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STEIN-WEISS INEQUALITIES FOR THE RIESZ POTENTIAL ON THE LAGUERRE HYPERGROUP

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Abstract. Let $\mathbb{K} = [0,\infty) \times \mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group, $|\cdot|$ its homogeneous norm and Q its homogeneous dimension. In this paper we study the Riesz potential operator $\mathcal{I}_\beta$, the fractional integral operator I_β and its modified version $\widetilde{I}_\beta$ in weighted Lebesgue spaces on $\mathbb{K}$, with weights of the form $|(x,t)|^\mu$. Necessary and sufficient conditions on the parameters for the boundedness of $\mathcal{I}_\beta$ and I_β from the spaces $L_{p,|\cdot|^\mu}(\mathbb{K})$ to $L_{q,|\cdot|^{-\lambda}}(\mathbb{K})$ for $1 < p \le q < \infty$, and from the spaces $L_{1,|\cdot|^\mu}(\mathbb{K})$ to the weak spaces $WL_{q,|\cdot|^{-\lambda}}(\mathbb{K})$ for $1 < q < \infty$ are proved. Moreover, in the limiting case $p = \frac{Q}{\beta-\mu-\lambda}$ conditions for the boundedness of the operator $\widetilde{I}_\beta$ acting from $L_{p,|\cdot|^\mu}(\mathbb{K})$ into $BMO_{|\cdot|^{-\lambda}}(\mathbb{K})$ are given.

რეზიუმე. ვთქვათ, $\mathbb{K} = [0,\infty) \times \mathbb{R}$ არის ლაგერის ჰიპერჯგუფი, რომელიც წარმოადგენს ჰეიზენბერგის ჯგუფებზე რადიალურ ფუნქციათა სივრცის ფუნდამენტურ მრავალნაირობას, $|\cdot|$ არის მისი ნორმა და Q კი აღნიშნავს მის განზომილებას. სტატიაში შესწავლილია სხვადასხვა ტიპის წილადური ინტეგრალები $\mathbb{K}$-ზე განსაზღვრული ლებეგის სივრცეებში $|(x,t)|^\mu$ სახის წონებით. დადგენილია ხსენებული ოპერატორების შემოსაზღვრულობისა და სუსტი ტიპის უტოლობების მართებულობის პირობები. განხილულია აგრეთვე ზღვრული შემთხვევა.

1. Introduction and Main Results

Let $\alpha \ge 0$ be a fixed number, $\mathbb{K} = [0,\infty) \times \mathbb{R}$ and m_α be the weighted Lebesgue measure on $\mathbb{K}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}, \quad \alpha \ge 0. \tag{1}$$

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The dilations on $\mathbb{K}$ are defined by

$$\delta_r(x,t) = (rx, r^2 t), \quad r > 0.$$

It is clear that the dilations are consistent with the structure of hypergroup. Note that $Q = 2\alpha + 4$ called the homogeneous dimension of Laguerre hypergroup and $dm_\alpha\big(\delta_r(x,t)\big) = r^Q\, dm_\alpha(x,t)$. We also have a homogeneous norm defined by $|(x,t)| = (x^4 + 4t^2)^{1/4}$, $(x,t) \in \mathbb{K}$. Then we can defined the ball centered at (x,t) of radius r, i.e., the set $B_r(x,t) = \{(y,s) \in \mathbb{K} : |(x-y, t-s)| < r\}$, $B_r = B_r(0,0)$, and by ${}^{\complement}B_r(x,t)$ denote its complement, i.e., the set ${}^{\complement}B_r(x,t) = \{(y,s) \in \mathbb{K} : |(x-y, t-s)| \geq r\}$. For any measurable set $E \subset \mathbb{K}$, let $m_\alpha(E) = \int_{\mathbb{K}} dm_\alpha(x,t)$.

For every $1 \leq p \leq \infty$, we denote by $L_p(\mathbb{K}) = L_p(\mathbb{K}; dm_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{K}$ such that,

$$\|f\|_{L_p(\mathbb{K})} = \Big(\int\limits_{\mathbb{K}} |f(x,t)|^p\, dm_\alpha(x,t) \Big)^{1/p} < \infty \quad \text{if} \quad p \in [1,\infty),$$

and

$$\|f\|_{L_\infty(\mathbb{K})} = \operatorname*{ess\,sup}_{(x,t)\in\mathbb{K}} |f(x,t)| \quad \text{if} \quad p = \infty.$$

The weak $L_p(\mathbb{K})$ spaces $WL_p(\mathbb{K})$, $1 \leq p < \infty$ is defined as the set of locally integrable functions $f(x,t)$, $(x,t) \in \mathbb{K}$ with the finite norm

$$\|f\|_{WL_p(\mathbb{K})} = \sup_{r>0} r \Big(m_\alpha \left\{ (x,t) \in \mathbb{K} \ : \ |f(x,t)| > r \right\} \Big)^{1/p}.$$

Let w be a weight function on $\mathbb{K}$, i.e., w is a non-negative and measurable function on $\mathbb{K}$, then for all measurable functions f on $\mathbb{K}$ the weighted Lebesgue space $L_{p,w}(\mathbb{K})$ and the weak weighted Lebesgue space $WL_{p,w}(\mathbb{K})$ are defined by

$$L_{p,w}(\mathbb{K}) = \{ f : \|f\|_{L_{p,w}(\mathbb{K})} = \|wf\|_{L_p(\mathbb{K})} < \infty \}$$

and

$$WL_{p,w}(\mathbb{K}) = \{ f : \|f\|_{WL_{p,w}(\mathbb{K})} = \|wf\|_{WL_p(\mathbb{K})} < \infty \},$$

respectively.

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. We consider on the Laguerre hypergroup the following partial differential operator

$$\mathcal{L} = -\Big(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x} \frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial t^2} \Big).$$

$\mathcal{L}$ is positive and symmetric in $L_2(\mathbb{K})$, and is homogeneous of degree 2 with respect to the dilations defined above. When $\alpha = n-1$, $n \in \mathbb{N} \backslash \{0\}$, the operator $\mathcal{L}$ is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$. We call $\mathcal{L}$ the generalized sublaplacian.

The potential and related topics in Laguerre hypergroup have been the research interests of many mathematicians such as Miloud Assal and Hacen Ben Abdallah [1], Miloud Assal and V.S.Guliyev [10], V. S. Guliyev and M. N. Omarova [11, 12], M.M. Nessibi and K. Trimeche [17] and others.

For $(x,t),(y,s)\in\mathbb{K}$ and $\theta\in[0,2\pi[$, $r\in[0,1]$ let

$$((x,t),(y,s))_{\theta,r}=\Big(\big(x^2+y^2+2xyr\cos\theta\big)^{1/2},t+s+xyr\sin\theta\Big).$$

The generalized translation operators $T^{(\alpha)}_{(x,t)}$ on the Laguerre hypergroup are given for a suitable function f by, acting according to the law

$$T^{(\alpha)}_{(x,t)}f(y,s)=$$

$$=\begin{cases}\dfrac{1}{2\pi}\int_0^{2\pi}f\left(((x,t),(y,s))_{\theta,1}\right)d\theta, \text{ if } \alpha=0,\\ \dfrac{\alpha}{\pi}\int_0^1\left(\int_0^{2\pi}f\left(((x,t),(y,s))_{\theta,r}\right)d\theta\right)r(1-r^2)^{\alpha-1}dr, \text{ if } \alpha>0.\end{cases}$$

We remark that the generalized shift operator $T^{(\alpha)}_{(x,t)}$ is closely connected with the Laguerre differential operator $\mathcal{L}$ (see [10, 11] for details). Furthermore, $T^{(\alpha)}_{(x,t)}$ generates the corresponding convolution product defined by

$$(h*g)(x,t)=\int_{\mathbb{K}}T^{(\alpha)}_{(x,t)}h(y,s)\,g(y,-s)\,dm_\alpha(y,s),\quad \text{for all}\quad (x,t)\in\mathbb{K}.$$

The Riesz potential on the Laguerre hypergroup is defined in terms of the generalized sublaplacian $\mathcal{L}$.

Definition 1. For $0<\beta<Q$, the Riesz potential $\mathcal{I}_\beta$ is defined, initially on the Schwartz space $S(\mathbb{K})$, by

$$\mathcal{I}_\beta f(x,t)=\mathcal{L}^{-\frac{\beta}{2}}f(x,t).$$

From Lemmas 2 and 3 (see, section 2) we get

$$|\mathcal{I}_\beta f(x,t)|\le CI_\beta f(x,t), \tag{2}$$

where

$$I_\beta f(x,t)=\int_{\mathbb{K}}T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q}f(y,s)dm_\alpha(y,s),\quad 0<\beta<Q,$$

is the fractional integral on the Laguerre hypergroup. Inequality (2) gives a suitable estimate for the Riesz potential on the Laguerre hypergroup. In this paper we study the Riesz potential, the fractional integral and the modified fractional integral

$$\tilde{I}_\beta f(x,t) = \int\limits_{\mathbb{K}} \Big(T^{(\alpha)}_{(y,s)} |(x,t)|^{\beta-Q} - |(y,s)|^{\beta-Q} \chi_{\complement_{B_1}}(y,s) \Big) f(y,s) dm_\alpha(y,s)$$

on the Laguerre hypergroup in weighted Lebesgues spaces $L_{p,|\cdot|^\mu}(\mathbb{K})$.

V. Kokilashvili and A. Meskhi [16] proved the Stein-Weiss inequality for the fractional integral operator defined on nonhomogeneous spaces. The strong and weak type Stein-Weiss inequalities for the fractional integral operators in Carnot groups proved by V. S. Guliyev, R. Ch. Mustafayev, A. Serbetci in [14].

In this article we study the Riesz potential operator $\mathcal{I}_\beta$ and the fractional integral operator I_β on the Laguerre hypergroup $\mathbb{K}$ in the weighted Lebesgue spaces $L_{p,|\cdot|^\mu}(\mathbb{K})$, where $|\cdot|$ is the homogeneous norm in $\mathbb{K}$. We establish the strong and weak version of Stein-Weiss inequalities for $\mathcal{I}_\beta$ and I_β, and obtain necessary and sufficient conditions on the parameters for the boundedness of $\mathcal{I}_\beta$ and I_β from the spaces $L_{p,|\cdot|^\mu}(\mathbb{K})$ to $L_{q,|\cdot|^{-\lambda}}(\mathbb{K})$ for $1 < p \le q < \infty$, and from the spaces $L_{1,|\cdot|^\mu}(\mathbb{K})$ to the weak spaces $WL_{q,|\cdot|^{-\lambda}}(\mathbb{K})$ for $1 < q < \infty$.

