in finding an ε-approximation for each $f \in F$. It is assuming that, in general, the element f is unknown. Instead, we have some information on f with the aid of $I(f)$. The problem under consideration fits in this setting if G is a linear metric space with a metric d over the field of real or complex numbers and $S : F \to G$. To verify that this is so, we define

$$S(f, \varepsilon) = \begin{cases} \{g \in G : d(S(f), g) < \varepsilon\} & \text{for} \quad \varepsilon > 0, \\ \{S(f)\} & \text{for} \quad \varepsilon = 0, \end{cases}$$

and repeat the reasoning from ([2], §3.2).

In the sequel, the operator S will be called a solution operator of some equation $Au = f$, if $u = Sf$. If there exists an inverse to A, then $S = A^{-1}$. Besides, the central (resp. linear, spline, optimal) algorithm, approximating the solution operator S [1], will be called the central (resp. linear, spline, optimal) algorithm for the equation $Au = f$.

The present paper is devoted to the construction of linear generalized central spline algorithms for an approximate solution of the computerized tomography problem. The results are based on the generalization of the best approximation and on the self-adjoint operator theories in Frechet spaces [3]. In [3]–[4], the case is considered in which a nonincreasing sequence of problem element sets is given in a linear space F_1. The linear problems are, in fact, considered for a sequence of solution operators. In [5], the solution operator acting from a metrizable lcs to the same-type space is investigated, which essentially generalize the cases considered in [1], [3] and [4]. A linear generalized central spline algorithm is constructed in [5] for equations containing inverse operators of harmonic oscillator, Lagrange, Legandre, Tricomi operators and also special types Sturm-Liouville operators for boundary value problems of second order differential and other operators. An optimal algorithm in an average case setting has been constructed by V.Tarieladze and N.Vakhania [6].

Section 1 of the present paper deals with the notions of a generalized interpolating spline and a generalized spline algorithm. These notions generalize the corresponding well-known notions from [1] for the case where in a linear space there exists not only the set of problem elements, but some nonincreasing sequnce of such sets. The generalized interpolating spline realizes a minimum not only with respect to the metric, but also with respect to the corresponding Minkowski functional. The conditions under which these minima are realized with respect to the metric constructed by D. Zarnadze, are established in [4]. Besides, the notions of a generalized spline and a generalized central algorithm are introduced for the solution operator acting from a Frechet space to the same type space.

In Section 2, the equation $Ku = f$ is considered in a Hilbert space H with a self-adjoint positive one-to-one compact operator with a dense image. In this case, the inverse operator K^{-1} is not continuous, and hence the solution of this problem is not stable. We introduce the Frechet space $D(K^{-\infty})$ and transfer this equation there. As a result. we obtain the equation $K_\infty u = f$, where K_∞ is the restriction of the operator $K^{\mathbb{N}}$ from the Frechet space $H^{\mathbb{N}}$ in $D(K^{-\infty})$. The latter operator maps the space $D(K^{-\infty})$ isomorphically onto, and the equation $K_\infty u = f$ has in this space a unique and stable solution. To solve this equation approximately, we construct a linear generalized central spline algorithm (Theorem 2). The obtained result is generalized for the operators acting in Hilbert spaces and having a singular value decomposition (Theorem 3).

In the concluding Section 3, using the results obtained in Section 2, we investigate the problem of an approximate solution of equations containing the Radon transform. The problem has, as is well-known, an important application in computerized tomography. Finally, using Theorem 3, we construct a linear generalized central spline algorithm for the computerized tomography problem (Theorem 4).

1. Generalized Spline Algorithm and the Condition for It to be Linear and Generalized Central

Let E be a locally convex metrizable space with a nondecreasing sequence of seminorms $\{\|\cdot\|_n\}$ generating the topology. It is well known that there exists a translations invariant metric d with absolutely convex balls $K_r = \{x \in E;\ d(x,0) \leq r\}$, such that (E,d) is a linear metric space. We denote by $q_r(\cdot)$ the Minkowski functional of the ball K_r, and by $|x| = d(x,0)$ the quasinorm. A metric with convex balls has been constructed by Albinus [7] and called a normlike metric. D.Zarnadze constructed a metric of the form ([8], see also [5])

$$d(x,y) = \begin{cases} \|x-y\|_1 \ , & \text{if} \quad \|x-y\|_1 \geq 1 \ , \\ 2^{-n+1} \ , & \text{if} \quad \|x-y\|_n \leq 2^{-n+1} \\ & \text{and} \quad \|x-y\|_{n+1} \geq 2^{-n+1} \ (n \in N) \ , \\ \|x-y\|_{n+1} \ , & \text{if } 2^{-n} \leq \|x-y\|_{n+1} \ < 2^{-n+1} \ (n \in N) \ , \\ 0, & \text{if } x-y=0 \ . \end{cases} \tag{3}$$

The Minkowski functionals $q_r(\cdot)$ for the balls K_r of the metric (3) depend on the initial seminorms by the following simple equality [8]:

$$q_r(\cdot) = r^{-1}\|\cdot\|_n, \text{ where } \quad r \in I_n = \begin{cases} [1,\infty[, & \text{if} \quad n=1, \\ [2^{-n+1}, 2^{-n+2}[, & \text{if } n \geq 2. \end{cases} \tag{4}$$

Thus $K_r = rV_n$, where $V_n = \{x \in E; \|x\|_n \leq 1\}$, if $r \in I_n$. For $V_1 = V_2 = \cdots = F$, we have $K_r = rF$, $|\cdot| = \mu_F(\cdot)$, where μ_F is the Minkowski functional of F.

Let $I : E \to \mathbb{R}^m$ be nonadaptive information (1) of cardinality $m, y \in I(E), I(f) = y$ for some $f \in E$ and $d(f, \mathrm{Ker}I) = r$. Then an element $\sigma = \sigma(y) = f - h^*$ is said to be a generalized spline interpolating y (or, briefly, a generalized spline) if $I(\sigma) = y$,

$$d(f, \mathrm{Ker}I) = d(f,h^*) = r = d(\sigma,0) = |\sigma| \tag{5}$$

and

$$\inf\{q_r(f-h) : h \in \mathrm{Ker}I\} = q_r(f-h^*) = q_r(\sigma). \tag{6}$$

The generalized interpolation spline σ minimizes not only the metric, but also the corresponding Minkowski functional. In other words, the generalized interpolating spline exists, iff the m-codimensional subspace $\mathrm{Ker}I$ is strongly proximal in the metric space (E,d). This notion has been introduced by us in [9]. Since for the normlike metrics the conditions (5) and (6) are equivalent and $q_r(\sigma) = 1$, the notion of a strong proximality coincides with the ordinary proximality.

