Proceedings of A. Razmadze
Mathematical Institute
Vol. **161** (2013), 97–116

# NEW REFINEMENTS OF HÖLDER AND MINKOWSKI INEQUALITIES WITH WEIGHTS

L. HORVÁTH, K. ALI KHAN AND J. PEČARIĆ

Abstract. In this paper, we present on new refinements of the discrete Jensen's inequality given in [3] and [4]. Our results are more general than the refinement results given in [5]. Also the parameter dependent results correspond to some new refinements of Hölder's and Minkowski's inequalities.

**რეზიუმე.** სტატიაში მოცემულია [3] და [4] ნაშრომებში მიღებული დისკრეტული იენსენის უტოლობის დაზუსტებები. ამასთანავე, პარამეტრზე დამოკიდებული შედეგი შეესაბამება ჰელდერისა და მინკოვსკის უტოლობების ახალ დაზუსტებებს.

## 1. Introduction and Preliminary Results

The well known discrete Jensen's inequality says: Let $U$ be a convex subset of a real linear space, and let $f : U \to \mathbb{R}$ be a convex function. If $x_i \in U$ $(1 \le i \le n)$ and $p_i \ge 0$ $(1 \le i \le n)$ are such that $\sum\limits_{i=1}^{n} p_i = 1$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

holds.

Let $I \subset \mathbb{R}$ be an interval, let $h : I \to \mathbb{R}$ be a continuous and strictly monotone function, let $\mathbf{a} = (a_1, \ldots, a_n) \in I^n$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative $n$-tuple such that $\sum\limits_{i=1}^{n} p_i = 1$. The quasi-arithmetic $h$-mean of $\mathbf{a}$ with weights $\mathbf{p}$ is defined by

$$h_n(\mathbf{a};\mathbf{p}) = h_n(a_i; 1 \le i \le n; \mathbf{p}) = h(\mathbf{a};\mathbf{p};n) := h^{-1}\left(\sum_{i=1}^{n} p_i h(a_i)\right).$$

If $p_i = \frac{1}{n}$ $(1 \le i \le n)$, then $\mathbf{p}$ will be ignored from the previous notations.

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2010 *Mathematics Subject Classification.* Primary 26D07, 26D15, 26D20, 26D99.
*Key words and phrases.* Convex function, mixed symmetric means, function of several variables.

The following hypothesis is utilized in [5] to extend Beck's results (see [1]):

($A_1$) Let $L_t : I_t \to \mathbb{R}$ $(t = 1, \ldots, m)$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_1 \times \cdots \times I_m \to I_N$ be a continuous function. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \in \mathbb{R}^n$ $(n \geq 2)$ be such that $\mathbf{x}^{(t)} := (x_1^{(t)}, \ldots, x_n^{(t)}) \in I_t^n$ for each $t = 1, \ldots, m$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative $n$-tuple such that $\sum_{i=1}^n p_i = 1$.

The following extension of Beck's result, given in [5], is a simple consequence of the discrete Jensen's inequality.

**Theorem 1.1.** *Assume* ($A_1$). *If $N$ is an increasing function, then the inequality*

$$f\left(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\right) \geq$$
$$\geq N^{-1}\left(\sum_{i=1}^n p_i N(f(x_i^{(1)}, \ldots, x_i^{(m)}))\right), \tag{2}$$

*holds for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ and $\mathbf{p}$, if and only if the function $H$ defined on $L_1(I_1) \times \cdots \times L_m(I_m)$ by*

$$H(t_1, \ldots, t_m) := N\left(f\left(L_1^{-1}(t_1), \ldots, L_m^{-1}(t_m)\right)\right)$$

*is concave. The inequality in* (2) *is reversed for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ and $\mathbf{p}$, if and only if $H$ is convex.*

Beck's original result was the special case of Theorem 1.1, where $m = 2$ and $I_1 = [k_1, k_2]$, $I_2 = [l_1, l_2]$ and $I_N = [n_1, n_2]$ (see [2], p. 249).

In the case $m = 2$ we shall use the following simplified form of ($A_1$):

($A_2$) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_K \times I_L \to I_N$ be a continuous function. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative $n$-tuple such that $\sum_{i=1}^n p_i = 1$.

Then (2) has the form

$$f\big(K_n(\mathbf{a}; \mathbf{p}), L_n(\mathbf{b}; \mathbf{p})\big) \geq N_n(f(\mathbf{a}, \mathbf{b}); \mathbf{p}), \tag{3}$$

where $f(\mathbf{a}, \mathbf{b}) := (f(a_1, b_1), \ldots, f(a_n, b_n))$.

The following results (see [5]) are important special cases of Theorem 1.1, and generalize the corresponding results of Beck [5]. The next hypothesis will be used:

($A_3$) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$ such that either $I_K + I_L \subset I_N$ and $f(x, y) = x + y$ $((x, y) \in I_K \times I_L)$ or $I_K, I_L \subset ]0, \infty[$, $I_K \cdot I_L \subset I_N$ and $f(x, y) = xy$ $((x, y) \in I_K \times I_L)$. Assume further that the functions $K$, $L$ and $N$ are twice continuously differentiable on the interior of their domains, respectively. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ be such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative $n$-tuple such that $\sum_{i=1}^n p_i = 1$.

$A^\circ$ means the interior of $A \subset \mathbb{R}$.

**Corollary 1.2.** *Assume* $(\mathrm{A}_3)$ *with* $f(x,y) = x + y$ $((x,y) \in I_K \times I_L)$, *and assume that* $K'$, $L'$, $N'$, $K''$, $L''$ *and* $N''$ *are all positive. Introducing* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (3) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

**Corollary 1.3.** *Assume* $(\mathrm{A}_3)$ *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* $I_K^\circ$, $I_L^\circ$ *and* $I_N^\circ$, *respectively. Assume further that* $K'$, $L'$, $N'$, $A$, $B$ *and* $C$ *are all positive. Then* (3) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

In [3], Mitrinović and Pečarić obtained a new inequality like (3), which is based on the following refinement of the discrete Jensen's inequality (see Pečarić and Volenec [9]):

**Lemma A.** *Let* $f$ *be a real valued convex function defined on a convex set* $U$ *from a real linear space. If* $x_1, \ldots, x_n \in U$, *and*

$$f_{k,n} = f_{k,n}(x_1, \ldots, x_n) := \\ = \begin{pmatrix} n \\ k \end{pmatrix}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{1}{k}\left(x_{i_1} + \cdots + x_{i_k}\right)\right), \quad 1 \leq k \leq n, \tag{4}$$

*then*

$$f\left(\sum_{i=1}^n \frac{1}{n} x_i\right) = f_{n,n} \leq \cdots \leq f_{k,n} \leq \cdots \leq f_{1,n} = \frac{1}{n} \sum_{i=1}^n f(x_i). \tag{5}$$