In the limiting case $p = \frac{Q}{\beta-\mu-\lambda}$ we prove that the modified fractional integral operator $\widetilde{I}_\beta$ is bounded from the spaces $L_{p,|\cdot|^\mu}(\mathbb{K})$ to the weighted BMO space $BMO_{|\cdot|^{-\lambda}}(\mathbb{K})$, where Q is the homogeneous dimension of $\mathbb{K}$.

As an application, in Theorem 5 we obtain boundedness of the operator $\mathcal{I}_\beta$ from the weighted Besov spaces $B^s_{p\theta,|\cdot|^\mu}(\mathbb{K})$ to $B^s_{q\theta,|\cdot|^{-\lambda}}(\mathbb{K})$.

Theorem 1. *Let* $0 < \beta < Q$, $1 < p \le q < \infty$, $\mu < \frac{Q}{p'}$, $\lambda < \frac{Q}{q}$, $\mu + \lambda \ge 0$ *(*$\mu+\lambda > 0$*, if* $p = q$*),* $\frac{1}{p} - \frac{1}{q} = \frac{\beta-\mu-\lambda}{Q}$ *and* $f \in L_{p,|\cdot|^\mu}(\mathbb{K})$. *Then* $I_\beta f \in L_{q,|\cdot|^{-\lambda}}(\mathbb{K})$ *and the following inequality holds:*

$$\Big(\int\limits_{\mathbb{K}} |(x,t)|^{-\lambda q} |I_\beta f(x,t)|^q dm_\alpha(x,t) \Big)^{1/q} \le$$
$$\le C \Big(\int\limits_{\mathbb{K}} |(x,t)|^{\mu p} |f(x,t)|^p dm_\alpha(x,t) \Big)^{1/p},$$

where C *is independent of* f.

Theorem 2. *Let* $0 < \beta < Q$, $1 < q < \infty$, $\mu \le 0$, $\lambda < \frac{Q}{q}$, $\mu + \lambda \ge 0$, $1 - \frac{1}{q} = \frac{\beta-\mu-\lambda}{Q}$ *and* $f \in L_{1,|\cdot|^\mu}(\mathbb{K})$. *Then* $I_\beta f \in WL_{q,|\cdot|^{-\lambda}}(\mathbb{K})$ *and the* *following inequality holds*

$$\Bigg(\int\limits_{\{(x,t)\in\mathbb{K}:|(x,t)|^{-\lambda}|I_\beta f(x,t)|>\tau\}} dm_\alpha(x,t)\Bigg)^{1/q} \le$$
$$\le \frac{C}{\tau}\int\limits_{\mathbb{K}} |(x,t)|^{\mu}|f(x,t)|dm_\alpha(x,t),$$

where C is independent of f.

In the following, by using Stein-Weiss type Theorems 1 and 2, we obtain main result about necessary and sufficient conditions on the parameters for the boundedness of the $\mathcal{I}_\beta$ and I_β from the spaces $L_{p,|\cdot|^\mu}(\mathbb{K})$ to $L_{q,|\cdot|^\lambda}(\mathbb{K})$, and from the spaces $L_{1,|\cdot|^\mu}(\mathbb{K})$ to the weak spaces $WL_{q,|\cdot|^\lambda}(\mathbb{K})$.

Theorem 3. *Let $0<\beta<Q$, $1\le p\le q<\infty$, $\mu<\frac{Q}{p'}$ ($\mu\le 0$, if $p=1$), $\lambda<\frac{Q}{q}$ $\beta>\mu+\lambda\ge 0$ ($\beta=\mu+\lambda>0$, if $p=q$).*

1) *If $1<p<\frac{Q}{\beta-\mu-\lambda}$, then the condition $\frac{1}{p}-\frac{1}{q}=\frac{\beta-\mu-\lambda}{Q}$ are necessary and sufficient for the boundedness of I_β and $\mathcal{I}_\beta$ from $L_{p,|\cdot|^\mu}(\mathbb{K})$ to $L_{q,|\cdot|^{-\lambda}}(\mathbb{K})$.*

2) *If $p=1$, then the condition $1-\frac{1}{q}=\frac{\beta-\mu-\lambda}{Q}$ are necessary and sufficient for the boundedness of I_β and $\mathcal{I}_\beta$ from $L_{1,|\cdot|^\mu}(\mathbb{K})$ to $WL_{q,|\cdot|^{-\lambda}}(\mathbb{K})$.*

If we take $p=q$ and $\mu=0$ in Theorem 3, then we get the following new result.

Corollary 1. *Let $0<\beta<Q$, $1\le p<\infty$, $\lambda<\frac{Q}{p}$, $\beta\ge\lambda>0$.*

1) *If $1<p<\infty$, then the condition $\beta=\lambda$ is necessary and sufficient for the boundedness of I_β from $L_p(\mathbb{K})$ to $L_{p,|\cdot|^{-\lambda}}(\mathbb{K})$.*

2) *If $p=1$, then the condition $\beta=\lambda$ is necessary and sufficient for the boundedness of I_β from $L_1(\mathbb{K})$ to $WL_{1,|\cdot|^{-\lambda}}(\mathbb{K})$.*

If we take $p=q$ and $\lambda=0$ in Theorem 3, then we get the following new result.

Corollary 2. *Let $0<\beta<Q$, $1\le p<\infty$, $\mu<\frac{Q}{p'}$, $\beta\ge\mu>0$.*

1) *If $1<p<\infty$, then the condition $\beta=\mu$ is necessary and sufficient for the boundedness of I_β from $L_{p,|\cdot|^{-\mu}}(\mathbb{K})$ to $L_p(\mathbb{K})$.*

2) *If $p=1$, then the condition $\beta=\mu$ is necessary and sufficient for the boundedness of I_β from $L_{1,|\cdot|^{-\mu}}(\mathbb{K})$ to $WL_1(\mathbb{K})$.*

The weighted BMO space on the Laguerre hypergroup BMO_w is defined as the set of locally integrable functions f with finite norm

$$\|f\|_{BMO_w(\mathbb{K})} =$$
$$= \sup_{(x,t)\in\mathbb{K},r>0} w(B_r)^{-1}\int\limits_{B_r} |T^{(\alpha)}_{(y,s)}f(x,t)-f_{B_r}(x,t)|dm_\alpha(y,s)<\infty,$$

and BMO space on the Laguerre hypergroup $BMO(\mathbb{K}) \equiv BMO_1(\mathbb{K})$, where

$$f_{B_r}(x,t) = m_\alpha\big(B_r\big)^{-1} \int\limits_{B_r} T^{(\alpha)}_{(y,s)} f(x,t) dm_\alpha(y,s).$$

Note that in the limiting case $1 < p = \frac{Q}{\beta-\mu-\lambda}$ statement 1) in Theorem 3 does not hold. Moreover, there exists $f \in L_{p,|\cdot|^\mu}(\mathbb{K})$ such that $I_\beta f(x,t) = \infty$ for all $(x,t) \in \mathbb{K}$. For example,

$$f(x,t) = \begin{cases} \frac{|(x,t)|^{-\beta+\mu+\lambda}}{\ln|(x,t)|}, & |(x,t)| \geq 2 \\ 0, & |(x,t)| < 2 \end{cases} \in L_p(\mathbb{K})$$

where $(x,t) \in \mathbb{K}$, $0 < \beta - \mu - \lambda < Q$ and $p = \frac{Q}{\beta-\mu-\lambda}$, but $I_\beta f(x,t) = \infty$ for all $(x,t) \in \mathbb{K}$. However, as will be proved, statement 1) in Theorem 3 holds for the modified fractional integral operator $\widetilde{I}_\beta$ if the space $L_{\infty,|\cdot|^{-\lambda}}(\mathbb{K})$ is replaced by a wider space $BMO_{|\cdot|^{-\lambda}}(\mathbb{K})$.

In the following theorem we obtain conditions ensuring that the operator $\widetilde{I}_\beta$ is bounded from the space $L_{p,|\cdot|^\mu}(\mathbb{K})$ to $BMO_{|\cdot|^{-\lambda}}(\mathbb{K})$, when $p = \frac{Q}{\beta-\mu-\lambda}$.

Theorem 4. *Let $\beta > \mu + \lambda \geq 0 \geq \lambda$ and $p = \frac{Q}{\beta-\mu-\lambda} > 1$, then the operator $\widetilde{I}_\beta$ is bounded from $L_{p,|\cdot|^\mu}(\mathbb{K})$ to $BMO_{|\cdot|^{-\lambda}}(\mathbb{K})$.*

Moreover, if the integral $I_\beta f$ exists almost everywhere for $f \in L_{p,|\cdot|^\mu}(\mathbb{K})$, then $I_\beta f \in BMO_{|\cdot|^{-\lambda}}(\mathbb{K})$ and the following inequality holds

$$\|I_\beta f\|_{BMO_{|\cdot|^{-\lambda}}(\mathbb{K})} \leq C \|f\|_{L_{p,|\cdot|^\mu}(\mathbb{K})},$$

where $C > 0$ is independent of f.

Corollary 3. 1) *Let $0 \leq \mu < \beta < Q$, $p = \frac{Q}{\beta-\mu} > 1$, then the operator $\widetilde{I}_\beta$ is bounded from $L_{p,|\cdot|^\mu}(\mathbb{K})$ to $BMO(\mathbb{K})$.*

2) *Let $\mu \geq 0$, $0 < \beta < Q$, $p = \frac{Q}{\beta}$, then the operator $\widetilde{I}_\beta$ is bounded from $L_{p,|\cdot|^\mu}(\mathbb{K})$ to $BMO_{|\cdot|^\mu}(\mathbb{K})$.*

3) *Let $0 < \beta < Q$, $p = \frac{Q}{\beta}$, then the operator $\widetilde{I}_\beta$ is bounded from $L_p(\mathbb{K})$ to $BMO(\mathbb{K})$.*

Schwartz's theory of Fourier transform and the Lebesgue spaces has been investigated by many authors in the study of Besov spaces on $\mathbb{R}^n$ ([3], [5], [19]). This theory has been generalized to different spaces, and was applied further to investigate spaces analogous to the classical Besov spaces ([2], [4], [18]). Besov spaces in the setting of the Laguerre hypergroups studied by M. Assal, Ben Abdallah [1] and V. S. Guliyev, M. Omarova [13].