For the metric (3), the above definition takes the form formulated as follows: $\sigma = \sigma(y) = f - h^*$ is said to be a generalized spline interpolating y if $I(\sigma) = y$,

$$d(f, \mathrm{KerI}) = \mathrm{d(f,h^*) = r = d(\sigma,0)} = |\sigma| \text{ if } \mathrm{r \in intI_n}, \tag{7}$$

and

$$\inf\{\|f-h\|_n : h \in \mathrm{KerI}\} = \|\mathrm{f-h^*}\|_\mathrm{n} = \|\sigma\|_\mathrm{n} \leq \mathrm{r} \text{ if } \mathrm{r} = 2^{-\mathrm{n}+1}\ (\mathrm{n} \in \mathbb{N}). \tag{8}$$

By the property (4) of the metric (3) we have that for $r \in \mathrm{int}I_n$ the fulfillment of the condition (7) is sufficient for σ to be a generalized interpolating spline. In the case $r = 2^{-n+1}$ $(n \in N)$, (7) follows from (8), but, in general, the best approximating element with respect to the metric may not have an analogous property with respect to $q_r(\cdot)$ (and therefore with respect to $\|\cdot\|_n$)[10].

We now give an example showing that if $d(x, G) = d(x, g_0) = 1$, then g_0 may be not a best approximation element with respect to the seminorm $\|\cdot\|_1$. Let $E = C(\mathbb{R})$ be the Frechet space of continuous real-valued functions with the compact convergence topology on $\mathbb{R}$, which is given by the nondecreasing sequence of seminorms $\|x\|_n = \max\{|x(t)|; t \in [-n,n]\},\ n \in \mathbb{N}$. Let $x(t) = t^2$, and let $G = \mathscr{P}_2$ be the subspace of polynomials of order at most 1. By the Chebyshev classical theorem,

$$\inf\{\|x-m\|_1,\ m \in \mathscr{P}_2\} = \inf\{\max\{|t^2-a_1t-a_2|,\ t \in [-1,1]\},\ a_1, a_2 \in \mathbb{R}\} =$$

$$\max\{|T_2(t)|/2;\ t \in [-1,1]\} = \max\{|t^2-1/2|;\ t \in [-1,1]\} = \|t^2-m_0\|_1 = 1/2,$$

where $T_2(t) = 2t^2 - 1$ is the Chebyshev polynomial of the first kind, and $m_0(t) \equiv 1/2$ is the unique best approximation element of x with respect to the seminorm $\|\cdot\|_1$. Thus $\|t^2 - m\|_1 \geq 1/2$ for all $m \in \mathscr{P}_2$. Furthermore, we have

$$\inf\{\|x-m\|_2,\ m \in \mathscr{P}_2\} = \inf\{\max\{|t^2 - a_1t - a_2|,$$

$$t \in [-2,2]\},\ a_1, a_2 \in \mathbb{R}\} \geq \|t^2 - 2\|_2 = 2.$$

This means that $\|t^2 - m\|_2 > 1$ for all $m \in \mathscr{P}_2$. Let us consider an element $m \in \mathscr{P}_2$ for which $1/2 < \|t^2 - m\|_1 \leq 1$. From the definition of the metric (3) it follows that the inequalities $\|t^2 - m\|_2 \geq 1$ and $\|t^2 - m\|_1 \leq 1$ hold for some elements of the subspace $\mathscr{P}_2$, and therefore $d(t^2, m) = d(t^2, \mathscr{P}_2) = 2^{-1+1} = 1$. This means that if $1/2 \leq \|t^2 - m\|_1 \leq 1$, then $m \in \mathscr{P}_2$ is a best approximation element with respect to the metric, but none of such elements is a best approximation element with respect to the seminorm $\|\cdot\|_1$. Only $m_0(t) = 1/2 \in \mathscr{P}_2$ is simultaneously a best approximation element for both d and $\|\cdot\|_1$.

The problem of proximality of hypersubspaces in Frechet spaces with respect to the normlike metrics was for the first time studied by G.Albinus in [7]. But the problem posed by him concerned with the construction of other metrics because the known normlike metrics were inconvenient for the study of fine geometrical problems of the best approximation. It was important to construct a new metric that would have certain advantages over normlike metrics. The new metric (3) was constructed by using the well-known Kakutani's metric. Defined by (4) balls K_r preserve the geometry of the initial space. Similarly to the case of a normed linear space, K_r are simply expressed with the unit balls of the topology generating seminorms (4).

The problem of proximality of hypersubspaces in Frechet spaces with respect to the above-mentioned metrics is connected with the well-known Jame's theorem for Banach spaces. The generalization of this theorem reads as follows [10]: in a Frechet space, every closed hypersubspace is proximal, iff the Frechet space is reflexive and strictly regular (quojection). Moreover, it is proved that for some classes of reflexive Frechet spaces there exists nonproximal closed hypersubspaces. Based on the above proximality properties, it is shown in [4] that in the Frechet space. a generalized interpolating spline for any nonadaptive information I cardinality 1 exists, iff this space is reflexive quojection. Also, it is proved that in some classes of reflexive Frechet spaces there exists a nonadaptive information I of cardinality 1 for which the generalized interpolating spline does not exist. Furthermore, it is proved that in some Frechet space the generalized interpolating spline exists for nonadaptive information of any cardinality.

Let us now define the notion of a generalized spline algorithm and generalized central algorithm for the solution operator $S : E \to G$ in the case of the metric (3). We consider the set $F = \{f \in E : \ d(f, 0) \le 1\}$. If the generalized spline exists and is unique, then the generalized spline algorithm is defined as in [1], using the equality $\varphi^s(y) = S\sigma(y),\ y \in I(F)$. Following [1], an algorithm φ that uses the information (1) is called linear if it has the form $\varphi(I(f)) = \sum\limits_{i=1}^{m} L_i(f)q_i,\ q_i \in G$.

Let I be a nonadaptive information (1) of cardinality $m \ge 1$ and $y = I(f),\ f \in F$. Then $d(f, 0) = r \in I_n$, i.e., $f \in V_n$ for some $n \in N$. We call the value

$$e_n(\varphi, I, y) = \sup\{d(S(f), \varphi(y));\ f \in I^{-1}(y) \cap V_n\}$$

the local error of the algorithm φ at a point y, where the last d denotes the metric in the space G. Denote by $r_n(I, y)$ the local radius of the nonadaptive information I at a point y defined by the equality

$$r_n(I, y) = \text{rad } (S(I^{-1}(y) \cap V_n)).$$

Here, the radius of the set $M \subset G$ is defined analogously to the case of a normed space by the equality $\mathrm{rad}(M) = \inf\{\sup\{d(a, g);\ a \in M\};\ g \in G\}$. The Chebyshev center $c \in G$ of a set $M \subset G$ is defined by the equality $\mathrm{rad}(M) = \sup\{d(a, c),\ a \in M\}$. It is not difficult to verify that $r_n(I, y) = \inf\{e_n(\varphi, I, y) : \varphi \in \Phi\}$, where Φ is the set of all algorithms. The global radius $r_n(I)$ of the nonadaptive information I is defined by the equality

$$r_n(I) = \sup\{r_n(I, y);\ y \in I(V_n)\}.$$

Let $y \in I(F) \subset R^m$, i.e., $y \in I(V_n)$ for some $n \in N$. Assume that the sets $S(I^{-1}(y) \cap V_k)$ have a Chebychev center $c = c(y)$ for all $y \in I(F)$ and $k \leq n$ if $y \in I(V_n)$. This means that for all $k \leq n$

$$\mathrm{rad}(S(I^{-1}(y) \cap V_k) = \inf\{\sup\{|S(f) - g|;\ f \in I^{-1}(y) \cap V_k\};\ g \in G\}) = \\ = \sup\{|S(f) - c(y)|;\ f \in I^{-1}(y) \cap V_k\}.$$

Then we call the algorithm $\varphi^c(y) = c(y)$ generalized central. If $V_1 = V_2 = \cdots = V_n = \cdots = F$, then the definition of generalized central algorithms coincides with the classical one.