Assume $(\mathrm{A}_2)$. We denote by $\alpha_i^k$ $(1 \leq i \leq v)$ and $\beta_i^k$ $(1 \leq i \leq v)$ the $k$-tuples of $\mathbf{a}$ and $\mathbf{b}$ respectively, where $v = \begin{pmatrix} n \\ k \end{pmatrix}$. Following [7], we introduce the mixed $N$-$K$-$L$ means of $\mathbf{a}$ and $\mathbf{b}$:

$$M(N,K,L;k) := N_v\big(f(K_k(\alpha_i^k), L_k(\beta_i^k));\, 1 \leq i \leq v\big), \quad 1 < k < n,$$

and

$$M(N,K,L;1) := N_n(f(\mathbf{a}, \mathbf{b})), \\ M(N,K,L;n) := f\left(K_n(\mathbf{a}), L_n(\mathbf{b})\right).$$

These means are studied in [7] (see also [8] page 195):

**Theorem A.** *Assume* $(A_2)$. *Let* $N$ *be an increasing* (*decreasing*) *function, and let*

$$H : K(I_K) \times L(I_L) \to \mathbb{R}, \quad H(s,t) := N\left(f\left(K^{-1}(s), L^{-1}(t)\right)\right)$$

*be a convex* (*concave*) *function. Then*

$$M(N,K,L;k+1) \leq M(N,K,L;k), \quad k = 1,\dots,n-1. \tag{6}$$

*If* $N$ *is increasing* (*decreasing*) *but* $H$ *is concave* (*convex*) *then the inequalities in* (6) *are reversed.*

In analogy of Corollary 1.2 and Corollary 1.3, the following consequences of Theorem A are given in [5, 7, 8].

**Corollary A.** *Assume* $(A_3)$ *with* $f(x,y) = x + y$ $((x,y) \in I_K \times I_L)$. *Assume further that* $K'$, $L'$, $N'$, $K''$, $L''$ *and* $N''$ *are all positive and* $E(x) + F(y) \leq G(x+y)$ $((x,y) \in I_K^\circ \times I_L^\circ)$, *where* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$. *Then* (6) *with reverse inequality is valid.*

**Corollary B.** *Assume* $(A_3)$ *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* $I_K^\circ$, $I_L^\circ$ *and* $I_N^\circ$, *respectively. If* $K'$, $L'$, $M'$, $A$, $B$ *and* $C$ *are all positive and* $A(x) + B(y) \leq C(xy)$ $((x,y) \in I_K^\circ \times I_L^\circ)$, *then* (6) *with reverse inequality is valid.*

The results given in [7] are without weights. By using the refinement of the discrete Jensen's inequality from [6], we gave results in [5] with weights, which cause the improvement of the results in [7]. But in this paper we work on the refinement given in [3] to establish the generalizations of the corresponding results given in [5]. Also we present some parameter dependent refinements of Hölder and Minkowski's inequalities with the help of [4]. First, we give the notations from [3]:

Let $X$ be a set. The power set of $X$ is denoted by $P(X)$. $|X|$ means the number of elements in $X$. For every nonnegative integer $d$, let

$$P_d(X) := \left\{Y \subset X \mid |Y| = d\right\}.$$

In the sequel we also need the following hypotheses:

$(H_1)$ Let $U$ be a convex set in $\mathbb{R}^m$, $\mathbf{x}_1, \dots, \mathbf{x}_n \in U$.

$(H_2)$ Let $\mathbf{p} := (p_1, \dots, p_n)$ be a positive $n$-tuple such that $\sum_{i=1}^n p_i = 1$.

$(H_3)$ Let $f : U \to \mathbb{R}$ be a convex function.

$(H_4)$ Let $S_1, \dots, S_n$ be finite, pairwise disjoint and nonempty sets, let

$$S := \bigcup_{j=1}^{n} S_j,$$

and let $c$ be a function from $S$ into $\mathbb{R}$ such that

$$c(s) > 0, \quad s \in S, \quad \text{and} \quad \sum_{s \in S_j} c(s) = 1, \quad j = 1, \ldots, n.$$

Let the function $\tau : S \to \{1, \ldots, n\}$ be defined by

$$\tau(s) := j, \quad \text{if} \quad s \in S_j.$$

($H_5$) Suppose $\mathcal{A} \subset P(S)$ is a partition of $S$ into pairwise disjoint and nonempty sets. Let

$$k := \max \left\{ |A| \mid A \in \mathcal{A} \right\},$$

and let

$$\mathcal{A}_l := \left\{ A \in \mathcal{A} \mid |A| = l \right\}, \quad l = 1, \ldots, k.$$

(We note that $\mathcal{A}_l$ $(l = 1, \ldots, k-1)$ may be the empty set, and of course, $|S| = \sum\limits_{l=1}^{k} l\,|\mathcal{A}_l|$.) The empty sum of numbers or vectors is taken to be zero.

The following refinement of the discrete Jensen's inequality is developed in [3]:

**Theorem B.** *If* ($H_1$)–($H_5$) *are satisfied, then*

$$f\left(\sum_{j=1}^{n} p_j \mathbf{x}_j\right) \le M_k \le M_{k-1} \le \cdots \le M_2 \le M_1 = \sum_{j=1}^{n} p_j f(\mathbf{x}_j),$$

*where*

$$M_k := \sum_{l=1}^{k} \left( \sum_{A \in \mathcal{A}_l} \left( \left( \sum_{s \in A} c(s) p_{\tau(s)} \right) f\left( \frac{\sum\limits_{s \in A} c(s) p_{\tau(s)} \mathbf{x}_{\tau(s)}}{\sum\limits_{s \in A} c(s) p_{\tau(s)}} \right) \right) \right), \tag{7}$$

*and for every* $1 \le d \le k-1$ *the number* $M_{k-d}$ *is given by*

$$M_{k-d} := \sum_{l=1}^{d} \left( \sum_{A \in \mathcal{A}_l} \left( \sum_{s \in A} c(s) p_{\tau(s)} f(\mathbf{x}_{\tau(s)}) \right) \right) + \sum_{l=d+1}^{k} \left( \frac{d!}{(l-1)\ldots(l-d)} \cdot \right.$$

$$\left. \cdot \sum_{A \in \mathcal{A}_l} \left( \sum_{B \in P_{l-d}(A)} \left( \left( \sum_{s \in B} c(s) p_{\tau(s)} \right) f\left( \frac{\sum\limits_{s \in B} c(s) p_{\tau(s)} \mathbf{x}_{\tau(s)}}{\sum\limits_{s \in B} c(s) p_{\tau(s)}} \right) \right) \right) \right). \tag{8}$$

A parameter dependent refinement of the discrete Jensen's inequality is obtained in [4].