In Theorem 5 we prove the boundedness of $\mathcal{I}_\beta$ in the weighted Besov spaces on $\mathbb{K}$

$$B^s_{p\theta,w}(\mathbb{K}) = \Big\{ f : \|f\|_{B^s_{p\theta,w}(\mathbb{K})} = \|f\|_{L_{p,w}(\mathbb{K})} + \\ + \Big(\int_{\mathbb{K}} \frac{\|T^{(\alpha)}_{(x,t)} f(\cdot) - f(\cdot)\|^\theta_{L_{p,w}(\mathbb{K})}}{|(x,t)|^{Q+sq}} \, dm_\alpha(x,t) \Big)^{\frac{1}{\theta}} < \infty \Big\} \qquad (3)$$

for a power weight w, $1 \le p, \theta \le \infty$ and $0 < s < 1$.

Theorem 5. *Let $0 < \beta < Q$, $1 < p \le q < \infty$, $\mu < \frac{Q}{p'}$, $\lambda < \frac{Q}{q}$, $\beta \ge \mu + \lambda \ge 0$ ($\mu + \lambda > 0$, if $p = q$).*

If $1 < p < \frac{Q}{\beta-\mu-\lambda}$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta-\mu-\lambda}{Q}$, $1 \le \theta \le \infty$ and $0 < s < 1$, then the operator $\mathcal{I}_\beta$ is bounded from $B^s_{p\theta,|\cdot|^\mu}(\mathbb{K})$ to $B^s_{q\theta,|\cdot|^{-\lambda}}(\mathbb{K})$. More precisely, there is a constant $C > 0$, such that,

$$\|\mathcal{I}_\beta f\|_{B^s_{q\theta,|\cdot|^{-\lambda}}(\mathbb{K})} \le C \|f\|_{B^s_{p\theta,|\cdot|^\mu}(\mathbb{K})}$$

holds for all $f \in B^s_{p\theta,|\cdot|^\mu}(\mathbb{K})$.

2. Preliminaries

Let $\Sigma = \Sigma_2$ be the unit sphere in $\mathbb{K}$. We denote by ω_2 the surface area of Σ and by Ω_2 its volume, $m_\alpha(B_1) = \Omega_2$ (see [8, 10]). Then $m_\alpha(B_r) = \Omega_2 r^Q$.

Lemma 1 ([8, 10])**.** *The following equalities are valid*

$$\omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}, \quad \Omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

Let $f \in L_1(\mathbb{K})$. Set $x = r(\cos\varphi)^{1/2}$, $t = r^2 \sin\varphi$. We get

$$\int_{\mathbb{K}} f(x,t) \, dm_\alpha(x,t) = \\ = \frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2} \int_0^\infty f\big(r(\cos\varphi)^{1/2}, r^2 \sin\varphi\big) \, r^{Q-1} (\cos\varphi)^\alpha dr d\varphi.$$

If f radial, i.e., there is a function ψ on $[0,\infty)$ such that $f(x,t) = \psi(|(x,t)|)$, then

$$\int\limits_{\mathbb{K}} f(x,t)\,dm_\alpha(x,t) =$$

$$= \frac{1}{2\pi\Gamma(\alpha+1)} \int\limits_{-\pi/2}^{\pi/2} (\cos\varphi)^\alpha d\varphi \int\limits_0^\infty \psi(r) r^{Q-1} dr =$$

$$= \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)} \int\limits_0^\infty \psi(r) r^{Q-1} dr.$$

Specifically,

$$m_\alpha(B_r) = \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}\Gamma(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)} r^Q.$$

Let

$$L_m^{(\alpha)}(x) = \sum_{j=0}^{m} \frac{\Gamma(m+\alpha+1)}{\Gamma(m-j+1)\Gamma(j+\alpha+1)} \frac{(-x)^j}{j!},$$

be the Laguerre polynomial of degree m and order α (see [1]) defined in terms of the generating function by

$$\sum_{m=0}^{\infty} s^m L_m^{(\alpha)}(x) = \frac{1}{(1-s)^{\alpha+1}} \exp\Big(-\frac{xs}{1-s}\Big). \tag{4}$$

For $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$, we put

$$\varphi_{\lambda,m}(x,t) = \frac{m!\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)} e^{i\lambda t} e^{-\frac{1}{2}|\lambda|x^2} L_m^{(\alpha)}(|\lambda|x^2).$$

The following proposition summarizes some basic properties of functions $\varphi_{(\lambda,m)}$.

Proposition 1. *The function $\varphi_{(\lambda,m)}$ satisfies that*
(a) $\|\varphi_{(\lambda,m)}\|_{\alpha,\infty} = \varphi_{(\lambda,m)}(0,0) = 1$,
(b) $\varphi_{(\lambda,m)}(x,t)\varphi_{(\lambda,m)}(y,s) = T^{(\alpha)}_{(x,t)}\varphi_{(\lambda,m)}(y,s)$,
(c) $L\varphi_{(\lambda,m)} = 2|\lambda|(2m+\alpha+1)\varphi_{(\lambda,m)}$.

Let $f \in L_1(\mathbb{K})$, the generalized Fourier transform of f is defined by

$$\mathcal{F}(f)(\lambda,m) = \int\limits_{\mathbb{K}} f(x,t)\varphi_{(-\lambda,m)}(x,t)dm_\alpha(x,t).$$

We have

$$\|\mathcal{F}(f)\|_{L_\infty(\mathbb{K})} \le \|f\|_{L_1(\mathbb{K})},$$

where

$$\|\mathcal{F}(f)\|_{L_\infty(\mathbb{K})} = \underset{(\lambda,m)\in\mathbb{K}}{ess\sup} |\mathcal{F}(f)(\lambda,m)|.$$

It is easy to see that

$$\mathcal{F}(T^{(\alpha)}_{(y,s)}f)(\lambda,m) = \mathcal{F}(f)(\lambda,m)\varphi_{(\lambda,m)}(y,s)$$

and

$$\mathcal{F}(f*g)(\lambda,m) = \mathcal{F}(f)(\lambda,m)\mathcal{F}(g)(\lambda,m).$$

Let $\{H^s : s>0\} = \{e^{-sL} : s>0\}$ be the heat semigroup generated by $\mathcal{L}$. There is an unique smooth function $h((x,t),s) = h_s(x,t)$ on $\mathbb{K}\times(0,+\infty)$ such that

$$H^s f(x,t) = f * h_s(x,t).$$

Further h_s is the heat kernel associated to the generalized Sublaplacian $\mathcal{L}$ and satisfies

$$\begin{aligned} \mathcal{F}(h_s(\lambda,m)) &= e^{-2|\lambda|(2m+\alpha+1)s}, \\ h_{s_1} * h_{s_2} &= h_{s_1+s_2}, \\ h_s(x,t) &= s^{-(\alpha+2)}h_1\left(\frac{x}{\sqrt{s}},\frac{t}{s}\right). \end{aligned}$$

Although the heat kernel $h_s(x,t)$ is not explicitly known, we do have a suitable estimate for $h_s(x,t)$ (see, for example [15]).

Lemma 2. *There exists $A>0$ such that*

$$0 < h_s(x,t) \le C s^{\alpha-2} e^{-\frac{A}{s}|(x,t)|^2}.$$

Remark 1. It is easy to see that

$$\mathcal{F}(\mathcal{I}_\beta f)(\lambda,m) = (2|\lambda|(2m+\alpha+1))^{-\frac{\beta}{2}}\mathcal{F}(f)(\lambda,m).$$

This means

$$\begin{aligned} \mathcal{I}_{\beta_1}(\mathcal{I}_{\beta_2}f) &= \mathcal{I}_{\beta_1+\beta_2}(f),\ \beta_1,\beta_2>0,\ \beta_1+\beta_2<Q, \\ L(\mathcal{I}_\beta f) &= \mathcal{I}_\beta(Lf) = \mathcal{I}_{\beta-2}(f),\ 2<\beta<Q. \end{aligned}$$

Lemma 3. *Let $h_r(x,t)$ be the heat kernel associated with $\mathcal{L}$ and $0<\beta<Q$. Then*

$$\mathcal{I}_\beta f(y,s) = \Gamma\Big(\frac{\beta}{2}\Big)^{-1} \int\limits_{\mathbb{K}} \Big(\int\limits_0^\infty r^{\frac{\beta}{2}-1} h_r(x,t)dr\Big) T^{(\alpha)}_{(y,s)} f(x,t)\, dm_\alpha(x,t).$$

3. Some Properties on the Laguerre Hypergroup

Lemma 4. *Let $0<\beta<Q$. Then for $2|(x,t)|\le|(y,s)|$, $(x,t),(y,s)\in\mathbb{K}$, the following inequality holds:*

$$\Big|T^{(\alpha)}_{(x,t)}|(y,s)|^{\beta-Q}-|(y,s)|^{\beta-Q}\Big|\le 2^{Q-\beta+1}|(y,s)|^{\beta-Q-1}|(x,t)|. \qquad (5)$$

Proof. We will show that

$$T^{(\alpha)}_{(x,t)}|(y,s)|^{\beta-Q}-|(y,s)|^{\beta-Q}=$$

$$=\begin{cases}\frac{1}{2\pi}\int\limits_0^{2\pi}\left[\Big|\Big((x,t),(y,s)\Big)_{\theta,1}\Big|^{\beta-Q}-|(y,s)|^{\beta-Q}\right]d\theta, \text{ if } \alpha=0,\\ \frac{\alpha}{\pi}\int\limits_0^1\left(\int\limits_0^{2\pi}\left[\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|^{\beta-Q}-|(y,s)|^{\beta-Q}\right]d\theta\right)r(1-r^2)^{\alpha-1}dr, \text{ if } \alpha>0.\end{cases}$$

From the mean value theorem we have

$$\Big|\Big|\Big((x,t),(y,s)\Big)_{\theta,1}\Big|^{\beta-Q}-|(y,s)|^{\beta-Q}\Big|\le$$
$$\le\Big|\Big|\Big((x,t),(y,s)\Big)_{\theta,1}\Big|-|(y,s)|\Big|\xi^{\beta-Q-1},$$

where,

$$\min\Big\{\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|,|(y,s)|\Big\}\le\xi\le\min\Big\{\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|,|(y,s)|\Big\}.$$

From the mean value theorem (see Lemma 3 of [12]) we have

$$\Big||(x,t)|-|(y,s)|\Big|\le\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|\le|(x,t)|+|(y,s)|,$$

Note that

$$\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|\le|(x,t)|+|(y,s)|\le\frac{3}{2}|(y,s)|,$$
$$\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|\ge\Big||(x,t)|-|(y,s)|\Big|\ge|(y,s)|-|(x,t)|\ge\frac{1}{2}|(y,s)|$$

and

$$\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|-|(y,s)|\le|(x,t)|+|(y,s)|-|(y,s)|\le|(x,t)|,$$
$$|(y,s)|-\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|\le|(y,s)|-|(x,t)|-|(y,s)|\le|(x,t)|.$$

Hence

$$\frac{1}{2}|(y,s)|\le\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|\le\frac{3}{2}|(y,s)|$$

and

$$\Big|\Big((x,t),(y,s)\Big)_{\theta,r}\Big|-|(y,s)|\le|(x,t)|. \qquad \square$$

We will need the following Hardy-type transforms defined on $\mathbb{K}$:

$$Hf(x,t) = \int\limits_{B_{|(x,t)|}} f(y,s)dm_\alpha(y,s),$$

and

$$H'f(x,t) = \int\limits_{\complement B_{|(x,t)|}} f(y,s)dm_\alpha(y,s).$$

The following two theorems related to the boundedness of these transforms were proved in [6] (see also [7], Section 1.1).