Assume that the topology of the Frechet space E is given by a sequence of Hilbertian seminorms $\{\|\cdot\|_n\}$, i.e., each seminorm $\|\cdot\|_n$ is generated by the semiscalar product $(x, y)_n$ and $V_n = \{x \in E;\ \|x\|_n \leq 1\}$. For such spaces, the notion of orthogonality is naturally defined as follows: the elements $x, y \in E$ are called orthogonal if $(x, y)_n = 0$ for each $n \in \mathbb{N}$. A subspace M possesses an orthogonal complement $M^{\perp}$ in E if each element $x \in E$ is represented as the sum $x = y + z$, where $y \in M,\ z \in M^{\perp}$ and $(y, z)_n = 0$ for each $n \in N$. This is equivalent to the fact that in the subspaces M and $M^{\perp}$, any element $x \in E$ has, respectively, a unique best approximation y and z with respect to all seminorms $\|\cdot\|_n$ generated by $(\cdot,\cdot)_n$. Note that the orthogonality in Frechet spaces differs essentially from that in Hilbert spaces. As was proved, in any Frechet space there exists a one-dimensional subspace that does not admit orthogonal complement. An example of such subspace in the space L^2_{loc} is given in [11]. The problem of the existence of an orthogonal complement for infinite dimensional subspaces in Frechet-Hilbert spaces is studied in [12].

The following theorem has been proved in [5].

Theorem 1. *Let E be a Fréchet space with a nondecreasing sequence of Hilbertian seminorms $\{\|\cdot\|_n\}$, $V_n = \{x \in E : \|x\|_n \leq 1\}$ and with the metric* (3). *Let $K_n : E \to E/\mathrm{Ker}\|\cdot\|_n$ be the canonical mapping, $X_n = (E/\mathrm{Ker}\|\cdot\|_n\ ,\ \widehat{\|\cdot\|_n})$ and G be a metrizable lcs, $S : E \to G$ be a linear solution operator, and I be a nonadaptive information of cardinality $m \geq 1$. Then the following assertions are valid:*

a) *If $K_n(\mathrm{Ker}I)$ is closed in the Hilbert space X_n, $n \in N$, then* Ker$I$ *is strongly proximal in E with respect to the metric* (3), *and for any $y \in I(E)$ there exists a generalized spline σ interpolating y.*

b) *If, moreover, the subspace* Ker$I$ *has an orthogonal complement in* E, *then for any* $y \in I(E)$ *there exists the unique generalized spline* σ *interpolating* y *such that* $(\sigma, h)_n = 0$ *for any* $n \in \mathbb{N}$ *and* $h \in$ Ker$I$. *If* $y \in I(V_1)$, *then* σ *is a center of all sets* $I^{-1}(y) \cap V_k$, *for which this intersections are nonempty. The corresponding spline algorithm* $\varphi^s(y) = S(\sigma)$ *is linear and generalized central.*

2. The Enlarged Ritz Method for Equations with Operators Admitting Singular Decompozition

Let K be a linear, compact, selfadjoint, positive, and one-to-one operator in a Hilbert space H with a dense image. Let $\{\varphi_k\}$ be an orthogonal sequence of eigenfunctions of K, and $\{\lambda_k\}$ be a sequence of eigenvalues corresponding to $\varphi_k,\ k \in N$. Then K has the form

$$Ku = \sum_{k=1}^{\infty} \lambda_k (\varphi_k, \varphi_k)^{-1} (u, \varphi_k) \varphi_k,\ \ \lambda_k \to 0,\ \ \lambda_k > 0.$$

It is easy to verify that $\{\varphi_k\}$ is a complete system in H. The inverse to the operator K is selfadjoint, positive definite and has the form

$$K^{-1} x = \sum_{k=1}^{\infty} \lambda_k^{-1} (x, \varphi_k)(\varphi_k, \varphi_k)^{-1} \varphi_k \ .$$

Indeed,

$$K^{-1}x = \sum_{k=1}^{\infty} (K^{-1}x, \varphi_k)(\varphi_k, \varphi_k)^{-1}\varphi_k = \sum_{k=1}^{\infty} \lambda_k^{-1} (K^{-1}x, K\varphi_k)(\varphi_k, \varphi_k)^{-1} \varphi_k =$$

$$= \sum_{k=1}^{\infty} \lambda_k^{-1} (KK^{-1}x, \varphi_k)(\varphi_k, \varphi_k)^{-1}\varphi_k = \sum_{k=1}^{\infty} \lambda_k^{-1} (x, \varphi_k)(\varphi_k, \varphi_k)^{-1} \varphi_k \ ,$$

and

$$(K^{-1}x, x) = \left(\sum_{k=1}^{\infty} \lambda_k^{-1} (x, \varphi_k)(\varphi_k, \varphi_k)^{-1} \varphi_k, \sum_{k=1}^{\infty} (x, \varphi_k)(\varphi_k, \varphi_k)^{-1} \varphi_k \right) =$$

$$= \sum_{k=1}^{\infty} \lambda_k^{-1} (x, \varphi_k)^2 (\varphi_k, \varphi_k)^{-1} \geq C_0 (x, x), \tag{9}$$

where $C_0 = \min\{\lambda_k^{-1}; k \in N\}$.