**Theorem C.** For any real number $\lambda \ge 1$, we suppose ($H_1$)–($H_3$) and consider the sets

$$T_k := \left\{ (i_1, \ldots, i_n) \in \mathbb{N}^n \,\middle|\, \sum_{j=1}^{n} i_j = k \right\}, \quad k \in \mathbb{N}. \tag{9}$$

Let

$$C_k(\lambda) = C_k(\mathbf{x}_1, \dots, \mathbf{x}_n; p_1, \dots, p_n; \lambda) :=$$

$$= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\dots,i_n)\in S_k} \frac{k!}{i_1! \dots i_n!} \left( \sum_{j=1}^{n} \lambda^{i_j} p_j \right) f \left( \frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j \mathbf{x}_j}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right), \quad (10)$$

for any $k \in \mathbb{N}$. Then

$$f\left( \sum_{j=1}^{n} p_j \mathbf{x}_j \right) = C_0(\lambda) \le C_1(\lambda) \le \dots \le C_k(\lambda) \le \dots \le \sum_{j=1}^{n} p_j f(\mathbf{x}_j), \quad k \in \mathbb{N}.$$

## 2. New Generalizations of Beck's Result

Assume $(A_1)$ with positive $n$-tuple $\mathbf{p}$, $(H_4)$ and $(H_5)$. Let

$$L_t(\mathbf{x}^{(t)}; c\mathbf{p}; B) = L_t^{-1} \left( \frac{\sum\limits_{s\in B} c(s) p_{\tau(s)} L_t(x^{(t)}_{\tau(s)})}{\sum\limits_{s\in B} c(s) p_{\tau(s)}} \right),$$

$$t = 1, \dots, m, \quad B \subset S,$$

and let

$$\mathbf{x}_i := \left( x_i^{(1)}, \dots, x_i^{(m)} \right), \quad i = 1, \dots, n.$$

Then weighted mixed means corresponding to (7) and (8) are defined in the following ways:

$$M_k^1 := M_k^1(L_1, \dots, L_m; \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}; c\mathbf{p}) :=$$

$$= N^{-1} \Bigg( \sum_{l=1}^{k} \Bigg( \sum_{A\in\mathcal{A}_l} \Bigg( \left( \sum_{s\in A} c(s) p_{\tau(s)} \right) \cdot$$

$$\cdot N\Big( f\big( L_1(\mathbf{x}^{(1)}; c\mathbf{p}; A), \dots, L_m(\mathbf{x}^{(m)}; c\mathbf{p}; A) \big) \Big) \Bigg) \Bigg) \Bigg),$$

and for $1 \le d \le k-1$

$$M_{k-d}^1 : = M_{k-d}^1 \big( L_1, \dots, L_m; \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}; c\mathbf{p} \big) :=$$

$$= N^{-1} \Bigg\{ \sum_{l=1}^{d} \left( \sum_{A\in\mathcal{A}_l} \left( \sum_{s\in A} c(s) p_{\tau(s)} N(f(\mathbf{x}_{\tau(s)})) \right) \right) +$$

$$+ \sum_{l=d+1}^{k} \Bigg( \frac{d!}{(l-1)\dots(l-d)} \sum_{A\in\mathcal{A}_l} \Bigg( \sum_{B\in P_{l-d}(A)} \Bigg( \left( \sum_{s\in B} c(s) p_{\tau(s)} \right) \cdot$$

$$\cdot N\Big(f\big(L_1(\mathbf{x}^{(1)}; c\mathbf{p}; B), \ldots, L_m(\mathbf{x}^{(m)}; c\mathbf{p}; B))\big)\Big)\Big)\Big)\Big\}.$$

Now, we get an interpolation of (2) by the direct application of Theorem B as follows.

**Theorem 2.1.** *Assume* (A$_1$) *with a positive $n$-tuple* $\mathbf{p}$*,* $(H_4)$ *and* $(H_5)$*. If $N$ is a strictly increasing* (*decreasing*) *function, then the inequalities*

$$f\left(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\right) \leq M_k^1 \leq M_{k-1}^1 \leq \cdots \leq$$
$$\leq M_2^1 \leq M_1^1 = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(\mathbf{x}_i))\right), \tag{11}$$

*hold for all possible* $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$*, if and only if the function $H$ defined in Theorem* 1.1 *is convex* (*concave*)*. If $N$ is a strictly increasing* (*decreasing*) *function, then the inequalities in* (11) *are reversed for all possible* $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$*, if and only if $H$ is concave* (*convex*)*.*

*Proof.* It follows from Theorem B and Theorem 1.1. We apply Theorem B to $m$-tuples

$$\left(L_1\left(x_i^{(1)}\right), \ldots, L_1\left(x_i^{(m)}\right)\right), \quad i = 1, \ldots, n,$$

and the function $H$ if either $H$ is convex and $N$ is strictly increasing or $H$ is concave and $N$ is strictly decreasing. $-H$ is used if either $H$ is convex and $N$ is strictly decreasing or $H$ is concave and $N$ is strictly increasing. □

The following applications of Theorem 2.1 are based on special cases of Theorem B from [3].

**Example 2.2.** Let $n \geq 1$ and $k \geq 1$ be fixed integers, and let $I_k \subset \{1, \ldots, n\}^k$ such that

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n,$$

where $\alpha_{I_k,i}$ means the number of occurrences of $i$ in the sequences $\mathbf{i}_k := (i_1, \ldots, i_k) \in I_k$. For $j = 1, \ldots, n$ we introduce the sets

$$S_j := \Big\{ ((i_1, \ldots, i_k), l) \mid (i_1, \ldots, i_k) \in I_k, \quad 1 \leq l \leq k, \quad i_l = j \Big\}.$$

Let $c$ be a positive function on $S := \bigcup\limits_{j=1}^{n} S_j$ such that

$$\sum_{((i_1,\ldots,i_k),l) \in S_j} c((i_1, \ldots, i_k), l) = 1, \quad j = 1, \ldots, n.$$

Assume ($A_1$) with a positive $n$-tuple $\mathbf{p}$. Then the corresponding weighted mixed means are

$$M_k^1 := N^{-1}\left( \sum_{(i_1,\dots,i_k)\in I_k} \left( \left( \sum_{l=1}^{k} c\left(\left(i_1,\dots,i_k\right),l\right) p_{i_l} \right) \cdot \right.\right.$$
$$\left.\left. \cdot N\Big(f\big(L_1(\mathbf{x}^{(1)};c\mathbf{p};\mathbf{i}_k),\dots,L_m(\mathbf{x}^{(m)};c\mathbf{p};\mathbf{i}_k)\big)\Big)\right)\right),$$

where

$$L_t(\mathbf{x}^{(t)};c\mathbf{p};\mathbf{i}_k)=L_t^{-1}\left( \frac{\sum_{l=1}^{k} c\left(\left(i_1,\dots,i_k\right),l\right) p_{i_l} L_t(x_{i_l}^{(t)})}{\sum_{l=1}^{k} c\left(\left(i_1,\dots,i_k\right),l\right) p_{i_l}} \right),$$
$$\mathbf{i}_k\in I_k,\ 1\le t\le m,$$