Theorem A. *Let $1 < q < \infty$. Suppose that ν and w are a.e. positive functions on $\mathbb{K}$. Then*

(a) *The operator H is bounded from $L_{1,w}(\mathbb{K})$ to $WL_{q,v}(\mathbb{K})$ if and only if*

$$A_1 \equiv \sup_{r>0} \Bigg(\int\limits_{\complement B_r} \nu^q(x,t)dm_\alpha(x,t) \Bigg)^{1/q} \sup_{B_r} w^{-1}(x,t) < \infty;$$

(b) *The operator H' is bounded from $L_{1,w}(\mathbb{K})$ to $WL_{q,v}(\mathbb{K})$ if and only if*

$$A_2 \equiv \sup_{r>0} \Bigg(\int\limits_{B_r} \nu^q(x,t)dm_\alpha(x,t) \Bigg)^{1/q} \sup_{\complement B_r} w^{-1}(x,t) < \infty.$$

Moreover, there exist positive constants a_j, $j = 1, \dots, 4$, depending only on q such that $a_1 A_1 \le \|H\| \le a_2 A_1$ and $a_3 A_2 \le \|H'\| \le a_4 A_2$.

Theorem B. *Let $1 < p \le q < \infty$. Suppose that ν and w are a.e. positive functions on $\mathbb{K}$. Then*

(a) *The operator H is bounded from $L_{p,w}(\mathbb{K})$ to $L_{q,v}(\mathbb{K})$ if and only if*

$$A_3 \equiv \sup_{r>0} \Bigg(\int\limits_{\complement B_r} \nu^q(x,t)dm_\alpha(x,t) \Bigg)^{1/q} \Bigg(\int\limits_{B_r} w^{-p'}(x,t)dm_\alpha(x,t) \Bigg)^{1/p'} < \infty,$$

$p' = p/(p-1)$;

(b) *The operator H' is bounded from $L_{p,w}(\mathbb{K})$ to $WL_{q,v}(\mathbb{K})$ if and only if*

$$A_4 \equiv \sup_{r>0} \Bigg(\int\limits_{B_r} \nu^q(x,t)dm_\alpha(x,t) \Bigg)^{1/q} \Bigg(\int\limits_{\complement B_r} w^{-p'}(x,t)dm_\alpha(x,t) \Bigg)^{1/p'} < \infty.$$

Moreover, there exist positive constants b_j, $j = 1, \dots, 4$, depending only on p and q such that $b_1 A_3 \le \|H\| \le b_2 A_3$ and $b_3 A_4 \le \|H'\| \le b_4 A_4$.

We will need the case that we substitute $L_{p,w}(X)$ with the homogeneous space (X, d, dm_α) in Theorems A and B in which $X = \mathbb{K}$, $d((x,t),(y,s)) = |(x-y, t-s)|$ and m_α be the weighted Lebesgue measure on $\mathbb{K}$, given by (1).

Definition 2. The weight function w belongs to the class $A_p(\mathbb{K})$ for $1 < p < \infty$, if

$$\sup_{(x,t)\in\mathbb{K},\ r>0}\left(m_\alpha(B_r(x,t))^{-1}\int\limits_{B_r(x,t)} w(y,s)dm_\alpha(y,s)\right)\times$$
$$\times\left(m_\alpha(B_r(x,t))^{-1}\int\limits_{B_r(x,t)} w^{-\frac{1}{p-1}}(y,s)dm_\alpha(y,s)\right)^{p-1} < \infty$$

and w belongs to $A_1(\mathbb{K})$, if there exists a positive constant C such that for any $(x,t) \in \mathbb{K}$ and $r > 0$

$$m_\alpha(B_r(x,t))^{-1}\int\limits_{B_r(x,t)} w(y,s)dm_\alpha(y,s) \le C \operatorname*{ess\,inf}_{(y,s)\in B_r(x,t)} w(y,s).$$

The properties of the class $A_p(\mathbb{K})$ are analogous to those of the Muckenhoupt classes. In particular, if $w \in A_p(\mathbb{K})$, then $w \in A_{p-\varepsilon}(\mathbb{K})$ for a certain sufficiently small $\varepsilon > 0$ and $w \in A_{p_1}(\mathbb{K})$ for any $p_1 > p$.

Note that $|(x,t)|^\beta \in A_p(\mathbb{K})$, $1 < p < \infty$, if and only if $-\frac{Q}{p} < \beta < \frac{Q}{p'}$; and $|(x,t)|^\beta \in A_1(\mathbb{K})$, if and only if $-Q < \beta \le 0$.

For the maximal function on the Laguerre hypergroup

$$Mf(x,t) = \sup_{r>0} m_\alpha\big(B_r\big)^{-1}\int\limits_{B_r} T^{(\alpha)}_{(y,s)}|f(x,t)|dm_\alpha(y,s)$$

the following analogue of Muckenhoupt theorem is valid.

Theorem C. 1. *If $f \in L_{1,w}(\mathbb{K})$ and $w \in A_1(\mathbb{K})$, then $Mf \in WL_{1,w}(\mathbb{K})$ and*

$$\|Mf\|_{WL_{1,w}(\mathbb{K})} \le C_{1,w}\|f\|_{L_{1,w}(\mathbb{K})}, \tag{6}$$

where $C_{1,w}$ depends only on k and n.

2. *If $f \in L_{p,w}(\mathbb{K})$ and $w \in A_p(\mathbb{K})$, $1 < p < \infty$, then $Mf \in L_{p,w}(\mathbb{K})$ and*

$$\|Mf\|_{L_{p,w}(\mathbb{K})} \le C_{p,w}\|f\|_{L_{p,w}(\mathbb{K})}, \tag{7}$$

where $C_{p,w}$ depends only on w, p, k and n.

Proof. Following [9], we define a maximal function on a space of homogeneous type. By this we mean a topological space X equipped with a continuous pseudometric d and a positive measure μ satisfying the doubling condition

$$\mu(E_{2r}(x,t)) \le c\mu(E_r(x,t)), \tag{8}$$

where c does not depend on $(x,t) \in X$ and $r > 0$. Here $E_r(x,t) = \{(y,s) \in X : |(x-y, t-s)| < r\}$. Denote

$$M_\mu f(x,t) = \sup_{r>0} \mu(E_r(x,t))^{-1} \int\limits_{E_r(x,t)} |f(y,s)| d\mu(y,s).$$

Let (X, d, μ) be a homogeneous type spaces. It is known that the maximal operator M_μ is weighted weak $(1,1)$ type, $w \in A_1(X)$, that is

$$\int\limits_{\{(x,t)\in X: M_\mu f(x,t) > \tau\}} w(x,t)\, d\mu(x,t) \leq \leq \Big(\frac{C_{1,w}}{\tau} \int\limits_X |f(x,t)| w(x,t)\, d\mu(x,t) \Big), \tag{9}$$

and is weighted (p,p) type, $1 < p \leq \infty$ and $w \in A_p(X)$, that is

$$\int\limits_X |M_\mu f(x,t)|^p w^p(x,t) d\mu(x,t) \leq C_{p,w} \int\limits_X |f(x,t)|^p w^p(x,t) d\mu(x,t). \tag{10}$$

In [11] it is proved that the following inequality

$$Mf(x,t) \leq C M_\mu f(x,t)$$

holds, where constant $C > 0$ does not depend on f and $(x,t) \in \mathbb{K}$.

If we take $X \equiv \mathbb{K}$, $d((x,t),(y,s)) = |(x-y, t-s)|$ and $d\mu(x,t) = dm_\alpha(x,t)$ then we have

$$\|Mf\|_{p,w} \leq C \|M_\mu f\|_{p,w} \leq C_{p,w} \|f\|_{p,\mu}, \quad 1 < p \leq \infty,$$

and for $p = 1$

$$\int\limits_{\{(x,t)\in \mathbb{K}: Mf(x,t) > \tau\}} w(x,t) dm_\alpha(x,t) \leq$$

$$\leq \int\limits_{\{(x,t)\in X: M_\mu f(x,t) > \frac{\tau}{C}\}} w(x,t)\, d\mu(x,t) \leq$$

$$\leq \frac{C_{1,w}}{\tau} \int\limits_X |f(x,t)| w(x,t)\, d\mu(x,t) =$$

$$= \frac{C_{1,w}}{\tau} \int\limits_{\mathbb{K}} |f(x,t)| w(x,t)\, dm_\alpha(x,t). \qquad \square$$

Remark 2. Note that in the nonweighted case Theorem C was proved in [10] and [15].

We will need the following Hardy-Littlewood-Sobolev theorem for I_β, which was proved in [11].

Theorem D. *Let* $0 < \beta < Q$ *and* $1 \le p < \frac{Q}{\beta}$. *Then*

1) *If* $1 < p < \frac{Q}{\beta}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{Q}$ *is necessary and sufficient for the boundedness of* I_β *from* $L_p(\mathbb{K})$ *to* $L_q(\mathbb{K})$.

2) *If* $p = 1$, *then the condition* $1 - \frac{1}{q} = \frac{\beta}{Q}$ *is necessary and sufficient for the boundedness of* I_β *from* $L_1(\mathbb{K})$ *to* $WL_q(\mathbb{K})$.