The sequence λ_k^{-1} is unbounded and tends to infinity. Therefore the selfadjoint operator K^{-1} has a discrete spectrum ([13], p. 98) and a dense domain. In [14], we have introduced the Frechet space $D(K^{-\infty}) = \bigcap_{n=1}^{\infty} D(K^{-n+1})$, where K^{-1} is the inverse to the operator K and $K^{-n} = K^{-1}(K^{-n+1}),\ n \in \mathbb{N}$. As a set, $D(K^{-\infty})$ is a part of the Hilbert space

H. The topology of the Frechet space $D(K^{-\infty})$ is given by the sequence of norms

$$\|x\|_n = (\|x\|^2 + \|K^{-1}x\|^2 + \cdots + \|K^{-n+1}x\|^2\ )^{1/2},\ n \in \mathbb{N},$$

generated by the scalar product

$$(x,y)_n = (x,y)+(K^{-1}x, K^{-1}y)+\cdots+(K^{-n+1}x, K^{-n+1}y),\ \ x,y \in D(K^{-\infty}).$$

The Frechet space $D(K^{-\infty})$ is topologically isomorphic to a subspace M of the Frechet-Hilbert space $H^{\mathbb{N}}$ considered in [14] by using the product topology. This isomorphism is realized by the mapping

$$D(K^{-\infty}) \ni x \to \mathrm{Orb}(K^{-1}, x) := \{x, K^{-1}x, \ldots, K^{-n+1}x, \ldots\} \in M \subset H^{\mathbb{N}}.$$

In [14] the operator $K^{-\infty} : D(K^{-\infty}) \to D(K^{-\infty})$ is also introduced as

$$K^{-\infty}(x) = \{K^{-1}x, K^{-2}x, \ldots, K^{-n}x, \ldots\}.$$

The topology of the space $H^{\mathbb{N}}$ is given by a sequence of seminorms $p_n(f) = (\|f_1\|^2 + \cdots + \|f_n\|^2)^{1/2},\ f = \{f_k\} \in H^{\mathbb{N}},\ n \in \mathbb{N}$. The isomorphism is also isometry.

In the space $D(K^{-\infty})$, we introduce the following sequence of norms:

$$[x]_n = (K^{-\infty}x, x)_n^{1/2} =$$

$$= ((K^{-1}x, x) + (K^{-2}x, K^{-1}x) + \cdots + (K^{-n}x, K^{-n+1}x))^{1/2}.$$

According to (9),

$$[x]_n^2 = (K^{-1}x, x) + (K^{-2}x, K^{-1}x) + \cdots + (K^{-n}x, K^{-n+1}x) \geq$$

$$\geq C_0(\|x\|^2 + \|K^{-1}x\|^2 + \cdots + \|K^{-n+1}x\|^2) = C_0\|x\|_n^2,\ n \in \mathbb{N}.$$

From [13] (Ch.2, Section 9) it follows that the identical map $(D(K^{-\infty}), [\cdot]_1) \to (D(K^{-\infty}), \|\cdot\|_1)$ is one-to-one and continuous. Hence we find that the mapping $(D(K^{-\infty}), [\cdot]_n) \to (D(K^{-\infty}), \|\cdot\|_n)$ is also one-to-one and continuous. This means that sequences of the norms $\{[\cdot]_n\}$ and $\{\|\cdot\|_n\}$ generate the comparable topologies in $D(K^{-\infty})$ ([15], Ch.1, Section 3). We call the space $D(K^{-\infty})$, endowed with the sequence of Hilbertian norms $\{[\cdot]_n\}$, the energetic space of the operator $K^{-\infty}$ and denote it by $E_{K^{-\infty}}$. Therefore, the Frechet spaces $E_{K^{-\infty}}$ and $D(K^{-\infty})$ are isomorphic. The operator $K^{-\infty}$ is symmetric and positive definite on the Frechet space $D(K^{-\infty})$ because the relations

$$(K^{-\infty}x, y)_n = (x, K^{-\infty}y)_n,\ (K^{-\infty}x, x)_n \geq C_0^{1/2}(x,x)_n,\ n \in \mathbb{N} \qquad (10)$$

hold for any $n \in N$ and $x, y \in D(K^{-\infty})$. The first of (10) follows from the symmetry of K^{-1}. Since $K^{-\infty}$ is symmetric and defined on the whole space $D(K^{-\infty})$, it is continuous and selfadjoint on this space [3]. From the above properties of $K^{-\infty}$ follows that it has the inverse $(K^{-\infty})^{-1}$ which is selfadjoint and continuous [3]. Therefore, the operator $K^{-\infty}$ is an isomorphism of the Frechet space $D(K^{-\infty})$ onto itself. We denote the operator $(K^{-\infty})^{-1}$ by K_∞. Thus

$$K_\infty u = (K^{-\infty})^{-1} u = \{Ku, u, K^{-1}u, \dots, K^{-n+2}u, \dots\}.$$

Indeed,

$$K^{-\infty} K_\infty u = K_\infty (K^{-\infty} u) = K_\infty \{K^{-1}u, K^{-2}u, \dots, K^{-n}u, \dots\} =$$

$$= \{u, K^{-1}u, \dots, K^{-n}u, \dots\} = u.$$

The equation $Ku = f$ in a Hilbert space H is ill-posed, and for $f \in D(K^{-\infty})$ we transfer it from H to the Frechet space $D(K^{-\infty})$. Namely, f should be replace by $\mathrm{Orb}(K^{-1}, f) = (f, K^{-1}f, \cdots)$ and Ku by $K_\infty u = \mathrm{Orb}(K^{-1}, Ku) = (Ku, u, K^{-1}u, \dots)$. For the sake of simplicity, we write the transferred equation in the form

$$K_\infty u = f, \tag{11}$$

which, we hope, will not lead to any confusion in the sequel. Since K_∞ is an onto isomorphism of $D(K^{-\infty})$, equation (11) has a unique and stable solution in the Frechet space $D(K^{-\infty})$, i.e., it is well-posed. The ill-posed equation $Ku = f$ has been transferred to the Frechet space $D(K^{-\infty})$. Let us consider the space $D(K^{-\infty})$ endowed with a sequence of Hilbert norms

$$\{x\}_n = (K_\infty x, x)_n^{1/2} = ((Kx, x) + (KK^{-1}x, K^{-1}x) +$$

$$+ \dots + (K^{-n+2}x, K^{-n+1}x))^{1/2}, \; n \in \mathbb{N} \tag{12}$$

which is generated by a sequence of scalar products

$$\{x, y\}_n = (K_\infty x, y)_n = (Kx, y) + (KK^{-1}x, K^{-1}y) +$$

$$+ \dots + (K^{-n+2}x, K^{-n+1}y), \; n \in \mathbb{N}.$$

According to (9), we have obtained for $n \in \mathbb{N}$ that

$$\{x\}_n^2 = (Kx, x) + (KK^{-1}x, K^{-1}x) + \dots + (K^{-n+2}x, K^{-n+1}x) \geq$$

$$\geq (Kx, x) + C_0\|x\|^2 + C_0\|K^{-1}x\|^2 + \dots + C_0\|K^{-n+2}x\|^2 =$$

$$= (Kx, x) + C_0\|x\|_{n-1}^2 \geq C_0\|x\|_{n-1}^2.$$

This also means that the sequences of norms $\{\{\cdot\}_n\}$ and $\{\|\cdot\|_n\}$ generate the comparable topologies on $D(K^{-\infty})$. We call the space $D(K^{-\infty})$ endowed with the sequence of Hilbertian norms $\{\{\cdot\}_n\}$ the energetic space of the operator K_∞ and denote it by E_{K_∞}. Therefore, the Frechet spaces $E_{K\infty}$ and $D(K^{-\infty})$ are isomorphic.