while for $1\le d\le k-1$,

$$M_{k-d}^1 := N^{-1}\left\{ \left( \frac{d!}{(k-1)\dots(k-d)} \cdot \right.\right.$$
$$\cdot \sum_{(i_1,\dots,i_k)\in I_k} \left( \sum_{1\le l_1<\dots<l_{k-d}\le k} \left( \left( \sum_{j=1}^{k-m} c\big((i_1,\dots,i_k),l_j\big)p_{i_{l_j}} \right) \cdot \right.\right.$$
$$\left.\left.\left.\left. \cdot N\Big(f\big(L_1(\mathbf{x}^{(1)};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d}),\dots,L_m(\mathbf{x}^{(m)};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})\big)\Big)\right)\right)\right)\right\},$$

where

$$L_t(\mathbf{x}^{(t)};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d}) = L_t^{-1}\left( \frac{\sum_{j=1}^{k-d} c\big((i_1,\dots,i_k),l_j\big)p_{i_{l_j}} L_t(x_{i_{l_j}}^{(t)})}{\sum_{j=1}^{k-d} c\big((i_1,\dots,i_k),l_j\big)p_{i_{l_j}}} \right),$$
$$1\le l_1<\dots<l_{k-d}\le k,\quad 1\le t\le m.$$

If $N$ is strictly increasing and the function $H$ defined in Theorem 1.1 is convex, then Theorem 2.1 gives

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\dots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \le M_k^1 \le M_{k-1}^1 \le \dots \le$$
$$\le M_2^1 \le M_1^1 = N^{-1}\left( \sum_{i=1}^{n} p_i N(f(x_i^{(1)},\dots,x_i^{(m)})) \right). \tag{12}$$

Taking

$$c((i_1,\dots,i_k),l) = \frac{1}{|S_j|} = \frac{1}{\alpha_{I_k,j}},\quad ((i_1,\dots,i_k),l)\in S_j,$$

in (12) we get Theorem 2.1 of [5].

**Example 2.3.** Let $n$, $d$, $r$ be fixed integers, where $n \geq 3$, $d \geq 2$ and $1 \leq r \leq n-2$. In this example, for every $i = 1, 2, \ldots, n$ and for every $l = 0, 1, \ldots, r$ the integer $i+l$ will be identified with the uniquely determined integer $j$ from $\{1, \ldots, n\}$ for which

$$l + i \equiv j \quad (mod\ n). \tag{13}$$

Introducing the notation

$$D := \{1, \ldots, n\} \times \{0, \ldots, r\},$$

let for every $j \in \{1, \ldots, n\}$

$$S_j := \Big\{ (i,l) \in D \mid i + l \equiv j \quad (mod\ n) \Big\} \bigcup \{j\},$$

and let $\mathcal{A} \subset P(S)$ $\big(S := \bigcup\limits_{j=1}^{n} S_j\big)$ contain the following sets:

$$A_i := \big\{(i,l) \in D \mid l = 0, \ldots, r\big\}, \quad i = 1, \ldots, n$$

and

$$A := \{1, \ldots, n\}.$$

Let $c$ be a positive function on $S$ such that

$$\sum_{(i,l)\in S_j} c\,(i,l) + c\,(j) = 1, \quad j = 1, \ldots, n.$$

A careful verification shows that the sets $S_1, \ldots, S_n$, the partition $\mathcal{A}$ and the function $c$ defined above satisfy the conditions ($H_4$) and ($H_5$),

$$\tau\,(i,l) = i + l, \quad (i,l) \in D,$$

(by the agreement (see (13)), $i+l$ is identified with $j$)

$$\tau\,(j) = j, \qquad j = 1, \ldots, n,$$
$$|S_j| = r + 2, \quad j = 1, \ldots, n,$$

and

$$|A_i| = r + 1, \quad i = 1, \ldots, n, \quad |A| = n.$$

Assume ($A_1$) with a positive $n$-tuple $\mathbf{p}$. If $N$ is increasing and the function $H$ defined in Theorem 1.1 is convex, then from Theorem 2.1 we get

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n), \ldots, L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \leq$$
$$\leq N^{-1}\left\{ \sum_{i=1}^{n} \left( \sum_{l=0}^{r} c(i,l) p_{i+l} \right) N\Big(f\big(L_1(\mathbf{x}^{(1)}, c\mathbf{p}; i), \ldots, L_m(\mathbf{x}^{(m)}, c\mathbf{p}; i)\big)\Big) + \right.$$
$$\left. + \left( \sum_{j=1}^{n} c(j) p_j \right) N\Big(f\big(L_1(\mathbf{x}^{(1)}, c\mathbf{p}), \ldots, L_m(\mathbf{x}^{(m)}, c\mathbf{p})\big)\Big) \right\} \leq$$

$$\leq N^{-1}\left(\sum_{i=1}^{n} p_i N\Big(f\big(x_i^{(1)},\ldots,x_i^{(m)}\big)\Big)\right),$$

where

$$L_t(\mathbf{x}^{(t)}, c\mathbf{p}; i) = L_t^{-1}\left(\frac{\sum\limits_{l=0}^{r} c\,(i,l)\, p_{i+l} L_t(x_{i+l}^{(t)})}{\sum\limits_{l=0}^{r} c\,(i,l)\, p_{i+l}}\right),$$

$$1 \leq i \leq n, \quad 1 \leq t \leq m,$$

and

$$L_t(\mathbf{x}^{(t)}, c\mathbf{p}) = L_t^{-1}\left(\frac{\sum\limits_{j=1}^{n} c(j) p_j L_t(x_j^{(t)})}{\sum\limits_{j=1}^{n} c(j) p_j}\right), \quad 1 \leq t \leq m.$$

**Example 2.4.** Let $n$ and $k$ be fixed positive integers. Let

$$D := \Big\{ (i_1,\ldots,i_n) \in \{1,\ldots,k\}^n \mid i_1 + \cdots + i_n = n + k - 1 \Big\},$$

and for each $j = 1,\ldots,n$, denote $S_j$ the set

$$S_j := D \times \{j\}.$$

For every $\mathbf{i}_n := (i_1,\ldots,i_n) \in D$ designate by $A_{(i_1,\ldots,i_n)}$ the set

$$A_{(i_1,\ldots,i_n)} := \{((i_1,\ldots,i_n), l) \mid l = 1,\ldots,n\}.$$

It is obvious that $S_j$ $(j = 1,\ldots,n)$ and $A_{(i_1,\ldots,i_n)}$ $((i_1,\ldots,i_n) \in D)$ are decompositions of $S := \bigcup\limits_{j=1}^{n} S_j$ into pairwise disjoint and nonempty sets, respectively. Let $c$ be a function on $S$ such that

$$c\,((i_1,\ldots,i_n), j) > 0, \quad ((i_1,\ldots,i_n), j) \in S$$

and

$$\sum_{(i_1,\ldots,i_n)\in D} c\,((i_1,\ldots,i_n), j) = 1, \quad j = 1,\ldots,n.$$