4. Proof of the Theorems

Proof of Theorem 1. We write

$$\Big(\int\limits_{\mathbb{K}} |(x,t)|^{-\lambda q} |I_\beta f(x,t)|^q dm_\alpha(x,t)\Big)^{1/q} \le I_1 + I_2 + I_3 \equiv$$

$$\equiv \Big(\int\limits_{\mathbb{K}} |(x,t)|^{-\lambda q} \Big| \int\limits_{B_{\frac{1}{2}|(x,t)|}} f(y,s) |T^{(\alpha)}_{(y,s)} |(x,t)|^{\beta - Q} dm_\alpha(y,s)\Big|^q dm_\alpha(x,t)\Big)^{1/q} +$$

$$+ \Big(\int\limits_{\mathbb{K}} |(x,t)|^{-\lambda q} \times$$

$$\times \Big| \int\limits_{B_{2|(x,t)|} \setminus B_{\frac{1}{2}|(x,t)|}} f(y,s) |T^{(\alpha)}_{(y,s)} |(x,t)|^{\beta - Q} dm_\alpha(y,s)\Big|^q dm_\alpha(x,t)\Big)^{1/q} +$$

$$+ \Big(\int\limits_{\mathbb{K}} |(x,t)|^{-\lambda q} \Big| \int\limits_{\complement B_{2|(x,t)|}} f(y,s) |T^{(\alpha)}_{(y,s)} |(x,t)|^{\beta - Q} dm_\alpha(y,s)\Big|^q dm_\alpha(x,t)\Big)^{1/q}.$$

It is easy to check that if $|(y,s)| < \frac{1}{2}|(x,t)|$, then $|(x,t)| \le |(y,s)| + |(x- \ y, t-s)| < \frac{1}{2}|(x,t)| + |(x-y, t-s)|$. Hence $\frac{1}{2}|(x,t)| < |(x-y,t-s)|$ and $T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q} \le (1 + 2^{Q-\beta})|(x,t)|^{\beta - Q}$. Indeed, from Lemma 4 we have

$$T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q} \le |(x,t)|^{\beta-Q} + 2^{Q-\beta+1}|(x,t)|^{\beta-Q-1}|(y,s)| \le \\ \le (1 + 2^{Q-\beta})|(x,t)|^{\beta-Q}. \qquad (11)$$

Then we get

$$I_1 \le (1 + 2^{Q-\beta}) \Big(\int\limits_{\mathbb{K}} |(x,t)|^{(\beta - Q - \lambda)q} (Hf(x,t))^q dm_\alpha(x,t)\Big)^{1/q}.$$

Further, taking into account the inequality $-\lambda q < (Q-\beta)q - Q$ (*i.e.*, $\beta < \frac{Q}{q'} + \lambda$) we have

$$\Big(\int\limits_{\complement B_r} |(x,t)|^{(-\lambda+\beta-Q)q} dm_\alpha(x,t)\Big)^{\frac{1}{q}} =$$

$$= \Big(\int\limits_{\sum}\int\limits_{r}^{\infty} \tau^{Q-1}\cdot\tau^{(-\lambda+\beta-Q)q} d\tau d\xi'\Big)^{\frac{1}{q}} =$$

$$= \Big(\frac{\omega_2}{Q-(Q-\beta+\lambda)q} r^{Q-(Q-\beta+\lambda)q}\Big)^{\frac{1}{q}} =$$

$$= \Big(\frac{\Omega_2}{\frac{\beta-\lambda}{Q}q - \frac{q}{q'}}\Big)^{\frac{1}{q}} r^{\frac{Q}{q}-Q+\beta-\lambda} = C_1 r^{\frac{Q}{q}-Q+\beta-\lambda},$$

where $C_1 = \Big(\frac{\Omega_2}{\frac{\beta-\lambda}{Q}q - \frac{q}{q'}}\Big)^{\frac{1}{q}}$. Similarly, by virtue of the condition $\mu p < Q(p-1)$ (*i.e.*, $\mu < \frac{Q}{p'}$) it follows that

$$\Big(\int\limits_{B_r} |(x,t)|^{-\mu p'} dm_\alpha(x,t)\Big)^{1/p'} = \Big(\int\limits_{\sum}\int\limits_{0}^{r} \tau^{Q-1}\cdot\tau^{-\mu p'} d\tau d\xi'\Big)^{\frac{1}{p'}} =$$

$$= \Big(\frac{\omega_2}{Q-\mu p'} r^{Q-\mu p'}\Big)^{\frac{1}{p'}} = \Big(\frac{\Omega_2}{1-\frac{\mu p'}{Q}}\Big)^{\frac{1}{p'}} r^{\frac{Q}{p'}-\mu} = C_2 r^{\frac{Q}{p'}-\mu},$$

where $C_2 = \Big(\frac{\Omega_2}{1-\frac{\mu p'}{Q}}\Big)^{\frac{1}{p'}}$.

Summarizing these estimates we find that

$$\sup_{r>0}\Big(\int\limits_{\complement B_r} |(x,t)|^{(-\lambda+\beta-Q)q} dm_\alpha(x,t)\Big)^{1/q}\Big(\int\limits_{\complement B_r} |(x,t)|^{-\mu p'} dm_\alpha(x,t)\Big)^{1/p'} =$$

$$= C_1C_2 \sup_{r>0} r^{\beta-\mu-\lambda+Q/q-Q/p} < \infty \Longleftrightarrow$$

$$\Longleftrightarrow \beta-\mu-\lambda = Q/p - Q/q.$$

Now the first part of Theorem B gives us the inequality

$$I_1 \le b_2 C_1 C_2 2^{Q-\beta}\Big(\int\limits_{\mathbb{K}} |(x,t)|^{\mu}|f(x,t)|^p dm_\alpha(x,t)\Big)^{1/p}.$$

If $|(y,s)| > 2|(x,t)|$, then $|(y,s)| \le \frac{1}{2}|(x,t)| + |(x-y,t-s)| < |(y,s)| + |(x-y,t-s)|$. Hence $\frac{1}{2}|(y,s)| < |(x-y,t-s)|$ and the inequality $T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q} \le$

$(\frac{1}{2}|(y,s)|)^{\beta-Q}$ can be shown immediately by similar method that of the inequality (11). Consequently, we get

$$I_3 \le 2^{Q-\beta}\Big(\int\limits_{\mathbb{K}} |(x,t)|^{-\lambda q}\Big(H'(|f(y,s)||(y,s)|^{\beta-Q})(|(x,t)|)\Big)^q dm_\alpha(x,t)\Big)^{1/q}.$$

Further, taking into account the inequality $-\lambda q > -Q$ (*i.e.*, $\lambda < Q/q$) we have

$$\Big(\int\limits_{B_r} |(x,t)|^{-\lambda q} dm_\alpha(x,t)\Big)^{\frac{1}{q}} = \Big(\int\limits_{\sum}\int\limits_0^r \tau^{Q-1}\cdot\tau^{-\lambda q} d\tau d\xi'\Big)^{\frac{1}{q}} =$$
$$= \Big(\frac{\omega_2}{Q-\lambda q} r^{Q-\lambda q}\Big)^{\frac{1}{q}} = \Big(\frac{\Omega_2}{1-\frac{\lambda q}{Q}}\Big)^{\frac{1}{q}} r^{\frac{Q}{q}-\lambda} = C_3 r^{\frac{Q}{q}-\lambda},$$

where $C_3 = \Big(\frac{\Omega_2}{1-\frac{\lambda q}{Q}}\Big)^{\frac{1}{q}}$. By the condition $\mu p > \beta p - Q$ (*i.e.*, $\beta < Q/p + \mu$) it follows that

$$\Big(\int\limits_{B_r} |(x,t)|^{-(\mu+Q-\beta)p'} dm_\alpha(x,t)\Big)^{\frac{1}{p'}} =$$
$$= \Big(\int\limits_{\sum}\int\limits_0^r \tau^{Q-1}\cdot\tau^{-(\mu+Q-\beta)p'} d\tau d\xi'\Big)^{\frac{1}{p'}} =$$
$$= \Big(\frac{\omega_2}{Q-(\mu+Q-\beta)p'}\ r^{Q-(\mu+Q-\beta)p'}\Big)^{\frac{1}{p'}} =$$
$$= \Big(\frac{\Omega_2}{(p'-1)+\frac{\mu-\beta}{Q}p'}\Big)^{\frac{1}{p'}}\ r^{\frac{Q}{p'}-\mu-Q+\beta} = C_4 r^{\frac{Q}{p'}-\mu-Q+\beta},$$

where $C_4 = \Big(\frac{\Omega_2}{(p'-1)+\frac{\mu-\beta}{Q}p'}\Big)^{\frac{1}{p'}}$.

Thus we find

$$\sup_{r>0}\Big(\int\limits_{B_r} |(x,t)|^{-\lambda q} dm_\alpha(x,t)\Big)^{1/q} \times$$
$$\times\Big(\int\limits_{\complement B_r} |(x,t)|^{-(\mu+Q-\beta)p'} dm_\alpha(x,t)\Big)^{1/p'} =$$
$$= C_3C_4 \sup_{r>0} r^{\beta-\mu-\lambda+Q/q-Q/p} < \infty \iff$$
$$\iff \beta-\mu-\lambda = Q/p - Q/q.$$

Now the second part of Theorem B gives us the inequality

$$I_3 \le b_4 C_3 C_4 2^{Q-\beta} \left(\int_{\mathbb{K}} |(x,t)|^{\mu} |f(x,t)|^p dm_\alpha(x,t) \right)^{1/p}.$$

To estimate I_2 we consider the cases $\beta < Q/p$ and $\beta > Q/p$, separately. If $\beta < Q/p$, then the condition

$$\beta = \mu + \lambda + Q/p - Q/q \ge Q/p - Q/q$$

implies $q \le p^*$, where $p^* = Qp/(Q - \beta p)$. Assume that $q < p^*$. In the sequel we use the notation

$$D_k \equiv \{(x,t) \in \mathbb{K} : 2^k \le |(x,t)| < 2^{k+1}\},$$

and

$$\widetilde{D_k} \equiv \{(x,t) \in \mathbb{K} : 2^{k-2} \le |(x,t)| < 2^{k+2}\}.$$