For an approximate solution of equation (11) we use the Ritz extended method in the space E_{K_∞}. The coefficients of the approximative solution $u_m = \sum\limits_{k=1}^{m} a_k\varphi_k$ are defined from the following system of equations:

$$\sum_{k=1}^{m} a_i\{\varphi_k,\varphi_i\}_n = (f,\varphi_k)_n, \quad i = 1,2,\dots,m,\ n\in\mathbb{N},$$

i.e.,

$$\sum_{k=1}^{m} a_i(K_\infty\varphi_k,\varphi_i)_n = (f,\varphi_k)_n, \quad i = 1,2,\dots,m,\ n\in\mathbb{N}.$$

In general, if $\{\varphi_k\}$ is an arbitrary linearly independent orthogonal sequence, the coefficients a_k depend on n. Let us prove that in our case, when φ_k are eigenfunctions of K, they do not depend on n. Indeed, we have

$$(K_\infty\varphi_k,\varphi_i)_n=(K\varphi_k,\varphi_i)+(KK^{-1}\varphi_k,K^{-1}\varphi_i)+\dots+(K^{-n+2}\varphi_k,K^{-n+1}\varphi_i)=$$
$$= \lambda_k(\varphi_k,\varphi_i)(1+\lambda_k^{-1}\lambda_i^{-1}+\dots+\lambda_k^{1-n}\lambda_i^{2-n}),$$

i.e.

$$(K_\infty\varphi_k,\varphi_i)_n = \begin{cases} 0, & \text{if} \quad k\neq i,\\ \lambda_k(\varphi_k,\varphi_k)(1+\lambda_k^{-2}+\dots+\lambda_k^{2-2n}), & \text{if} \quad k=i.\end{cases}$$

Moreover

$$(f,\varphi_k)_n = (f,\varphi_k)+(K^{-1}f,K^{-1}\varphi_k)+\dots+(K^{-n+1}f,K^{-n+1}\varphi_k) =$$
$$= (f,\varphi_k)+(f,K^{-2}\varphi_k)+\dots+(f,K^{-2n+2}\varphi_k) =$$
$$= (1+\lambda_k^{-2}+\dots+\lambda_k^{2-2n})(f,\varphi_k).$$

Hence it follows that

$$a_k = \lambda_k^{-1}(f,\varphi_k)(\varphi_k,\varphi_k)^{-1}.$$

Therefore an approximate solution of equation (11) obtained by the Ritz extended method, takes the form

$$u_m = \sum_{k=1}^{m}(f,\varphi_k)((\varphi_k,\varphi_k)\lambda_k)^{-1}\varphi_k. \tag{13}$$

Let $y = I(f) = [L_1(f),\ L_2(f),\dots,L_m(f)]$ be a nonadaptive information of cardinality m on $D(K^{-\infty})$, where $L_i(f) = (f,\varphi_i)$. Ker$I$ is a finite codimensional subspace in $D(K^{-\infty})$ and $(\mathrm{Ker}I)^\perp = \mathrm{span}\{\varphi_1,\varphi_2,\dots,\varphi_m\}$. A generalized spline σ_m interpolating y has the form [5]

$$\sigma_m = \sum_{k=1}^{m}(f,\varphi_k)(\varphi_k,\varphi_k)^{-1}\varphi_k. \tag{14}$$

The solution operator for the equation $K_\infty u = f$ is $S = (K_\infty)^{-1} = K^{-\infty}$ and it realizes an isomorphism of the space $D(K^{-\infty})$ onto itself. It follows from (14) that

$$S\sigma_m = \sum_{k=1}^{m}(f,\varphi_k)(\varphi_k,\varphi_k)^{-1}S\varphi_k = \sum_{k=1}^{m}(f,\varphi_k)(\varphi_k,\varphi_k)^{-1}(K_\infty)^{-1}\varphi_k =$$
$$= \sum_{k=1}^{m}(f,\varphi_k)(\varphi_k,\varphi_k)^{-1}K^{-\infty}\varphi_k = \sum_{k=1}^{m}\lambda_k^{-1}(f,\varphi_k)(\varphi_k,\varphi_k)^{-1}\varphi_k = u_m,$$

because

$$K^{-\infty}(\varphi_k) = \{K^{-1}\varphi_k, K^{-2}\varphi_k, \dots, K^{-n}\varphi_k, \dots\} =$$
$$= \lambda_k^{-1}\{\varphi_k, K^{-1}\varphi_k, \dots, K^{-1}\varphi_k, \dots\} = \lambda_k^{-1}\varphi_k.$$

$S\sigma_m = u_m$ is a best approximation element for $Sf = (K_\infty)^{-1}f$ in the subspace $(\mathrm{Ker}I)^\perp$ with respect to the energetic norms $\{\cdot\}_n$ of the energetic space E_{K_∞} of the operator K_∞ for all $n \in \mathbb{N}$. Indeed, a unique the best approximation element of $Sf = (K_\infty)^{-1}f$ in the subspace $(\mathrm{Ker}I)^\perp$ with respect to the Hilbertian norm $\{\cdot\}_n$ in the prehilbertian space $(E_{K_\infty}, \{\cdot\}_n)$ has the form ([16], §2.4)

$$\sum_{k=1}^{m}\{(K_\infty)^{-1}f,\varphi_k\}_n\{\varphi_k,\varphi_k\}_n^{-1}\varphi_k =$$
$$= \sum_{k=1}^{m}(K_\infty(K_\infty)^{-1}f,\varphi_k)_n\{\varphi_k,\varphi_k\}_n^{-1}\varphi_k =$$
$$= \sum_{k=1}^{m}(f,\varphi_k)((\varphi_k,\varphi_k)\lambda_k)^{-1}\varphi_k = u_m.$$

Therefore this best approximation element does not depend on n. Hence the subspace $\mathrm{Ker}I$ admits an orthogonal complement subspace in the Frechet space E_{K_∞}. This kind of the best approximation in locally convex spaces was considered by many mathematicians (see the survey in [17]). Such problem of the best approximation has also been considered by N.Vakhania and S.Chobanyan in [18]–[19].