In summary we have that the conditions $(\mathrm{H}_5)$ and $(\mathrm{H}_6)$ are valid, and

$$\tau\,((i_1,\ldots,i_n), j) = j, \quad ((i_1,\ldots,i_n), j) \in S.$$

Assume $(\mathrm{A}_1)$ with positive $n$-tuple $\mathbf{p}$. If $N$ is strictly increasing and the function $H$ defined in Theorem 1.1 is convex, then from Theorem 2.1 we get

$$f\Big(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\Big) \leq$$

$$\leq N^{-1}\left(\sum_{(i_1,\ldots,i_n)\in D}\left(\left(\sum_{l=1}^{n} c\big((i_1,\ldots,i_n), l\big) p_l\right)\right.\right.\cdot$$

$$\cdot N\Big(f\big(L_1(\mathbf{x}^{(1)}, c\mathbf{p}; \mathbf{i}_n), \ldots, L_m(\mathbf{x}^{(m)}, c\mathbf{p}; \mathbf{i}_n)\big)\Big)\Bigg) \le$$

$$\le N^{-1}\Big(\sum_{i=1}^{n} p_i N\big(f(x_i^{(1)}, \ldots, x_i^{(m)})\big)\Big),$$

where

$$L_t(\mathbf{x}^{(t)}, c\mathbf{p}; \mathbf{i}_n) = L_t^{-1}\left(\frac{\sum_{l=1}^{n} c\left((i_1, \ldots, i_n), l\right) p_l L_t(x_l^{(t)})}{\sum_{l=1}^{n} c\left((i_1, \ldots, i_n), l\right) p_l}\right),$$
$$\mathbf{i}_n \in D,\ 1 \le t \le m.$$

Now assume ($\mathrm{A}_1$), consider a real number $\lambda \ge 1$, and let $S_k$ be the set defined in (9). Then the mixed means corresponding to (10) are

$$M_k^2(\lambda) := M_k^2\big(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}; \lambda\big) :=$$
$$= N^{-1}\Bigg(\frac{1}{(n+\lambda-1^k} \sum_{i_1,\ldots,i_n \in S_k} \Bigg(\frac{k!}{i_1! \ldots i_n!}\Big(\sum_{j=1}^{n} \lambda^{i_j} p_j\Big) \cdot$$
$$\cdot N\Big(f\big(L_1(\mathbf{x}^{(1)}; \mathbf{p}; \mathbf{i}_{n,k}; \lambda), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; \mathbf{i}_{n,k}; \lambda)\big)\Big)\Bigg)\Bigg),$$

where

$$L_t(\mathbf{x}^{(t)}; \mathbf{p}; \mathbf{i}_{n,k}; \lambda) = L_t^{-1}\left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j L_t(x_j^{(t)})}{\sum_{j=1}^{n} \lambda^{i_j} p_j}\right),$$
$$\mathbf{i}_{n,k} \in S_k, \quad 1 \le t \le m.$$

In this case Theorem C gives another interpolation of (2) as follows:

**Theorem 2.5.** *Assume* ($\mathrm{A}_1$)*, let $\lambda \ge 1$ be a real number, and let $S_k$ be the set defined in* (9)*. If $N$ is a strictly increasing* (*decreasing*) *function, then the inequalities*

$$f\Big(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\Big) = M_0^2(\lambda) \le M_1^2(\lambda) \le \cdots \le$$
$$\le M_k^2(\lambda) \le \cdots \le N^{-1}\Big(\sum_{i=1}^{n} p_i N\Big(f\big(x_i^{(1)}, \ldots, x_i^{(m)}\big)\Big)\Big), \quad k \in \mathbb{N}, \tag{14}$$

*hold for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ and $\mathbf{p}$, if and only if the function $H$ defined in Theorem* 1.1 *is convex* (*concave*)*. If $N$ is an increasing* (*decreasing*) *function, then the inequalities in* (14) *are reversed for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ and $\mathbf{p}$, if and only if $H$ is concave* (*convex*).

*Proof.* Similar to the proof of Theorem 2.1. □

## 3. New Generalizations of the Consequences of Beck's Result

Assume ($A_2$) with positive $n$-tuple $\mathbf{p}$, ($H_4$) and ($H_5$). Then for $m = 2$, the reverse of (11) can be written as

$$f\big(K_n(\mathbf{a};\mathbf{p}), L_n(\mathbf{b};\mathbf{p})\big) \geq M_k^1 \geq M_{k-1}^1 \geq \cdots \geq M_1^1 = = N^{-1}\bigg(\sum_{j=1}^{n} p_j N\big(f(a_j,b_j)\big)\bigg). \tag{15}$$

Analogous to the results of Corollary A and Corollary B (see [7] and also [8], p. 195), we have immediately from Theorem 2.1 and Corollaries 1.2, 1.3 that

**Corollary 3.1.** *Assume* ($A_3$) *with* $f(x,y) = x+y$ $((x,y) \in I_K \times I_L)$ *and with positive $n$-tuple* $\mathbf{p}$*, assume* ($H_4$)*–*($H_5$)*, and assume that* $K'$*,* $L'$*,* $N'$*,* $K''$*,* $L''$ *and* $N''$ *are all positive. Introducing* $E := \frac{K'}{K''}$*,* $F := \frac{L'}{L''}$*,* $G := \frac{N'}{N''}$*,* (15) *holds for all possible* $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

*In this case*

$$M_k^1 := M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}) := N^{-1}\Bigg(\sum_{l=1}^{k}\Bigg(\sum_{A\in\mathcal{A}_l}\Bigg(\bigg(\sum_{s\in A} c(s)p_{\tau(s)}\bigg)\cdot \cdot N\big((K(\mathbf{a};c\mathbf{p};A) + L(\mathbf{b};c\mathbf{p};A))\big)\Bigg)\Bigg)\Bigg), \tag{16}$$

*and for* $1 \leq d \leq k-1$

$$\begin{aligned} M_{k-d}^1 &:= M_{k-d}^1\big(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}\big) := \\ &= N^{-1}\Bigg\{\sum_{l=1}^{d}\Bigg(\sum_{A\in\mathcal{A}_l}\bigg(\sum_{s\in A} c(s)p_{\tau(s)}N(a_{\tau(s)} + b_{\tau(s)})\bigg)\Bigg) + \\ &+ \sum_{l=d+1}^{k}\Bigg(\frac{d!}{l-1)\ldots(l-d)}\sum_{A\in\mathcal{A}_l}\Bigg(\sum_{B\in P_{l-d}}(A)\Bigg(\bigg(\sum_{s\in B} c(s)p_{\tau(s)}\bigg)\cdot \\ &\cdot N\Big(K(\mathbf{a};c\mathbf{p};B) + L(\mathbf{b};c\mathbf{p};B)\Big)\Bigg)\Bigg)\Bigg)\Bigg\}. \end{aligned} \tag{17}$$