By Hölder's inequality with respect to the exponent p^*/q and Theorem D we get

$$I_2 = \Bigg(\int_{\mathbb{K}} |(x,t)|^{-\lambda q} \Bigg(\int_{B_{2|(x,t)|} \setminus B_{\frac{1}{2}|(x,t)|}} |f(y,s)| \times$$

$$\times T^{(\alpha)}_{(y,s)} |(x,t)|^{\beta - Q} dm_\alpha(y,s) \Bigg)^q dm_\alpha(x,t) \Bigg)^{1/q} =$$

$$= \Bigg(\sum_{k\in\mathbb{Z}} \int_{D_k} |(x,t)|^{-\lambda q} \Bigg(\int_{B_{2|(x,t)|} \setminus B_{\frac{1}{2}|(x,t)|}} |f(y,s)| \times$$

$$\times T^{(\alpha)}_{(y,s)} |(x,t)|^{\beta - Q} dm_\alpha(y,s) \Bigg)^q dm_\alpha(x,t) \Bigg)^{1/q} \le$$

$$\le \Bigg(\sum_{k\in\mathbb{Z}} \Bigg(\int_{D_k} \Bigg(\int_{B_{2|(x,t)|} \setminus B_{\frac{1}{2}|(x,t)|}} |f(y,s)| \times$$

$$\times T^{(\alpha)}_{(y,s)} |(x,t)|^{\beta - Q} dm_\alpha(y,s) \Bigg)^{p^*} dm_\alpha(x,t) \Bigg)^{q/p^*} \times$$

$$\times \Bigg(\int_{D_k} |(x,t)|^{\frac{-\lambda q p^*}{p^* - q}} dm_\alpha(x,t) \Bigg)^{\frac{p^*-q}{p^*}} \Bigg)^{1/q} \le$$

$$\le C_5 \Bigg(\sum_{k\in\mathbb{Z}} 2^{k[-\lambda q + \frac{p^*-q}{p^*} Q]} \Bigg(\int_{D_k} \left| I_\beta \left(f \chi_{\widetilde{D_k}} \right)(x,t) \right|^{p^*} dm_\alpha(x,t) \Bigg)^{q/p^*} \Bigg)^{1/q} \le$$

$$\le C_6 \left(\sum_{k\in\mathbb{Z}} 2^{k[-\lambda q + \frac{p^*-q}{p^*}Q]} \Bigg(\int\limits_{\widetilde{D_k}} |f(x,t)|^p dm_\alpha(x,t) \Bigg)^{q/p} \right)^{1/q} \le$$

$$\le C_7 \Bigg(\int\limits_{\mathbb{K}} |(x,t)|^{\mu} |f(x,t)|^p dm_\alpha(x,t) \Bigg)^{1/p}.$$

If $q = p^*$, then $\mu + \lambda = 0$. By using directly Theorem D we get

$$I_2 \le C_8 \left(\sum_{k\in\mathbb{Z}} 2^{k\mu p^*} \int\limits_{D_k} \left| I_\beta \left(f\chi_{\widetilde{D_k}} \right)(x,t) \right|^{p^*} dm_\alpha(x,t) \right)^{1/p^*} \le$$

$$\le C_9 \left(\sum_{k\in\mathbb{Z}} 2^{k\mu p^*} \Bigg(\int\limits_{\widetilde{D_k}} |f(x,t)|^p dm_\alpha(x,t) \Bigg)^{p^*/p} \right)^{1/p^*} \le$$

$$\le C_{10} \Bigg(\int\limits_{\mathbb{K}} |(x,t)|^{\mu p} |f(x,t)|^p dm_\alpha(x,t) \Bigg)^{1/p}.$$

For $\beta > Q/p$ by Hölder's inequality with respect to the exponent p we get the following inequality

$$I_2 \le \Bigg(\int\limits_{\mathbb{K}} |(x,t)|^{-\lambda q} \Bigg(\int\limits_{B_{2|(x,t)|}\setminus B_{\frac{1}{2}|(x,t)|}} |f(y,s)|^p dm_\alpha(y,s) \Bigg)^{q/p} \times$$

$$\times \Bigg(\int\limits_{B_{2|(x,t)|}\setminus B_{\frac{1}{2}|(x,t)|}} \left(T^{(\alpha)}_{(y,s)} |(x,t)|^{\beta-Q} \right)^{p'} dm_\alpha(y,s) \Bigg)^{q/p'} dm_\alpha(x,t) \Bigg)^{1/q}.$$

On the other hand by using (2) and the inequality $\beta > Q/p$, we obtain

$$\int\limits_{B_{2|(x,t)|}\setminus B_{\frac{1}{2}|(x,t)|}} \left(T^{(\alpha)}_{(y,s)} |(x,t)|^{\beta-Q} \right)^{p'} dm_\alpha(y,s) \le$$

$$\le \int\limits_{B_{2|(x,t)|}\setminus B_{\frac{1}{2}|(x,t)|}} |(x-y,t-s)|^{(\beta-Q)p'} dm_\alpha(y,s) \le$$

$$\le \int\limits_0^\infty m_\alpha \Big(B_{2|(x,t)|} \cap B_{\tau^{\frac{1}{(\beta-Q)p'}}}(x,t) \Big) d\tau \le$$

$$\leq \int_0^{|(x,t)|^{(\beta-Q)p'}} m_\alpha\Big(B_{2|(x,t)|}\Big)d\tau + \int_{|(x,t)|^{(\beta-Q)p'}}^{\infty} m_\alpha\Big(B_{\tau^{\frac{1}{(\beta-Q)p'}}}(x,t)\Big)d\tau \leq$$

$$\leq C_{11}|(x,t)|^{(\beta-Q)p'+Q} + C_{12}\int_{|(x,t)|^{(\beta-Q)p'}}^{\infty} \tau^{\frac{Q}{(\beta-Q)p'}}d\tau =$$

$$= C_{13}|(x,t)|^{(\beta-Q)p'+Q},$$

where the positive constant C_{13} does not depend on $(x,t)\in\mathbb{K}$. The latter estimate yields

$$I_2 \leq C_{14}\Bigg(\sum_{k\in\mathbb{Z}}\int_{D_k}|(x,t)|^{-\lambda q+[(\beta-Q)p'+Q]q/p'}\times$$

$$\times\Bigg(\int_{B_{2|(x,t)|}\setminus B_{\frac{1}{2}|(x,t)|}}|f(y,s)|^p dm_\alpha(y,s)\Bigg)^{q/p} dm_\alpha(x,t)\Bigg)^{1/q} \leq$$

$$\leq C_{14}\Bigg(\sum_{k\in\mathbb{Z}}\int_{D_k}\Bigg(\int_{\widetilde{D}_k}|f(y,s)|^p dm_\alpha(y,s)\Bigg)^{q/p}\times$$

$$\times|(x,t)|^{-\lambda q+[(\beta-Q)p'+Q]q/p'}dm_\alpha(x,t)\Bigg)^{1/q} \leq$$

$$\leq C_{14}\Bigg(\sum_{k\in\mathbb{Z}}2^{k(-\lambda+\beta-Q+Q/p'+Q/q)q}\Bigg(\int_{\widetilde{D}_k}|f(y,s)|^p dm_\alpha(y,s)\Bigg)^{q/p}\Bigg)^{1/q} \leq$$

$$\leq C_{14}\Bigg(\sum_{k\in\mathbb{Z}}2^{k\mu q}\Bigg(\int_{\widetilde{D}_k}|f(x,t)|^p dm_\alpha(x,t)\Bigg)^{q/p}\Bigg)^{1/q} \leq$$

$$\leq C_{15}\Bigg(\int_{\mathbb{K}}|(x,t)|^{\mu p}|f(x,t)|^p dm_\alpha(x,t)\Bigg)^{q/p}. \qquad \square$$

Proof of Theorem 2. Let

$$E = \Bigg\{(x,t)\in\mathbb{K}:$$

$$|(x,t)|^{-\lambda}\int_{B_{2|(x,t)|}\setminus B_{\frac{1}{2}|(x,t)|}}|f(y,s)|\,T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q}dm_\alpha(y,s) > \tau/3\Bigg\}.$$

We write

$$\Bigg(\int\limits_{\{(x,t)\in\mathbb{K}\colon |(x,t)|^{-\lambda}|I_\beta f(x,t)|>\tau\}} dm_\alpha(x,t)\Bigg)^{1/q} \le J_1+J_2+J_3 \equiv$$

$$\equiv \Bigg(\int\limits_{\{(x,t)\in\mathbb{K}\colon |(x,t)|^{-\lambda}\int_{B_{\frac{1}{2}|(x,t)|}}|f(y,s)|\,T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q}dm_\alpha(y,s)>\tau/3\}} dm_\alpha(x,t)\Bigg)^{1/q}+$$

$$+\Bigg(\int\limits_E dm_\alpha(x,t)\Bigg)^{1/q}+$$

$$+\Bigg(\int\limits_{\{(x,t)\in\mathbb{K}\colon |(x,t)|^{-\lambda}\int_{\complement_{B_{2|(x,t)|}}}|f(y,s)|\,T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q}dm_\alpha(y,s)>\tau/3\}} dm_\alpha(x,t)\Bigg)^{1/q}.$$

Then it is clear that

$$J_1 \le \Bigg(\int\limits_{\{(x,t)\in\mathbb{K}\colon 2^{Q-\beta}|(x,t)|^{\beta-Q-\lambda}Hf(x,t)>\tau/3\}} dm_\alpha(x,t)\Bigg)^{1/q}.$$

Further, taking into account the inequality $-\lambda q<(Q-\beta)q-Q$ (*i.e.*, $\beta< Q-Q/q+\lambda$) we have

$$\int\limits_{\complement_{B_r}} |(x,t)|^{(-\lambda+\beta-Q)q}dm_\alpha(x,t) = C_1^q\, r^{(-\lambda+\beta-Q)q+Q}.$$

By the condition $\mu\le 0$ it follows that $\sup\limits_{B_r}|(x,t)|^{-\mu}=r^{-\mu}$.

Summarizing these estimates we find that

$$\sup_{r>0}\Bigg(\int\limits_{\complement_{B_r}} |(x,t)|^{(-\lambda+\beta-Q)q}dm_\alpha(x,t)\Bigg)^{1/q}\sup_{B_r}|(x,t)|^{-\mu}=$$
$$=C_1\sup_{r>0} r^{Q/q-\lambda+\beta-Q-\mu}<\infty \Longleftrightarrow$$
$$\Longleftrightarrow \beta-\mu-\lambda=Q-Q/q.$$

Now in the case $p=1$ the first part of Theorem A gives us the inequality

$$J_1\le \frac{C_{16}}{\tau}\int\limits_{\mathbb{K}} |(x,t)|^{\mu}|f(x,t)|^p dm_\alpha(x,t),$$

where the positive constant C_{16} does not depend on f.