By part b) of Theorem 1, the above-considered generalized spline algorithm is linear and generalized central. Moreover, the completeness of the system φ_k implies that the sequence of the constructed best approximation elements $\{u_m\}$ tends to $(K_\infty)^{-1}f$ in the energetic space E_{K_∞} ([16], §2.5), which is a pre-Hilbert with respect to its every norm. The above reasoning gives rise to

Theorem 2. *Let K be a compact, selfadjoint, positive and one-to-one operator in a Hilbert space H with a dense image and an orthogonal sequence of eigenfunctions φ_j. Let λ_j be the eigenvalue correspondong to the* *eigenfunction* φ_j*, and let* u_m *be defined by* (13)*. Then* $\varphi^s(I(f)) = u_m$ *is the linear generalized spline and generalized central algorithm for the solution operator* $S = K_\infty^{-1}$ *and information* $I(f) = [(f, \varphi_1),\ (f, \varphi_2), \dots, (f, \varphi_m)]$*. Moreover, the sequence of approximate solutions* $\{u_m\}$ *converges to a solution of equation* (11) *in the energetic space* E_{K_∞} *of the operator* K_∞ *with respect to the norms* (12)*, and also in the space* $D(K^{-\infty})$*.*

Let now H and M be Hilbert spaces and $\{\varphi_k\}$, $\{\psi_k\}$ be orthogonal systems in H and M, respectively. For the sake of simplicity, we use the same notation $(\cdot, \cdot)$ for the inner product in H and M. Let, further, A be the operator acting from H to M, having a singular value decomposition [20]

$$Au = \sum_{k=1}^{m} \sigma_k (u, \varphi_k) \psi_k\ ,\ u \in H,\ \sigma_k > 0. \tag{15}$$

In the latter case, we also say that $\{\psi_k, \varphi_k, \sigma_k\},\ k \in \mathbb{N}$, is a singular system for A. Numbers σ_k are called singular numbers of the operator A. It is usually assumed that $\{\varphi_k\}$ and $\{\psi_k\}$ are orthonormal systems [20], but, without loss of generality, they can be assumed to be only orthogonal. If the equation

$$Au = f, \tag{16}$$

is ill-posed, we seek for a generalized solution of (16) in the sense of Mourie-Penrose ([20], Ch. IV). This means that if $f \in \mathrm{ImA} + \mathrm{ImA}^{\perp}$, a generalized solution is represented by an element $A^{+}f$ which minimizes the norm $\|Au - f\|$. If there exists a set of such elements, among which we choose any with a minimal norm. This generalized solution satisfies the equation

$$A^{*}Au = A^{*}f \tag{17}$$

and belongs to the set $(\mathrm{Ker}A)^{\perp} = \overline{\mathrm{Im}A^{*}}$, where $A^{*} : M \to H$ is the conjugate operator to A in the sense of Hilbert spaces and

$$A^{*}f = \sum_{k=1}^{\infty} \sigma_k (f, \psi_k) \varphi_k. \tag{18}$$

The operator $A^{*}A : H \to H$ has the form

$$A^{*}Au = \sum_{k=1}^{m} \sigma_k^2 (u, \varphi_k)(\psi_k, \psi_k) \varphi_k,\ u \in H. \tag{19}$$

From (15), (18) and (19), we find that $A\varphi_k = \sigma_k(\varphi_k, \varphi_k)\psi_k$, $A^{*}\psi_k = \sigma_k(\psi_k, \psi_k)\varphi_k$, $A^{*}A\varphi_k = \sigma_k^2(\psi_k, \psi_k)(\varphi_k, \varphi_k)\varphi_k$. Using the latter formulas, it can be proved, similarly to [20], that if A has the decomposition (15), then the unique solution u^{+} of (16) is given in the sense of Mourie-Penrose by the formula

$$u^+ = \sum_{k=1}^{\infty} (\sigma_k(\psi_k, \psi_k)(\varphi_k, \varphi_k))^{-1}(f, \psi_k)\varphi_k. \tag{20}$$

The operator A^*A is symmetric and positive. The positiveness follows from the equality

$$(A^*Au, u) = \sum_{k=1}^{\infty} \sigma_k^2(\psi_j, \psi_j)(u, \varphi_k)^2.$$

Let us further assume that the operator A is one-to-one on the whole space H. It follows from the formula $\mathrm{Ker}A^{\perp} = \overline{\mathrm{Im}A^*}$ that under the above conditions the equality $\overline{\mathrm{Im}A^*A} = H$ is fulfilled. Therefore the operator A^*A is selfadjoint and has positive eigenvalues $\sigma_k^2(\psi_k, \psi_k)(\varphi_k, \varphi_k)$, which correspond to the functions φ_k.

If the systems $\{\varphi_k\}$, $\{\psi_k\}$ contained in (15) are orthonormal and $\sigma_k \to 0$, then the operator A is compact (see e.g., [15], Ch.1, Section 2). Hence it follows that if these systems are only orthogonal and

$$\lim_{k\to\infty} \sigma_k(\varphi_k, \varphi_k)(\psi_k, \psi_k) = 0 \ , \tag{21}$$

then A is compact. In this case, A^*A is also compact, and we can apply the results of Section 2 to the operator $K := A^*A$. By (13) and (19), the approximate solution u_m of (13) takes the form

$$u_m = \sum_{k=1}^{m} (\sigma_k^2(\psi_k, \psi_k)(\varphi_k, \varphi_k)^2)^{-1}(A^*f, \varphi_k)\varphi_k =$$

$$= \sum_{k=1}^{m} (\sigma_k^2(\psi_k, \psi_k)(\varphi_k, \varphi_k)^2)^{-1}(f, A\varphi_k)\varphi_k =$$

$$= \sum_{k=1}^{m} (\sigma_k(\psi_k, \psi_k)(\varphi_k, \varphi_k))^{-1}(f, \psi_k)\varphi_k \ .$$

This means that the set u_m, which is constructed for an approximate solution of equation (17), coincides with the m-th partial sum of the Moorie–Penrose generalized solution defined by (20). We take this fact into account and replace the operator K in Theorem 2 by A^*A, where A admits the singular value decomposition (15). In this case,

$$D((A^*A)^{-\infty}) \ni x \to \mathrm{Orb}((A^*A)^{-1}, x) = \{x, (A^*A)^{-1}x, \dots\} \in M \subset H^{\mathbb{N}},$$

$$(A^*A)^{-1}x = \{(A^*A)^{-1}x, (A^*A)^{-2}x, \dots\},$$

and

$$(A^*A)_{\infty} = ((A^*A)^{-\infty})^{-1}$$

is the onto isomorphism of the space $D((A^*A)^{-\infty})$. The space $D((A^*A)^{-\infty})$ is equipped with a sequence of energetic norms of the operator $(A^*A)_\infty$ which have the form

$$\{x\}_n = ((A^*A)_\infty x, x)_n^{1/2} = ((A^*Ax, x) + (x, (A^*A)^{-1}x)+ \\ + \cdots + ((A^*A)^{-n+2}x, (A^*A)^{-n+1}x),\ n \in \mathbb{N}.$$

With the aid of this sequence of norms we obtain the energetic Frechet space $E_{(A^*A)_\infty}$ of the operator $(A^*A)_\infty$. We come to

Theorem 3. *Let H and M be Hilbert spaces and A be an operator admitting the singular value decomposition* (15)*, where $\{\varphi_k\}$, $\{\psi_k\}$ are orthogonal systems in the spaces H and M, respectively, and the condition* (21) *be satisfied. Then $\varphi^s(I(f)) = \sum\limits_{k=1}^{m} (\sigma_k(\psi_k,\psi_k)(\varphi_k,\varphi_k))^{-1}(f,\psi_k)\varphi_k$ is the linear generalized spline and the generalized central algorithm for the solution operator $S = (A^*A)_\infty^{-1}$ and information $I(f) = [(f,\varphi_1),\dots,(f,\varphi_m)]$. Moreover, these approximate solutions converge to the solution of the equation* (12) (*in the sense of Moorie-Penrose*) *in the energetic space $E_{(A^*A)_\infty}$ of the operator $(A^*A)_\infty$, and also in the space $D((A^*A)^{-\infty})$.*

Problem. It is not known under what conditions the space $D((A^*A)^{-\infty})$ is nuclear, Montel, Schwartz or isomorphic to the space of rapidly converging sequences. In particular, what properties has the space $D((\mathfrak{R}^*\mathfrak{R})^{-\infty})$, where $\mathfrak{R}$ is the Radon transform.