**Corollary 3.2.** *Assume* ($H_4$)*,* ($H_5$) *and consider* ($A_3$) *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$ *and with positive $n$-tuple* $\mathbf{p}$*. Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$*,* $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* $I_K^\circ$*,* $I_L^\circ$ *and* $I_N^\circ$ *respectively. Assume further that* $K'$*,* $L'$*,* $M'$*,* $A$*,*

*$B$ and $C$ are all positive. Then* (15) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x)+B(y)\leq C(xy),\quad (x,y)\in I_K^\circ\times I_L^\circ.$$

*In this case*

$$M_k^1 := M_k^1\big(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}\big) := N^{-1}\left(\sum_{l=1}^{k}\left(\sum_{A\in\mathcal{A}_l}\left(\left(\sum_{s\in A}c(s)p_{\tau(s)}\right)\cdot\right.\right.\right.$$

$$\left.\left.\left.\cdot N\big(K(\mathbf{a};c\mathbf{p};A)L(\mathbf{b};c\mathbf{p};A)\big)\right)\right)\right),\tag{18}$$

*and for* $1\leq d\leq k-1$,

$$M_{k-d}^1 := M_{k-d}^1\big(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}\big) :=$$

$$= N^{-1}\left\{\sum_{l=1}^{d}\left(\sum_{A\in\mathcal{A}_l}\left(\sum_{s\in A}c(s)p_{\tau(s)}N(a_{\tau(s)}b_{\tau(s)})\right)\right)+\right.$$

$$+\sum_{l=d+1}^{k}\left(\frac{d!}{(l-1)\dots(l-d)}\sum_{A\in\mathcal{A}_l}\left(\sum_{B\in P_{l-d}(A)}\left(\left(\sum_{s\in B}c(s)p_{\tau(s)}\right)\cdot\right.\right.\right.$$

$$\left.\left.\left.\left.\cdot N\big(K(\mathbf{a};c\mathbf{p};B)L(\mathbf{b};c\mathbf{p};B)\big)\right)\right)\right)\right\}.\tag{19}$$

Under the considerations of examples in Section 2, we show some special cases of the Corollaries 3.1 and 3.2.

*Remark* 3.3. Under the settings of Example 2.2, if $f(x_1,x_2)=x_1+x_2$, then (16) becomes

$$M_k^1 := M_k^1\big(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}\big) :=$$

$$= N^{-1}\left(\sum_{(i_1,\dots,i_k)\in I_k}\left(\left(\sum_{l=1}^{k}c\big((i_1,\dots,i_k),l\big)p_{i_l}\right)\cdot\right.\right.$$

$$\left.\left.\cdot\ N\big(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k)+L(\mathbf{b};c\mathbf{p};\mathbf{i}_k)\big)\right)\right),$$

and for $1\leq d\leq k-1$ (17) becomes

$$M_{k-d}^1 := M_k^1\big(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}\big) :=$$

$$= N^{-1}\left(\left(\frac{d!}{(k-1)\dots(k-d)}\cdot\right.\right.$$

$$\cdot\sum_{(i_1,\dots,i_k)\in I_k}\left(\sum_{1\leq l_1<\dots<l_{k-d}\leq k}\left(\left(\sum_{j=1}^{k-m}c\big((i_1,\dots,i_k),l_j\big)p_{i_{l_j}}\right)\cdot\right.\right.$$

$$\cdot N\Big(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})+L(\mathbf{b};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})\Big)\Bigg)\Bigg)\Bigg).$$

Under the conditions of Corollary 3.1, we have

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{a};\mathbf{p})\geq M_k^1\geq M_{k-1}^1\geq\cdots\geq M_1^1=$$
$$=N^{-1}\left(\sum_{i=1}^{n}p_iN(a_i+b_i)\right). \quad (20)$$

Similarly, if $f(x_1,x_2)=x_1x_2$, then from (18) we have

$$M_k^1:=M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}):=$$
$$=N^{-1}\left(\sum_{(i_1,\ldots,i_k)\in I_k}\left(\left(\sum_{l=1}^{k}c((i_1,\ldots,i_k),l)p_{i_l}\right)\cdot\right.\right.$$
$$\left.\left.\cdot N\Big(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k)L(\mathbf{b};c\mathbf{p};\mathbf{i}_k)\Big)\right)\right),$$

and for $1\leq d\leq k-1$, we have from (19)

$$M_{k-d}^1:=M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}):=$$
$$=N^{-1}\left(\frac{d!}{(k-1)\ldots(k-d)}\cdot\right.$$
$$\cdot\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{1\leq l_1<\ldots<l_{k-d}\leq k}\left(\left(\sum_{j=1}^{k-m}c((i_1,\ldots,i_k),l_j)p_{i_{l_j}}\right)\cdot\right.\right.$$
$$\left.\left.\left.\cdot N\Big(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})L(\mathbf{b};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})\Big)\right)\right)\right).$$

Under the conditions of Corollary 3.2, we have

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{a};\mathbf{p})\geq M_k^1\geq M_{k-1}^1\geq\cdots\geq M_1^1=N^{-1}\left(\sum_{i=1}^{n}p_iN(a_ib_i)\right). \quad (21)$$

Taking

$$c((i_1,\ldots,i_k),l)=\frac{1}{|S_j|}=\frac{1}{\alpha_{I_k,j}},\quad ((i_1,\ldots,i_k),l)\in S_j,$$

in (20) and (21), we get Corollary 3.1 and Corollary 3.2 of [5], respectively.

*Remark* 3.4. We consider Example 2.3. If $f(x_1,x_2)=x_1+x_2$, then under the conditions of Corollary 3.1 we have

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p})\geq$$
$$\geq N^{-1}\left\{\sum_{i=1}^{n}\left(\sum_{l=0}^{r}c(i,l)p_{i+l}\right)N\Big(K_r(\mathbf{a},c\mathbf{p};i)+L_r(\mathbf{b},c\mathbf{p};i)\Big)+\right.$$

$$+\left(\sum_{j=1}^{n} c(j)p_j\right) N\big(K_n(\mathbf{a}; c\mathbf{p}) + L_n(\mathbf{b}; c\mathbf{p})\big)\Bigg\} \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i)\right).$$

Similarly, if $f(x_1, x_2) = x_1 x_2$, then under the conditions of Corollary 3.2 we have

$$K_n(\mathbf{a}; \mathbf{p})L_n(\mathbf{b}; \mathbf{p}) \geq$$
$$\geq N^{-1}\Bigg\{\sum_{i=1}^{n}\left(\sum_{l=0}^{r} c(i,l)p_{i+l}\right) N\big(K_r(\mathbf{a}; c\mathbf{p}; i)L_r(\mathbf{b}; c\mathbf{p}; i)\big)+$$
$$+\left(\sum_{j=1}^{n} c(j)p_j\right) N\big(K_n(\mathbf{a}; c\mathbf{p})L_n(\mathbf{b}; c\mathbf{p})\big)\Bigg\} \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i)\right).$$