Further, we have

$$J_3 \le \left(\int\limits_{\{(x,t)\in\mathbb{K}:\, 2^{Q-\beta}|(x,t)|^{-\lambda}H'(|f(y,s)||(y,s)|^{\beta-Q})(x,t)>\tau/3\}} dm_\alpha(x,t) \right)^{1/q}.$$

Taking into account the inequality $-\lambda q > -Q$ (*i.e.*, $\lambda < Q/q$) we get

$$\int\limits_{B_r} |(x,t)|^{-\lambda q} dm_\alpha(x,t) = C_{17}^q r^{-\lambda q+Q},$$

where the positive constant C_{17} depends only on β and λ. Analogously, by virtue of the condition $\mu \ge \beta - Q$ it follows that

$$\sup_{\complement B_r} |(x,t)|^{-\mu+\beta-Q} = r^{-\mu+\beta-Q}.$$

Then we obtain

$$\sup_{r>0} \left(\int\limits_{B_r} |(x,t)|^{-\lambda q} dm_\alpha(x,t) \right)^{1/q} \sup_{\complement B_r} = |(x,t)|^{-\mu+\beta-Q} =$$
$$= C_{17} \sup_{r>0} r^{Q/q-\lambda+\beta-Q-\mu} < \infty \Longleftrightarrow$$
$$\Longleftrightarrow \beta - \mu - \lambda = Q - Q/q.$$

Now in the case $p = 1$, from the second part of Theorem A we get the inequality

$$J_3 \le \frac{C_{18}}{\tau} \int\limits_{\mathbb{K}} |(x,t)|^\mu |f(x,t)| dm_\alpha(x,t),$$

where the positive constant C_{18} does not depend on f.

We now estimate J_2. Let

$$E_{1,k} = \Bigg\{ (x,t) \in D_k :$$
$$|(x,t)|^{-\lambda} \int\limits_{B_{2|(x,t)|}\setminus B_{\frac{1}{2}|(x,t)|}} |f(y,s)|\, T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q} dm_\alpha(y,s) > \tau/3 \Bigg\}$$

and

$$E_{2,k} = \Bigg\{ (x,t) \in D_k :$$
$$\int\limits_{B_{2|(x,t)|}\setminus B_{\frac{1}{2}|(x,t)|}} |f(y,s)||(y,s)|^\mu\, T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-\mu-\lambda-Q} dm_\alpha(y,s) > c\tau \Bigg\}.$$

From $\mu+\lambda \geq 0$ and Theorem D, we get

$$J_2 = \left(\sum_{k\in\mathbb{Z}} \int\limits_{E_{1,k}} dm_\alpha(x,t) \right)^{1/q} \leq$$

$$\leq \left(\sum_{k\in\mathbb{Z}} \int\limits_{E_{2,k}} dm_\alpha(x,t) \right)^{1/q} \leq$$

$$\leq \left(\sum_{k\in\mathbb{Z}} \int\limits_{\{(x,t)\in D_k : \left| I_{\beta-\mu-\lambda}\left(f\,|\cdot|^{\mu}\,\chi_{\widetilde{D_k}}\right)(x,t)\right| > c\tau\}} dm_\alpha(x,t) \right)^{1/q} \leq$$

$$\leq \left(\sum_{k\in\mathbb{Z}} \left(\frac{C_{19}}{\tau} \int\limits_{\widetilde{D_k}} |f(x,t)||(x,t)|^{\mu}\, dm_\alpha(x,t) \right)^{q} \right)^{1/q} \leq$$

$$\leq \left(\frac{C_{20}}{\tau} \int\limits_{\mathbb{K}} |(x,t)|^{\mu} |f(x,t)| dm_\alpha(x,t) \right)^{1/q}. \qquad \square$$

Proof of Theorem 3. Sufficiency part of Theorem 3 follows from Theorems 1 and 2.

Necessity. 1) Suppose that the operators $\mathcal{I}_\beta$ and I_β are bounded from $L_{p,|\cdot|^{\mu}}$ to $L_{q,|\cdot|^{-\lambda}}(\mathbb{K})$ and $1<p<Q/(\beta-\mu-\lambda)$.

Define $f^r(x,t) =: f(\delta_r(x,t))$ for $r>0$. Then it can be easily shown that

$$\|f^r\|_{L_{p,|\cdot|^{\mu}}(\mathbb{K})} = r^{-\frac{Q}{p}-\mu} \|f\|_{L_{p,|\cdot|^{\mu}}(\mathbb{K})},$$
$$(I_\beta f^r)(x,t) = r^{-\beta} I_\beta f(\delta_r(x,t)),$$
$$(\mathcal{I}_\beta f^r)(x,t) = r^{-\beta} \mathcal{I}_\beta f(\delta_r(x,t)),$$

and

$$\|I_\beta f^r\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})} = r^{-\beta-\frac{Q}{q}+\lambda} \|I_\beta f\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})}$$
$$\|\mathcal{I}_\beta f^r\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})} = r^{-\beta-\frac{Q}{q}+\lambda} \|\mathcal{I}_\beta f\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})}.$$

From the boundedness of $\mathcal{I}_\beta$, we have

$$\|\mathcal{I}_\beta f\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})} \leq C\|f\|_{L_{p,|\cdot|^{\mu}}(\mathbb{K})},$$

where C does not depend on f. Then we get

$$\|\mathcal{I}_\beta f\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})} = r^{\beta+Q/q-\lambda} \|\mathcal{I}_\beta f^r\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})} \leq$$
$$\leq C r^{\beta+Q/q-\lambda} \|f^r\|_{L_{p,|\cdot|^{\mu}}(\mathbb{K})} =$$
$$= C r^{\beta+Q/q-\lambda-Q/p-\mu} \|f\|_{L_{p,|\cdot|^{\mu}}(\mathbb{K})}.$$

If $\frac{1}{p}-\frac{1}{q}<\frac{\beta-\mu-\lambda}{Q}$, then for all $f\in L_{p,|\cdot|^{\mu}}$ we have $\|\mathcal{I}_{\beta}f\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})}=0$ as $r\to 0$.

If $\frac{1}{p}-\frac{1}{q}>\frac{\beta-\mu-\lambda}{Q}$, then for all $f\in L_{p,|\cdot|^{\mu}}$ we have $\|\mathcal{I}_{\beta}f\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})}=0$ as $r\to\infty$.

Therefore we obtain the equality $\frac{1}{p}-\frac{1}{q}=\frac{\beta-\mu-\lambda}{Q}$. Analogously we get the last equality for I_{β}.

2) The proof of necessity for the case 2) is similar to that of the case 1); therefore we omit it.

3) Let $f\in L_{p,|\cdot|^{\mu}}$, $1<p=Q/(\beta-\mu-\lambda)$. For given $r>0$ we denote

$$f_1(x,t)=\big(f\chi_{B_{2r}}\big)(x,t),\quad f_2(x,t)=f(x,t)-f_1(x,t), \tag{12}$$

where $\chi_{B_{2r}}$ is the characteristic function of the set B_{2r}. Then

$$\widetilde{I}_{\beta}f(x,t)=\widetilde{I}_{\beta}f_1(x,t)+\widetilde{I}_{\beta}f_2(x,t)=F_1(x,t)+F_2(x,t),$$

where

$$F_1(x,t)=\int\limits_{B_{2r}}\Big(T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q}-|(y,s)|^{\beta-Q}\chi_{\complement_{B_1}}(y,s)\Big)f(y,s)dm_{\alpha}(y,s),$$

and

$$F_2(x,t)=\int\limits_{\complement_{B_{2r}}}\Big(T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q}-|(y,s)|^{\beta-Q}\chi_{\complement_{B_1}}(y,s)\Big)f(y,s)dm_{\alpha}(y,s).$$

Note that the function f_1 has compact (bounded) support and thus

$$a_1=-\int\limits_{B_{2r}\setminus B_{\min\{1,2r\}}}|(y,s)|^{\beta-Q}f(y,s)dm_{\alpha}(y,s)$$

is finite.

Note also that

$$\begin{aligned}F_1(x,t)-a_1&=\int\limits_{B_{2r}}T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q}f(y,s)dm_{\alpha}(y,s)-\\&-\int\limits_{B_{2r}\setminus B_{\min\{1,2r\}}}|(y,s)|^{\beta-Q}f(y,s)dm_{\alpha}(y,s)+\\&+\int\limits_{B_{2r}\setminus B_{\min\{1,2r\}}}|(y,s)|^{\beta-Q}f(y,s)dm_{\alpha}(y,s)=\\&=\int\limits_{\mathbb{K}}T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q}f_1(y,s)dm_{\alpha}(y,s)=I_{\beta}f_1(x,t).\end{aligned}$$

Therefore

$$|F_1(x,t)-a_1| \le \int_{\mathbb{K}} |(y,s)|^{\beta-Q} T^{(\alpha)}_{(y,s)} |f_1(x,t)| \, dm_\alpha(y,s) =$$

$$= \int_{B_{2r}(x,t)} |(y,s)|^{\beta-Q} T^{(\alpha)}_{(y,s)} |f(x,t)| dm_\alpha(y,s).$$

Further, for $(x,t) \in B_r$, $(y,s) \in B_{2r}(x,t)$ we have

$$|(y,s)| \le |(x,t)| + |(x-y,t-s)| < 3r.$$

Consequently, for all$(x,t) \in B_r$ we have

$$|F_1(x,t)-a_1| \le \int_{B_{3r}} |(y,s)|^{\beta-Q} T^{(\alpha)}_{(y,s)} |f(x,t)| dm_\alpha(y,s). \tag{13}$$