3. A Linear Generalized Central Spline Algorithm of Computerized Tomography

The main problem of computerized tomography consists in reconstructing the function by means of its integrals over hyperplanes. This mathematical problem frequently arises in medicine, science, technology, and, in general, in situations in which the inner structure of an object is investigated by means of certain kinds of radiation.

Let us consider a beam of x-rays directed along the line L passing through some object. Let $f(t)$ be the factor of X-ray absorption by biotissues at a point t, and $I(t)$ be the beam intensity at t. A relative decrease of the beam intensity over a small distance Δt at the point t is $\Delta I/I = f(t)\Delta t$. If the initial intensity of the beam L is I_0 and its intensity after it passes through the body is I_1, then $I_1/I_0 = \exp\{-\int_L f(t)dt\}$. This means that as a result of scanning we obtain linear integrals along every line L. The problem is to reconstruct f by using the set of these integrals. When a spatial body is investigated, linear integrals are replaced by integrals along hyperplanes. The mapping $\mathfrak{R}$, which for a function f given on $\mathbb{R}^n$ is defined as the integrals of f along all hyperplanes, is called the Radon transform. In the previous paragraphs, by n we denoted the indices of seminorms of the

Frechet space. Below, by n we will denote the dimension of the Euclidean space $\mathbb{R}^n$. We hope that this does not lead us to any confusion in the sequel.

We study the problem of an approximate inversion of the Radon transform $\mathfrak{R}$ in the n-dimensional Euclidean space $\mathbb{R}^n$. We use the standard parametrization of a hyperplane by means of the normal unit vector ω and its distance s from the origin. The Radon transform $\mathfrak{R}$ maps the density function u onto its integrals over all hyperplanes and is defined by the formula

$$\mathfrak{R}u(\omega, s) = \int\limits_{(t,\omega)=s} u(t)dt = \int\limits_{\omega^{\perp}} u(s\omega + t)dt, \tag{22}$$

where $\omega \in \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \ |x| = 1\}$ and u belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing functions in $\mathbb{R}^n$. In case $n = 2$, the integrals in (22) is taken along straight lines. From this definition it follows that $\mathfrak{R}$ is an even function, namely, $\mathfrak{R}u(-\omega, -s) = \mathfrak{R}u(\omega, s)$ for all $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}^+$. By the well-known Schwartz theorem [20], $\mathfrak{R}$ is the one-to-one operator acting from $\mathscr{S}(\mathbb{R}^n)$ to the Schwartz space $\mathscr{S}(Z)$, where Z is the cylinder $Z = \mathbb{S}^{n-1} \times \mathbb{R}$.

It is an important fact that in addition to many practical areas the inverse of the Radon transform is used in computerized tomography. The Radon transform has been studied in a lot of scientific works. Here it is relevant to refer to the papers of F. Natterer [20], A. Louis [21]–[22], R. Dietz [23] and P. Maas [24].

The Radon transform, which is defined by (22) only for the functions belonging to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, admits a continuous extension in some weighted L_2-spaces. Let $W_\nu(x) = (1 - |x|^2)^{\nu - n/2}$ be the weight function defined in the unit ball $\Omega^n = \{x \in \mathbb{R}^n : \ |x| \le 1\}$ and let $w_\nu(s) = (1-s^2)^{\nu-1/2}, \ s \in [-1, 1]$ be the weight function defined on the cylinder Z. It is proved in [20] that the Radon transform $\mathfrak{R}$ is a continuous operator acting from the space $H := L_2(\Omega^n, W_\nu^{-1})$ to the space $M := L_2(Z, w_\nu^{-1})$, which are endowed with usual norms. This fact is also proved by A. Cormack in [25] for $\nu = 1, \ n = 2$. When $\nu > n/2 - 1$, the operator $\mathfrak{R}$ acting in these spaces admits a singular value decomposition with respect to the products of Gegenbauer polynomials and spherical harmonics, which is obtained by A. Louis [21]. The case $\nu = n/2$ has also been considered in [20].

We consider the problem of constructing a linear central spline algorithms for an approximate solution of the equation

$$\mathfrak{R}u = f, \tag{23}$$

where the Radon transform $\mathfrak{R}$ acts from the above-mentioned space H in M. Towards this end, we use the singular value decomposition of $\mathfrak{R}$ given in [23] and the results of Section 2.

First we need the following notation:

$P_m^{(\alpha,\beta)}$ is the Jacobi polynomial of order m and indices α, β; C_m^ν is the Gegenbauer polynomial of degree m and index ν; Γ is the first kind Euler integral;

$\{Y_{lk},\ k = 1, \dots, N(n,l)\}$ is the orthonormal basis of spherical functions defined on $\mathbb{S}^{n-1}$, where $l = 0, 1, \dots,$ and $N(n,l) = \frac{(2l+n-2)(n+l-3)!}{l!(n-2)!},\ n \geq 2$;

$$v_{mlk}^\nu(x) = W_\nu(x)|x|^l P_{(m-l)/2}^{(\nu-n/2, l+n/2-1)}(2|x|^2 - 1) Y_{lk}(x/|x|); \tag{24}$$

$u_{mlk}^\nu(\omega, s) = d_{ml} w_\nu(s) C_m^\nu(s) Y_{lk}(\omega)$, where

$$d_{ml} = \pi^{n/2-1} 2^{2\nu-1} \frac{\Gamma((m-l)/2 + \nu - n/2 + 1)\Gamma(m+1)\Gamma(\nu)}{\Gamma((m-l)/2+1)\Gamma(m+2\nu)}; \tag{25}$$

$$\sigma_{mlk}^2 = $$

$$= \frac{2^{2\nu}\Gamma((m+l)/2+\nu)\Gamma((m-l)/2+\nu-n/2+1)\Gamma(m+1)}{\pi^{1-n}\Gamma((m+l+n)/2)\Gamma((m-l)/2+1)\Gamma(m+2\nu)} = \sigma_{ml}^2; \tag{26}$$

Note that in (24)–(26), $P_0^{(\alpha,\beta)} \equiv 1,\ C_0^\lambda \equiv 1$ and $Y_{0k} \equiv 1$.