*Remark* 3.5. We now consider Example 2.4. If $f(x_1, x_2) = x_1 + x_2$, then under the conditions of Corollary 3.1 we have

$$K_n(\mathbf{a}; \mathbf{p}) + L_n(\mathbf{b}; \mathbf{p}) \geq$$
$$\geq N^{-1}\left(\sum_{(i_1,\dots,i_n)\in D}\left(\left(\sum_{l=1}^{n} c((i_1,\dots,i_n),l)p_l\right)\cdot\right.\right.$$
$$\left.\left.\cdot N\big(K_n(\mathbf{a}; c\mathbf{p}, \mathbf{i}_n + L_n(\mathbf{b}; c\mathbf{p}, \mathbf{i}_n)\big)\right)\right) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i + b_i)\right).$$

Similarly, if $f(x_1, x_2) = x_1 x_2$, then under the conditions of Corollary 3.2 we have

$$K_n(\mathbf{a}; \mathbf{p})L_n(\mathbf{b}; \mathbf{p}) \geq$$
$$\geq N^{-1}\left(\sum_{(i_1,\dots,i_n)\in D}\left(\left(\sum_{l=1}^{n} c\big((i_1,\dots,i_n),l\big)p_l\right)\cdot\right.\right.$$
$$\left.\left.\cdot N\big(K_n(\mathbf{a}, c\mathbf{p}, \mathbf{i}_n)L_n(\mathbf{b}, c\mathbf{p}, \mathbf{i}_n)\big)\right)\right) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i\right).$$

Next, assume ($\mathrm{A}_2$), let $\lambda \geq 1$, and let $T_k$ be the set defined in (9). Then for $m = 2$, the reverse of (14) becomes

$$f\big(K_n(\mathbf{a}; \mathbf{p}), L_n(\mathbf{b}; \mathbf{p})\big) = M_0^2(\lambda) \geq M_1^2(\lambda) \geq \cdots \geq M_k^2(\lambda) \geq \cdots \geq$$
$$\geq N^{-1}\left(\sum_{i=1}^{n} p_i N\big(f(a_i, b_i)\big)\right), \quad k \in \mathbb{N}, \tag{22}$$

where

$$M_k^2(\lambda) := M_k^2(K, L; \mathbf{a}, \mathbf{b}; \mathbf{p}; \lambda) :=$$

$$= N^{-1}\Bigg(\frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\dots,i_n)\in S_k}\Bigg(\frac{k!}{i_1!\dots i_n!}\Bigg(\sum_{j=1}^{n}\lambda^{i_j}p_j\Bigg)\cdot$$
$$\cdot N\big(f(K_n(\mathbf{a};\mathbf{p};\mathbf{i}_{n,k};\lambda),L_n(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda))\big)\Bigg)\Bigg).$$

By using Theorem 2.5 (for $m=2$) and Corollaries 1.2, 1.3, we get parameter dependent generalizations of Beck's results.

**Corollary 3.6.** *Assume* $(A_3)$ *with* $f(x,y)=x+y$ $((x,y)\in I_K\times I_L)$, *let* $\lambda\geq 1$, *and let* $T_k$ *be the set defined in* (9). *Assume further that* $K'$, $L'$, $N'$, $K''$, $L''$ *and* $N''$ *are all positive. Introducing* $E:=\frac{K'}{K''}$, $F:=\frac{L'}{L''}$, $G:=\frac{N'}{N''}$, (22) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x)+F(y)\leq G(x+y),\quad (x,y)\in I_K^\circ\times I_L^\circ.$$

*In this case for* $k\in\mathbb{N}$, *we have*

$$M_k^2(\lambda):=M_k^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p};\lambda):=$$
$$= N^{-1}\Bigg(\frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\dots,i_n)\in S_k}\Bigg(\frac{k!}{i_1!\dots i_n!}\Bigg(\sum_{j=1}^{n}\lambda^{i_j}p_j\Bigg)\cdot$$
$$\cdot N\big(K_n(\mathbf{a};\mathbf{p};\mathbf{i}_{n,k};\lambda)+L_n(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda))\big)\Bigg)\Bigg).$$

**Corollary 3.7.** *Assume* $(A_3)$ *with* $f(x,y)=xy$ $((x,y)\in I_K\times I_L)$, *let* $\lambda\geq 1$, *and let* $T_k$ *be the set defined in* (9). *Suppose the functions* $A(x):=\frac{K'(x)}{K'(x)+xK''(x)}$, $B(x):=\frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x):=\frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* $I_K^\circ$, $I_L^\circ$ *and* $I_N^\circ$ *respectively. Assume further that* $K'$, $L'$, $M'$, $A$, $B$ *and* $C$ *are all positive. Then* (22) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x)+B(y)\leq C(xy),\quad (x,y)\in I_K^\circ\times I_L^\circ.$$

*In this case for* $k\in\mathbb{N}$, *we have*

$$M_k^2(\lambda):=M_k^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p};\lambda):=$$
$$= N^{-1}\Bigg(\frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\dots,i_n)\in S_k}\Bigg(\frac{k!}{i_1!\dots i_n!}\Bigg(\sum_{j=1}^{n}\lambda^{i_j}p_j\Bigg)\cdot$$
$$\cdot N\big(K_n(\mathbf{a};\mathbf{p};\mathbf{i}_{n,k};\lambda)L_n(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda))\big)\Bigg).$$

## 4. Generalization of Minkowski's Inequality

We need the following hypothesis:

$(A_4)$ Let $I$ be an interval in $\mathbb{R}$, and let $M:I\to\mathbb{R}$ be a continuous and strictly monotone function. Let $\mathbf{x}_i\in I^m$ $(i=1,\dots,n)$, let $\mathbf{p}=(p_1,\dots,p_n)$

be a positive $n$-tuple such that $\sum_{i=1}^{n} p_i = 1$, and let $\mathbf{w} = (w_1, \dots, w_m)$ be a nonnegative $m$-tuple such that $\sum_{i=1}^{m} w_i = 1$.

We give a generalization of the Minkowski's inequality by using Theorem B.