By Theorem C and inequality (13), for $(\beta-\mu-\lambda)p = Q$ we have

$$r^{-Q-\lambda} \int_{B_r} \left| T^{(\alpha)}_{(z,l)} F_1(x,t) - a_1 \right| dm_\alpha(z,l) \le$$

$$\le Cr^{-Q-\lambda} \int_{B_r} T^{(\alpha)}_{(z,l)} \Bigg(\int_{B_{3r}} |(y,s)|^{\beta-Q} T^{(\alpha)}_{(y,s)} |f(x,t)| dm_\alpha(y,s) \Bigg) dm_\alpha(z,l) \le$$

$$\le Cr^{\beta-Q-\lambda} \cdot r^{Q/p'} \Bigg(\int_{B_r} T^{(\alpha)}_{(z,l)} \left(M(f(x,t))\right)^p dm_\alpha(z,l) \Bigg)^{1/p} \le$$

$$\le Cr^{\mu} \Bigg(\int_{B_r} T^{(\alpha)}_{(z,l)} \left(M(f(x,t))\right)^p dm_\alpha(z,l) \Bigg)^{1/p} \le$$

$$\le C \Bigg(\int_{B_r} |(z,l)|^{\mu p}_{\mathbb{K}} \; T^{(\alpha)}_{(z,l)} \left(M(f(x,t))\right)^p dm_\alpha(z,l) \Bigg)^{1/p} =$$

$$= C \Bigg(\int_{\mathbb{K}} T^{(\alpha)}_{(z,l)} \left(\chi_{B_r} |\cdot|^{\mu p}\right)(x,t) \;\; \left(M(f(z,l))\right)^p dm_\alpha(z,l) \Bigg)^{1/p} \le$$

$$\le C \Bigg(\int_{\mathbb{K}} |(z,l)|^{\mu p}_{\mathbb{K}} \left(M(f(z,l))\right)^p dm_\alpha(z,l) \Bigg)^{1/p} \le$$

$$\le C \|f\|_{L_{p,|\cdot|^{\mu}}(\mathbb{K})}. \tag{14}$$

Denote

$$a_2 = \int_{B_{\max\{1,2r\}} \setminus B_{2r}} |(y,s)|^{\beta-Q} f(y,s) dm_\alpha(y,s)$$

and estimate $|F_2(x,t)-a_2|$ for $(x,t)\in B_r$:

$$|F_2(x,t)-a_2| \leq \int\limits_{\complement B_{2r}} |f(y,s)| \left|T^{(\alpha)}_{(y,s)}|(x,t)|^{\beta-Q} - |(y,s)|^{\beta-Q}\right| dm_\alpha(y,s).$$

Applying Lemma 4 and Hölder's inequality we get

$$\begin{aligned}|F_2(x,t)-a_2| &\leq 2^{Q-\beta+1}|(x,t)| \int\limits_{\complement B_{2r}} |f(y,s)||(y,s)|^{\beta-Q-1} dm_\alpha(y,s) \leq \\ &\leq 2^{Q-\beta+1}|(x,t)|\Bigg(\int\limits_{\complement B_r} |(y,s)|^{\mu p}|f(y,s)|^p dm_\alpha(y,s)\Bigg)^{1/p} \times \\ &\times \Bigg(\int\limits_{\complement B_r} |(y,s)|^{(-\mu+\beta-Q-1)p'} dm_\alpha(y,s)\Bigg)^{1/p'} \leq \\ &\leq C|(x,t)| r^{\beta-\mu-1-Q/p}\|f\|_{L_{p,|\cdot|^\mu}(\mathbb{K})} \leq \\ &\leq C|(x,t)| r^{\lambda-1}\|f\|_{L_{p,|\cdot|^\mu}(\mathbb{K})} \leq \\ &\leq C r^{\lambda}\|f\|_{L_{p,|\cdot|^\mu}(\mathbb{K})}.\end{aligned}$$

Note that if $|(x,t)|<r$ and $|(z,l)|_{\mathbb{K}}<2r$, then $T^{(\alpha)}_{(z,l)}|(x,t)| \leq |(x,t)| + |(z,l)|_{\mathbb{K}} \leq 3r$. Thus for $(\beta-\mu-\lambda)p = Q$ we obtain

$$\begin{aligned}\left|T^{(\alpha)}_{(z,l)}F_2(x,t)-a_2\right| &\leq T^{(\alpha)}_{(z,l)}\,|F_2(x,t)-a_2| \leq \\ &\leq C r^{\lambda}\|f\|_{L_{p,|\cdot|^\mu}(\mathbb{K})} \leq \\ &\leq C|(x,t)|^{\lambda}\|f\|_{L_{p,|\cdot|^\mu}(\mathbb{K})}.\end{aligned} \tag{15}$$

Denote

$$a_f \equiv a_1 + a_2 = \int\limits_{B_{\max\{1,2r\}}} |(y,s)|^{\beta-Q} f(y,s)\, dm_\alpha(y,s).$$

Finally, from (14) and (15) we have

$$\sup_{(x,t)\in\mathbb{K}, r>0} r^{-Q-\lambda} \int\limits_{B_r} \left|T^{(\alpha)}_{(y,s)}\widetilde{I}_\beta f(x,t) - a_f\right| dm_\alpha(y,s) \leq C\|f\|_{L_{p,|\cdot|^\mu}(\mathbb{K})}.$$

Thus

$$\begin{aligned}\left\|\widetilde{I}_\beta f\right\|_{BMO_{|\cdot|^{-\lambda}}(\mathbb{K})} &\leq 2C \sup_{(x,t)\in\mathbb{K}, r>0} r^{-Q-\lambda}\int\limits_{B_r}\left|T^{(\alpha)}_{(y,s)}\widetilde{I}_\beta f(x,t)-a_f\right| dm_\alpha(y,s) \leq \\ &\leq C\|f\|_{L_{p,|\cdot|^\mu}(\mathbb{K})}.\end{aligned}$$

□

Proof of Theorem 5. By the definition of the weighted B-Besov spaces it suffices to show that

$$\|T^{(\alpha)}_{(y,s)} I_\beta f - I_\beta f\|_{L_{q,|\cdot|^{-\lambda}}(\mathbb{K})} \leq C \|T^{(\alpha)}_{(y,s)} f - f\|_{L_{p,|\cdot|^{\mu}}(\mathbb{K})}.$$

It is easy to see that $T^{(\alpha)}_{(y,s)}$ commutes with I_β, i.e., $T^{(\alpha)}_{(y,s)} I_\beta f = I_\beta(T^{(\alpha)}_{(y,s)} f)$. Hence we obtain

$$|T^{(\alpha)}_{(y,s)} I_\beta f - I_\beta f| = |I_\beta(T^{(\alpha)}_{(y,s)} f) - I_\beta f| \leq I_\beta(|T^{(\alpha)}_{(y,s)} f - f|).$$

Taking $L_{q,|\cdot|^{-\lambda}}(\mathbb{K})$-norm on both sides of the last inequality, we obtain the desired result by using the boundedness of I_β from $L_{p,|\cdot|^{\mu}}(\mathbb{K})$ to $L_{q,|\cdot|^{-\lambda}}(\mathbb{K})$.

From Theorem 5 we get the following result on the boundedness of I_β on the B-Besov spaces $B^s_{p\theta}(\mathbb{K}) \equiv B^s_{p\theta,1}(\mathbb{K})$ on the Laguerre hypergroups $\mathbb{K}$. □

Corollary 4. *Let* $0 < \beta < Q$, $1 < p < \frac{Q}{\beta}$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{Q}$, $1 \leq \theta \leq \infty$ *and* $0 < s < 1$. *Then the operator* I_β *is bounded from* $B^s_{p\theta}(\mathbb{K})$ *to* $B^s_{q\theta}(\mathbb{K})$. *More precisely, there is a constant* $C > 0$, *such that,*

$$\|I_\beta f\|_{B^s_{q\theta}(\mathbb{K})} \leq C \|f\|_{B^s_{p\theta}(\mathbb{K})}$$

holds for all $f \in B^s_{p\theta}(\mathbb{K})$.

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References

1. Miloud Assal and Hacen Ben Abdallah, Generalized Besov type spaces on the Laguerre hypergroup. *Ann. Math. Blaise Pascal*, **12** (2005), No. 1, 117–145.
2. H. Bahouri, P. Gerard and C. J. Xu, Espaces de Besov et estimations de Strichartz generalisees sur le groupe de Heisenberg. (French) *J. Anal. Math.* **82** (2000), 93–118.
3. O. V. Besov, Investigation of a class of function spaces in connection with imbedding and extension theorems. (Russian) *Trudy. Mat. Inst. Steklov.* **60** (1961), 42–81.
4. J. J. Betancor and L. Rodriguez Mesa, On the Besov-Hankel spaces. *J. Math. Soc. Japan*, **50** (1998), No. 3, 781–788.
5. G. Bourdaud, Realisations des espaces de Besov homogenes. (French) *Ark. Mat.* **26** (1988), No. 1, 41–54
6. D. E. Edmunds, V. M. Kokilashvili and A. Meskhi, Boundedness and compactness of Hardy-type operators in Banach function spaces. *Proc. A. Razmadze Math. Inst.* **117** (1998), 7–30.
7. D. E. Edmunds, V. M. Kokilashvili and A. Meskhi, Bounded and compact integral operators. Mathematics and its Applications, 543. *Kluwer Academic Publishers, Dordrecht*, 2002.
8. V. S. Guliyev, Polar coordinates in Laguerre hypergroup. *Khazar Journal of Mathematics*, **2 (3)** (2006), 3–11.
9. V. S. Guliyev, On maximal function and fractional integral, associated with the Bessel differential operator. *Math. Inequal. Appl.* **6** (2003), No. 2, 317–33.

10. V. S. Guliyev and Miloud Assal, On maximal function on the Laguerre hypergroup. *Fract. Calc. Appl. Anal.* **9** (2006), No. 3, 307–318.
11. V. S. Guliyev and M. Omarova, On fractional maximal function and fractional integral on the Laguerre hypergroup. *J. Math. Anal. Appl.* **340** (2008), No. 2, 1058–1068.
12. V. S. Guliyev and M. Omarova, (L_p, L_q) boundedness of the fractional maximal operator on the Laguerre hypergroup. *Integral Transforms Spec. Funct.* **19** (2008), No. 9-10, 633–641.
13. V. S. Guliyev and M. Omarova, On the boundedness of the maximal operators in Besov spaces on the Laguerre hypergroup, *XIII International conference of Mathematics, Baku, November*, 20-23 (2007).
14. V. S. Guliyev, R. Ch. Mustafayev and A. Serbetci, Stein-Weiss inequalities for the fractional integral operators in Carnot groups and applications. *Complex Var. Elliptic Equ.* **55** (2010), No. 8-10, 847–863.
15. J. Huang and H. Liu, The weak type $(1,1)$ estimates of maximal functions on the Laguerre hypergroup. *Canad. Math. Bull.* **53** (2010), No. 3, 491–502.
16. V. Kokilashvili and A. Meskhi, On some weighted inequalities for fractional integrals on nonhomogeneous spaces. *Z. Anal. Anwendungen*, **24** (2005), No. 4, 871–885.
17. M. M. Nessibi and K. Trimeche, Inversion of the Radon transform on the Laguerre hypergroup by using generalized wavelets. *J. Math. Anal. Appl.* **208** (1997), No. 2, 337–363.
18. L. Skrzypczak, Besov spaces and Hausdorff dimension for some Carnot-Caratheodory metric spaces. *Canad. J. Math.* **54** (2002), No. 6, 1280–1304.
19. H. Triebel, Theory of function spaces. Monographs in Mathematics. 78. *Birkhauser Verlag, Basel*, 1983.

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