Proposition 1 ([23])**.** *The system* $\{v_{mlk}^\nu,\ u_{mlk}^\nu,\ \sigma_{ml}\}$, $m \geq 0$, $0 \leq l \leq m$, $k = 1, \dots, N(n,l)$, *where* v_{mlk}^ν , u_{mlk}, σ_{ml}, *are defined by* (24)–(26), *is a singular system for the Radon transform* $\mathfrak{R}$ *acting from* $L_2(\Omega^n, W_\nu^{-1})$ *to* $L_2(Z, w_\nu^{-1})$. *In other words,*

$$\mathfrak{R}u(\omega, s) = \sum_{m=0}^{\infty} {\sum_{l \leq m}}' \sigma_{ml} \sum_{k=1}^{N(n,l)} (u, v_{mlk}^\nu)_{L_2(\Omega^n, W_\nu^{-1})} \cdot u_{mlk}^\nu(\omega, s)$$

where Σ' *means that the summation is taken only for even* $m + l$.

Proposition 2. *If* $v_{mlk}^\nu, u_{mlk}^\nu, \sigma_{ml}$, $l \leq m$, $1 \leq k \leq N(n,l)$ *are represented by* (24)–(26), *then*

$$\lim_{m \to \infty} \sigma_{ml} \|u_{mlk}^\nu\| \cdot \|v_{mlk}^\nu\| = 0.$$

Proof. Using the well-known inequality (see e.g. [26], Ch. VI, Section 3)

$$\frac{\Gamma(m+\alpha+\beta)}{\Gamma(m+\alpha)} = O(m^\alpha),\ m \in N,\ \alpha > -1,\ \beta > -1, \tag{27}$$

after some calculations, we obtain

$$\sigma_{ml} = O(m^{(1-n)/2}), \tag{28}$$

$$d_{ml} = O(m^{(1-\nu-n/2)/2}). \tag{29}$$

Further, using (25), we have

$$\|u_{mlk}^\nu\|^2 = d_{ml}^2 \int_{S^{n-1}} Y_{lk}^2(\omega) d\omega \int_{-1}^{1} w_\nu(s)[C_m^\nu(s)]^2 ds. \tag{30}$$

Since the system $\{Y_{lk}\}$ is orthonormal on S^{n-1}, the first integral in (30) is equal to 1. If to the second integral we apply the equality ([27], 7.3–7.4)

$$\int_{-1}^{1} w_\nu(s)[C_m^\nu(s)]^2 ds = \frac{\pi 2^{1-2\nu}\Gamma(2\nu+m)}{m!(m+\nu)(\Gamma(\nu))^2} \,,$$

then by (27)–(30), we obtain

$$\|u_{mlk}^\nu\| = O(m^{(1-n)/2}). \tag{31}$$

Further, by (24),

$$\|v_{mlk}^\nu\|^2 = \int_{\Omega^n} W_\nu(x)|x|^{2l}[P_{(m-l)/2}^{(\nu-n/2,l+n/2-1)}(2|x|^2)-1]^2 Y_{lk}^2(x/|x|)dx =$$

$$= \int_{S^{n-1}} Y_{lk}^2(x/|x|)dS^{n-1} \int_0^1 (1-s^2)^{\nu-n/2}s^{2l}[P_{(m-l)/2}^{(\nu-n/2,l+n/2-1)}(2s^2-1)]s^{n-1}ds.$$

By the change of the variable $t=2s^2-1$, the last formula transforms to

$$\|v_{mlk}^\nu\|^2 = 2^{-\nu-n/2-l}\int_{-1}^{1}(1-t)^{\nu-n/2}(1+t)^{l+n/2-1}[P_{(m-l)/2}^{(\nu-n/2,l+n/2-1}(t))]^2 dt.$$

Performing some transformations of the integrand in the above equality, using the well-known formulas 7.3–7.4 from [27] and applying (27), we obtain

$$\|v_{mlk}^\nu\|^2 = \frac{2^{\nu+l}\Gamma(\nu-n/2+(m-l)/2+1)}{2^{\nu+n/2+l-1}(m-l)!(\nu-n/2+l+n/2-1+1+m-l)}$$

$$\frac{\Gamma(l+(n+m-l)/2)}{\Gamma(\nu-n/2+l+n/2-1+1+(m-l)/2)} = O(1). \tag{32}$$

By (28), (31) and (32), we conclude that Proposition 2 is proved. □

Summarizing the results obtained in this section and applying Theorem 3 to the equation $(\mathfrak{R}^*\mathfrak{R})_\infty u = f$, we come to

Theorem 4. *Let $\{v_{rlk}^\nu, u_{rlk}^\nu, \sigma_{rl}\}$, $l\le r$, $1\le k\le N(n,l)$ be a singular system for the Radon transform $\mathfrak{R}$ acting from $L_2(\Omega^n, W_\nu^{-1})$, $\nu>n/2-1$, to the space $L_2(Z,w_\nu^{-1})$. Then the algorithm*

$$\varphi^s(I(f))(x) = \sum_{r=0}^{m}\sum_{l\le r}{}'\sigma_{rl}\sum_{k=1}^{N(n,l)}(f,u_{rlk}^\nu)_{L_2(Z,w_\nu^{-1})}v_{rlk}^\nu(x),\ x\in\Omega^n, \tag{33}$$

where $\sum'$ means that the summation is taken only for even $m+l$, is the linear generalized spline and the generalized central algorithm for the solution operator $S=(\mathfrak{R}^*\mathfrak{R})_\infty^{-1}$ and nonadaptive information $I(f)=[(f,u_{001}^\nu),\dots,(f,u_{mmN(n,m)}^\nu)]$. Moreover, these approximate solutions tend to a solution *of equation* (23) (*in the sense of Moorie-Penrose*) *in the energetic space* $E_{(\mathfrak{R}^*\mathfrak{R})_\infty}$, *and also in the space* $D((\mathfrak{R}^*\mathfrak{R})^{-\infty})$.

We can rewrite (33) in the form

$$\varphi^s(I(f))(x) = W_\nu(x)\sum_{r=1}^{m} q_r(x),$$

where

$$q_r(x) = {\sum_{l\le r}}' h_{rl}|x|^l P_{(r-l)/2}^{(\nu-n/2,l+n/2-1)}(2|x|^2-1)Y_{lk}(x/|x|),$$

$$h_{rl} = d_{rl}\sigma_{rl}\sum_{k=1}^{N(n,l)}(f, w_\nu(s)C_r^\nu(s)Y_{lk}(\omega))_{L_2(Z,w_\nu^{-1})},$$

and $\sum'$ means that the summation is taken only for even $r+l$.

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Authors' address:

Niko Muskhelishvili Institute of Computational
Mathematics, Georgian Technical University
Georgia
E-mail: duglasugu@yahoo.com; zarnadzedavid@yahoo.com