**Theorem 4.1.** *Assume* $(A_4)$, $(H_4)$ *and* $(H_5)$. *Further, assume that the quasi-arithmetic mean function*

$$\mathbf{x} \to M_m(\mathbf{x}; \mathbf{w}), \quad \mathbf{x} \in I^m \tag{23}$$

*is convex. Then*

$$M_m\left(\sum_{r=1}^{n} p_r \mathbf{x}_r; \mathbf{w}\right) \le A_k \le A_{k-1} \le \dots \le A_2 \le A_1 = \sum_{r=1}^{n} p_r M_m(\mathbf{x}_r; \mathbf{w}),$$

*where*

$$A_k := \sum_{l=1}^{k} \left( \sum_{A \in \mathcal{A}_l} \left( \left( \sum_{s \in A} c(s) p_{\tau(s)} \right) M_m \left( \frac{\sum\limits_{s \in A} c(s) p_{\tau(s)} \mathbf{x}_{\tau(s)}}{\sum\limits_{s \in A} c(s) p_{\tau(s)}}; \mathbf{w} \right) \right) \right), \tag{24}$$

*and for* $1 \le d \le k-1$

$$\begin{aligned} A_{k-d} :&= \sum_{l=1}^{d} \left( \sum_{A \in \mathcal{A}_l} \left( \sum_{s \in A} c(s) p_{\tau(s)} M_m(\mathbf{x}_{\tau(s)}; \mathbf{w}) \right) \right) + \\ &+ \sum_{l=d+1}^{k} \left( \frac{d!}{(l-1)\dots(l-d)} \cdot \sum_{A \in \mathcal{A}_l} \left( \sum_{B \in P_{l-d}(A)} \left( \left( \sum_{s \in B} c(s) p_{\tau(s)} \right) \cdot \right.\right.\right. \\ &\left.\left.\left. \cdot M_m \left( \frac{\sum\limits_{s \in B} c(s) p_{\tau(s)} \mathbf{x}_{\tau(s)}}{\sum\limits_{s \in B} c(s) p_{\tau(s)}; \mathbf{w}} \right) \right) \right) \right). \end{aligned} \tag{25}$$

*Proof.* We apply Theorem B to the convex function $M_m(\cdot; \mathbf{w})$ and the vectors $\mathbf{x}_i$ $(i = 1, \dots, n)$. We get $A_d$ $(k \ge d \ge 1)$ in (24) and (25) from (7) and (8) respectively. □

Similarly, by using Theorem C we get

**Theorem 4.2.** *Let* $\lambda \ge 1$ *be a real number, assume* $(A_4)$ *and suppose* $T_k$ $(k \in \mathbb{N})$ *is the set given in* (9). *If the quasi-arithmetic mean function* 23 *is convex, then*

$$\begin{aligned} M_m\left(\sum_{r=1}^{n} p_r \mathbf{x}_r; \mathbf{w}\right) = C_0(\lambda) \le C_1(\lambda) \le \dots \le C_k(\lambda) \le \dots \le \\ \le \sum_{r=1}^{n} p_r M_m(\mathbf{x}_r; \mathbf{w}), \quad k \in \mathbb{N}, \end{aligned}$$

*where*

$$C_k(\lambda) = C_k(\mathbf{x}_1, \dots, \mathbf{x}_n; p_1, \dots, p_n; \lambda) :=$$
$$= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\dots,i_n)\in S_k} \frac{k!}{i_1!\dots i_n!}\left(\sum_{j=1}^{n} \lambda^{i_j} p_j\right)\cdot$$
$$\cdot M_m\left(\frac{\sum_{j=1}^{n}\lambda^{i_j} p_j \mathbf{x}_j}{\sum_{j=1}^{n}\lambda^{i_j} p_j}; \mathbf{w}\right), \quad k \in \mathbb{N}.$$

The following result gives a necessary and sufficient condition for the quasi-arithmetic mean function to be convex (see [8], p. 197):

**Theorem D.** *If $M : [m_1, m_2] \to \mathbb{R}$ has continuous derivatives of second order and it is strictly increasing and strictly convex, then the quasi-arithmetic mean function $M_m(\cdot; w)$ is convex if and only if $M'/M''$ is a concave function.*

($A_5$) Let $M :]0,\infty[\to]0,\infty[$ be a continuous and strictly monotone function such that $\lim_{x\to 0} M(x) = \infty$ or $\lim_{x\to\infty} M(x) = \infty$. Let $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{w} = (w_1, \dots, w_m)$ be positive $m$-tuples such that $w_i \geq 1$ $(i = 1, \dots, m)$. Let $\mathbf{p} = (p_1, \dots, p_n)$ be a positive $n$-tuple such that $\sum_{i=1}^{n} p_i = 1$.

Then we define

$$\widetilde{M}_m(\mathbf{x}; \mathbf{w}) = M^{-1}\left(\sum_{i=1}^{m} w_i M(x_i)\right). \tag{26}$$

The following result is also given in ([8], page 197):

**Theorem E.** *If $M :]0,\infty[\to]0,\infty[$ has continuous derivatives of second order and it is strictly increasing and strictly convex, then $\widetilde{M}_m(\cdot; w)$ is a convex function if $M/M'$ is a convex function.*

By using (26) we have

**Theorem 4.3.** *Assume* ($A_5$) *and let*

$$\mathbf{x} \to \widetilde{M}_m(\mathbf{x}; \mathbf{w}), \quad \mathbf{x} \in ]0,\infty[^m$$

*be a convex function.*

(a) *Consider* ($H_4$) *and* ($H_5$). *Then Theorem* 4.1 *remains valid for* $\widetilde{M}_m(\mathbf{x}; \mathbf{w})$ *instead of* $M_m(\mathbf{x}; \mathbf{w})$.

(b) *Consider $\lambda \in \mathbb{R}$ such that $\lambda \geq 1$ and suppose $T_k$ $(k \in \mathbb{N})$ is the set defined in* (9). *Then Theorem* 4.2 *also remains valid for* $\widetilde{M}_m(\mathbf{x}; \mathbf{w})$ *instead of* $M_m(\mathbf{x}; \mathbf{w})$.

*Remark* 4.4. All special cases (as given in Section 2) can also be considered for Theorem 4.1, Theorem 4.2 and Theorem 4.3.

## Acknowledgement

This research was partially funded by Higher Education Commission, Pakistan. The research of the first author was supported by the Hungarian National Foundation for Scientific Research, Grant No. K101217, and research of the third author was supported by the Croatian Ministry of Science, Education and Sports under Research Grant 117-1170889-0888.

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(Received 16.05.2012; revised 2.12.2012)

Authors' addresses:

L. Horváth

Department of Mathematics, University of Pannonia
Egyetem u.10, 8200, Veszprém, Hungary
E-mail: lhorvath@almos.uni-pannon.hu

K. Ali Khan

Abdus Salam School of Mathematical Sciences, GC University
68-B, New Muslim, Town, Lahore 54600, Pakistan,
Department of mathematics, University of Sargodha
Sargodha 40100, Pakistan
E-mail: Khuramsms@gmail.com

J. Pečarić

Abdus Salam School of Mathematical Sciences, GC University
68-B, New Muslim, Town, Lahore 54600, Pakistan,
Faculty of Textile Technology, University of Zagreb, 10000 Zagreb
Croatia
E-mail: pecaric@mahazu.hazu.hr